State of Iowa
1964

MATHEMATICS FOR IOWA SCHOOLS

GRADES 7, 8, and 9

Published by the
STATE OF IOWA
Des Moines

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Iowa Cooperative Curriculum Development Program

Issued by

Iowa State Department of Public Instruction

Published by the
STATE OF IOWA
Des Moines

Price $1.00 per copy. Order from the State Department of Public Instruction, State Office Building, Des Moines, Iowa 50319

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FOREWORD

Anyone who has been responsible for mathematics programs in recent years is aware of the increased interest and activity that has characterized this most important part of the elementary- and secondary-school curriculum. This attention has resulted in changes and revisions as well as proposed changes and proposed revisions that have implications for the mathematics education of the youth of Iowa.

The final responsibility for providing sound programs in this area of the curriculum rests with the local administrator and his staff. This publication has been prepared for the purpose of aiding and guiding them as they study their local programs to revise and upgrade both the mathematics offerings and methodology of their respective schools.

PAUL F. JOHNSTON
State Superintendent of Public Instruction

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Introduction

Modern mathematics is many things. It includes the current ideas about the nature of mathematics, the patterns of thinking developed by examining mathematics as a logical structure, the consequent reorganization of the content and approaches to teaching mathematics. Modern mathematics is contrasted with social mathematics and the rote mathematics that preceded social mathematics. Although mathematics must be applied to everyday things, the applications do not dictate the best organization for its study.

This outline is developed as one part of a K-12 program. Its major purpose is to acquaint teachers and administrators with the principal concepts of modern mathematics, and to provide a guide for the introduction of new materials into the school curriculum. The materials are designed to provide a basic structure for the development of mathematics programs in the classrooms of Iowa, each school adapting the guide to its individual needs. Perhaps teachers will want to adopt topics from this outline for use in present courses. Many new ideas are explored here that can be extremely valuable as a supplement and guide to the enrichment of the regular mathematics curriculum.

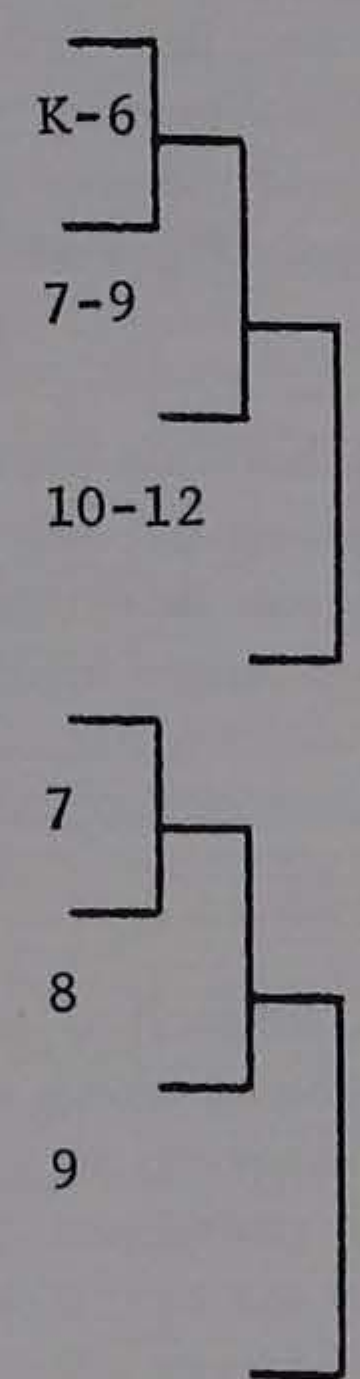
Every teacher is faced with the problem of challenging his pupils to work to the level of their individual abilities. This goal is not readily attainable in the junior high school when pupils study mathematics with an emphasis on social utility. The new approach to the teaching of mathematics has served to promote active interest and participation on the part of pupils and a sense of accomplishment in teachers.

A major problem of schools in building curriculum is that of providing for ability grouping. One of the basic concepts followed in the writing of these materials should be extremely helpful in this regard. The committee believes that all junior high school pupils should follow the same track of sound, well structured mathematics. They should, however, proceed along this track at different rates of speed according to their individual abilities.

It is anticipated that pupils of higher ability may complete this program at the end of the ninth grade, including the suggested supplementary materials.

It has been found that lower-level pupils also appreciate the sound, logical approach which is inherent in modern mathematics. It is the goal of the committee that all pupils will study the materials through TOPIC X by the end of the eighth grade. At this point it is recommended that a separate program be developed for the slower pupils. This general mathematics program, as envisioned by the committee should at least be a two-year sequence of mathematics at the ninth and twelfth grade levels.

The major outline for the K-12 program is developed with the following pattern in mind.



Note the allowance for variation in the amount of work completed. This same idea is used within the structure of the junior high school materials.

The normal program for seventh grade includes TOPICS I through VI. TOPICS VII through XI are included in the eighth-grade program. The balance of the TOPICS XII through XVI are planned for ninth-grade study.

In making use of this material, teachers should keep in mind that it was written as a guide, emphasizing the structure of a sound program in modern mathematics. The topics have been thoughtfully ordered to take advantage of the logical development of our system of mathematics. The references have been carefully selected from the limited materials that are available dealing with the new mathematics. Discussion questions have been suggested as an aid in the teaching of the topics. Examples of appropriate problems and exercises have been planned to be most beneficial to teachers trying the new approach for the first time. While it is possible to select certain topics from this guide to supplement conventional courses in mathematics, this procedure is not likely to allow the teacher to fully appreciate the power which new programs instill in school pupils. To profit, most pupils must study mathematics as a logically structured system, and teachers must take advantage of modern learning theory which is best described as *discovery learning.*

It is always enjoyable to teach a curriculum which is inspiring and meaningful to school youth. We believe that this is such a program. More than that, it provides a valuable enrichment for both teachers and pupils. The mathematics of this outline is developed as part of a logically interrelated whole. Those who pursue this program of modern mathematics will gain confidence, understanding, and enjoyment.

TOPIC I

Systems For Recording Numbers*

Suggested time allotment: 2-3 weeks

This first topic for junior high mathematics is an exploratory one, a study of different numeration systems. By investigating the properties of numeration systems other than base ten, it is felt that pupils will gain a better understanding of the decimal (base ten) system of recording numbers.

Additional benefits from this topic would be:

1. To implement the pupil's concept of the difference between "number" and "numeral"
2. To discover anew the meaning of place-value
3. To help pupils like mathematics through arousing curiosity and interest
4. To provide a "painless" review of addition, multiplication, subtraction, and division
5. To help pupils appreciate the place of mathematics in the humanities as they study historically the early numeration systems
6. To stimulate pupils to do creative projects related to numeration systems
7. To convince pupils that in many phases today's mathematics is a reapplication of the concepts of the ancients
8. To require pupils to think the way mathematicians think about mathematics; to investigate patterns and to attempt to make valid generalizations about these patterns
9. To help pupils see a distinction between the properties of "number" systems as contrasted with "numeration" systems

The Use of Exponents

The use of exponents can be introduced by using the number 2 as the base for the exponent. Ultimately, the pupil should be guided to accept the idea that any number, $x \neq 0$ to the zero power, is 1 ($x^0=1$).

Having used exponents with a base of 2, pupils will be better able to understand our decimal system of notation. Much time should be spent expressing numbers as sums of appropriate powers of ten. This should further aid the pupil to understand the decimal system. Here also is an appropriate place for thorough review of fundamental arithmetic operations. Various ways of performing the basic computations such as different methods of multiplication and the subtractive method division should be illustrated. These methods can be found in several of the more recent elementary mathematics texts listed in the bibliography for this topic.

It is recommended that work be done in base five and base two numeration. It is not essential that five be used as a base, but base two certainly should be used to point out the application of the binary system in digital computers.

TOPIC I has much historical implication. The pupils should see how the "name changing" processes of computation are easier with one system than another. For example the Hindu arabic numeral has some advantage over the Roman numeral. Likewise, the base ten numeration system has advantage over the Roman system.

Before the topic is finished, various number and exponential patterns should be pointed out or observed. The computations within any system are basically similar. Commutativity and associativity of operations should be seen. (These names need not be used at this time.) The foregoing properties should be observed in multiplication and addition tables for every numeration system. The value of pointing them out at this time will become apparent to the pupils as the topic progresses and the power of these generalizations is developed through the study of various numeration systems.

*Topic References: Numbers 18, 30, 36

Suggested Developmental Questions

The following are some questions which the teacher may ask the pupils during the development of this topic:

1. What is another way of saying
   $5+5+5$?
   $3+3+3+3$?
   in general terms:
   $A+A$?
   $A+A+A$?
2. Can you find a new way of saying
   4×4?
   $5\times5\times5$?
   $8\times8\times8\times8\times8$?
   $a\times a\times a$?
3. What is another name for
   2^4; 2^3; 2^2; 2^1; 2^0?
   2×4; 2×3; 2×2; 2×1; 2×0?
   5^4; 5^3; 5^2; 5^1; 5^0?
   5×4; 5×3; 5×2; 5×1; 5×0?
4. What is another name for
   $2^4\times2^3$; $2^5\times2^0$; $2^3\times2^2$; $2^1\times2^6$?
   $2^6\div2^1$; $2^8\div2^2$; $2^4\div2^3$; $2^7\div2^7$?
5. What is another name for
   10^0; 10^1; 10^2; 10^3; 10^4; 10^5?
6. Express the following numbers in a different way.
   $10^2+10^1+10^0=$
   $(2\times10^2)+(3\times10^1)+(2\times10^0)=$
   $=54$
   $=431$
   $=6{,}205$
7. How many number symbols are used to form all the numbers we use? What are they? Why do you suppose there are ten?
8. How many weeks and days are there in 8 days? 13 days? 21 days? 26 days?
9. Complete the following chart by writing the missing name for the number in the blank.

Base Ten	Base Five	Base Ten	Base Five
0	0	8	—
1	1	16	—
2	2	—	42
3	3	23	—
4	4	24	44
5	10	25	100
—	11	28	—
—	12	—	110

10. Write the base ten name for each of these numbers.
    1. 41_5
    2. 14_5
    3. 102_5
    4. 422_5
    5. 210_5
    6. $1{,}010_5$
11. Divide each by 5.
    1. 21
    2. 9
    3. 27
    4. 112
    5. 55
    6. 130
12. How many symbols would we have using a base two for our numeration system?
    Count to 25 using the base two numeration system.
13. Could you use the numerial '12' as a base for our decimal numeration system?
    What difficulties would you encounter?

Discussion of Developmental Questions

The following is a possible development of the ideas involved in questions three and four listed previously.

Before this topic is started, it is essential that the pupil know the meaning of an exponent and how it can be used. The problems listed are illustrative. The purposes of this section should be clear:

1. To establish the value of x^0 through investigation of patterns.
2. To show that x^n is not generally equal to $n \cdot x$.
3. To show that exponents can be a useful aid in many calculations.

To achieve the first purpose, it is suggested that the teacher tabulate the answers to the following problems. The pupil can readily be led to see the pattern of the powers.

$2^4=16$	$3^4=81$	$1^{10}=1$
$2^3=8$	$3^3=27$	$1^8=1$
$2^2=4$	$3^2=9$	$1^5=1$
$2^1=2$	$3^1=3$	$1^2=1$
$2^0=$?	$3^0=$?	$1^0=$?

It should become apparent that each answer is obtained from the previous answer through division by 2, 3, and 1, respectively. What then must we have as an answer to 2^0, 3^0, 1^0, 10^0, $1{,}500^0$, x^0?

Here is a real opportunity to illustrate to pupils how mathematics evolves. The answer to new problems is made consistent with the information obtained from previous problems. The bright pupil might easily ask at this time—what is the value of 2^{-1}, 2^{-2}, . . .?

To achieve the second purpose listed, the teacher tabulates answers to a series of problems:

$2^2=4$ $2\times2=4$ $3^2=9$ $3\times2=6$
$2^3=8$ $2\times3=6$ $3^4=81$ $3\times4=12$
$2^4=16$ $2\times4=8$

Numerous exercises illustrating this idea should be given at this time.

The third purpose listed is not quickly achieved but should be pointed out to the pupils throughout the topic study.

By observing powers of 2 as listed, the pupil should see how multiplication and division could be done.

$2^0=1$	$2^6=64$	$2^{12}=4{,}096$
$2^1=2$	$2^7=128$	$2^{13}=8{,}192$
$2^2=4$	$2^8=256$	$2^{14}=16{,}384$
$2^3=8$	$2^9=512$	$2^{15}=32{,}768$
$2^4=16$	$2^{10}=1{,}024$	
$2^5=32$	$2^{11}=2{,}048$	

Problems to be worked:

$2^5\times2^2=128\ =2^7$ $\qquad 2^{12}\div2^3=512=2^9$

$2^8\times2^3=2{,}048=2^{11}$ $\qquad 2^3\div2^3=\ 1=2^0$

$2^6\times2^6=4{,}096=2^{12}$ $\qquad 2^{11}\div2^5=\ 64=2^6$

Here the pupils are using the method of logarithmic computation. The amount of time spent on the study of computation at this time will vary according to class interest and ability, but all should be able to gain some insights into this method of multiplication and division.

Question six will provide a method of review of basic arithmetical skills through the study of scientific notation in the decimal numeration system.

A few examples are given below:

$727=7\times10^2+2\times10^1+7\times10^0$

$4{,}021=4\times10^3+0\times10^2+2\times10^1+1\times10^0$

$84=8\times10^1+4\times10^0$

Investigating these patterns leads naturally into standard notation. Standard notation is not only an excellent device for giving the pupil further understanding of exponents but also provides a tool for his work in science. Only positive whole numbers and zero should be used as exponents at this time. Later, after negative numbers have been introduced in TOPIC VII, negative exponents can be introduced.

Both very large and very small numbers should be investigated and represented by standard notation. To express a number in standard notation, use two factors. The first factor is a number between 1 and 10. The second factor is a power of ten in exponent form.

$400{,}000{,}000\ =4\ \times \qquad 100{,}000{,}000=4\ \times10^8$

$30{,}600{,}000\ =3.06\times \qquad 10{,}000{,}000=3.06\times10^7$

$.0000034\ =3.4\ \times \qquad \frac{1}{1{,}000{,}000}=3.4\ \times\frac{1}{10^6}$

$.0000000000593=5.93\times \qquad \frac{1}{100{,}000{,}000{,}000}=5.93\times\frac{1}{10^{11}}$

Examples from science should be used for practice in computation using standard notation. For example: The mass of the earth is estimated at 6×10^{21} tons. The mass of the sun is about 3.3×10^5 times that of the earth. What is $(3.3\times10^5)\times(6\times10^{21})$? What property of numbers allows us to rewrite this example as follows?

$$(3.3\times6)\times(10^5\times10^{21})$$

Carry out the computation and express the mass of the sun in standard notation.

Following is a possible development of the ideas involved in questions nine, ten, and eleven. The work on these questions should show the pupil how to make changes from one numeration system to another. In the introduction to this work, it may be wise to ask the pupil such a question as how he would count if man had had only one hand. Intuitively, the pupil should soon be able to fill in the missing parts of question nine. This work should be extended further in base five and other bases of numeration.

The work of question ten and eleven is intended to show the pupil how to change from a base ten numeral to a base five numeral. Here again much practice will be needed and the division process can be used to further strengthen the pupil's efficiency in the operation of division.

Using the above ideas, the pupil can gain a firmer understanding of arithmetic computation.

Consider the following examples:

$$\begin{aligned}68+95&=(6\times10^1+8\times10^0)+(9\times10^1+5\times10^0)\\&=(6\times10^1+8\times10^0)+9\times10^1+5\times10^0\\&=9\times10^1+(6\times10^1+8\times10^0)+5\times10^0\\&=(9\times10^1+6\times10^1)+8\times10^0+5\times10^0\\&=[(9+6)\times10^1]+[(8+5)\times10^0]\\&=15\times10^1+13\times10^0\\&=150+13\\&=163\end{aligned}$$

Teachers will recognize the use of the commutative and associative properties of addition used above. At this time, most pupil work ought to be done as follows:

$$\begin{aligned}302&=\ 3\times10^2+0\times10^1+2\times10^0\\&=\ \underline{7\times10^2+2\times10^1+5\times10^0}\\&=\ 10\times10^2+2\times10^1+7\times10^0\\&=1{,}000+20\ +7\\&=1{,}027\end{aligned}$$

Here is a possible method of multiplication:

$$\begin{aligned}69\times57&=(60+9)\times(50+7)\\&=(60+9)\times50+(60+9)\times7\\&=60\times50+9\times50+60\times7+9\times7\\&=3{,}000+450+420+63\\&=3{,}933\end{aligned}$$

Obviously, not all computational work needs to be done as illustrated here. However, the methods demonstrated will give the pupils deeper understanding of the operations of arithmetic.

It is quite natural that some computations should be done with base five numerals. Addition and multiplication tables may be set up for the pupil's use.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Now many simple problems can be done by the pupils.

$$\begin{array}{r} 43_5 \\ +\;311_5 \\ \hline 404_5 \end{array} \qquad \begin{array}{r} 323_5 \\ 42_5 \\ \hline 1201 \\ 2402 \\ \hline 30{,}221_5 \end{array}$$

These calculations can be checked using base ten numerals. Most pupils should be fascinated by the operations in a different numeration system. This work will also aid slow pupils in the strengthening of their arithmetic skills by giving them insights in the development of a system, its numeration, and operations.

It is also possible to do some work with subtraction and division with base five numerals. However, not too much time should be spent on any of these operations. The main idea is to build better understanding of the base ten numeration system and the operations with base ten numerals.

It should be noticed that, in the construction of any addition and multiplication table, the commutative properties are well illustrated. The pupils should be led to see these generalizations.

The topic described above is rich in mathematical content. It is not intended that every class should do everything suggested. However, every pupil in every class should gain important understandings. They should understand the meaning and notation of exponents. They should understand our decimal system of numeration and its operations. They should know that man is not restricted to base ten in expressing number values.

In this topic the foundation is laid for further understanding of the basic properties of our number system (commutativity, associativity, and distributivity, etc.).

Numerous materials are available for extensive study of this topic. Certainly the teacher should study and use these materials to further aid the pupil in his appreciation of and learning about mathematics.

TOPIC II

Sentences Which Describe Sets of Natural Numbers and Zero*

Suggested time allotment: 6-7 weeks

Mathematical sentences such as $n+4=6$, $n+7>7$, $n+5\neq10$, and $n-2<6$ are introduced early in the seventh grade, primarily to give the pupil a tool for problem solving. The sentence (equation or inequality) in mathematical symbolism is a visually apparent representation of the structure of the problem. Finding true replacements from a meaningful set of numbers gives the pupil a solution satisfying the conditions of the problem.

By investigating open sentences (containing variables) that are true for *some* number replacements, for *no* number replacements, and for *every* number replacement, the pupils realize the importance of those which are true for every replacement. The ideas expressed by these sentences are called *properties* of the number system. Names are given to these properties and they are used in proofs in this topic. In fact, all through the mathematics program these properties give reasons for steps which formerly seemed to be mechanical manipulations.

Another reason for using mathematical sentences is to give the pupil an appreciation for the structure of our man-made number system. Certain groups of sentences create a need for "new" sets of numbers. This progressive expansion of the universal set of numbers continues throughout the pupil's mathematical program in the junior- and senior-high school. These "new" sets of numbers are introduced in TOPICS II, IV, VII, and XIV. Sentences like the following lead to a study of particular sets of numbers.

Seventh grade:

$n+4=6$	natural numbers	(TOPIC II)
$n+7=7$	zero	(TOPIC II)
$2n=1$	rational numbers	(TOPIC IV)

Eighth grade:

$n+5=3$	negative integers	(TOPIC VII)
$2n={}^{-}1$	negative rationals	(TOPIC VII)
$n \cdot n=4$	positive and negative rationals	(TOPIC VII)
$n \cdot n=3$	positive and negative irrationals	(TOPIC VII)

Ninth grade:

$n \cdot n=3$	positive and negative irrationals	(TOPIC XIV)

Graphical representation of sentences can be made on the number line. The universal set under consideration should be precisely stated and limited to the set whose members are plausible for the stated conditions of the problem situation.

Throughout this topic and the topics to follow, the concepts, language, and symbolism of sets are employed. Not only do the symbols and vocabulary of sets give a precise way of stating conditions about numbers or problem situations, but also the ideas of sets are fundamental ideas in mathematics. The concept of a set, for example, provides a precise meaning to the concept of a mathematical relation. A number system is considered as a set of elements with operations defined on the elements. Sets of points and lines are studied in geometry. Formulas are viewed as special kinds of mathematical sentences that express certain relations.

The pupil should be expected early to become accustomed to using the set builder notation (i.e., $\{x \mid x \text{ is an odd number}\}$). Any symbols such as frames (△, □) or *, or letters of the Greek or English alphabet may be used to hold a place open for a replacement from a specified and meaningful universe.

*Topic References: Numbers 2, 23, 29, 31, 32, 33, 34, 36, 37

Mathematical sentences have no truth value until their placeholders are given replacements.

Examples are: $x+3=15$, $2>n+4$, $3\times\square\neq\square$.
In order for sentences to express true or false ideas, a replacement for the placeholder must be made. This replacement is always a member of the universe being considered.

Those replacements from a specified universal set which make the sentence true are members of the *solution set*. The graphic representation of solution sets can be made on the number line of natural numbers and zero for this topic.

$\{n \mid 2n=6\}$ is graphed as:

```
·  ·  ·  •  ·  ·  ·  ·  ·  ·  ·  ·
0  1  2  3  4  5  6  7  8  9  10
```

$\{n \mid 2n>6\}$ is graphed as:

```
·  ·  ·  ⊙  •  •  •  •  •  •  •  •  •  •
0  1  2  3  4  5  6  7  8  9  10
```

$\{n \mid 2n<6\}$ is graphed as:

```
•  •  •  ⊙  ·  ·  ·  ·  ·  ·  ·
0  1  2  3  4  5  6  7  8  9  10
```

By comparing the three graphs the pupil is led to see the solution set of the equation in relation to the solution sets of the corresponding inequalities. The *union* of the three solution sets is the set of all natural numbers and zero.

Later this same concept carries over into the two-dimensional work on the coordinate plane when the pupil views the graph of $y=2x+3$ as the boundary between the graphs of $y>2x+3$ and $y<2x+3$.

$\{a \mid a=15-2\}$	$\{n \mid 12+n<20\}$
$\{x \mid x<25\}$	$\{n \mid n-3>5\}$
$\{x \mid 3x<25\}$	$\{n \mid n\cdot n<3\}$
$\{x \mid 2x>5x\}$	$\{x \mid x\cdot x\not<0\}$
$\{y \mid y>9+2\}$	$\{n \mid \frac{1}{2}n<10\}$
$\{* \mid 13-4>*\}$	$\{y \mid \frac{2}{3}y<20\}$
$\{\square \mid 13-4\not>\square\}$	$\{\square \mid \square\neq10\}$
$\{n \mid 13-n=4\}$	

Using a finite universal set such as $U=\{3, 6, 9, 12\}$, replacements could be made in sentences such as $2x+1>19$, $2+3x=8$, and $4x+1>8$.

In all of these exercises, sentences should be included which have empty (null) solution sets. When the replacement set for x is the universe of natural numbers and zero, the solution sets $\{x \mid 2x<0\}$ and $\{x \mid x+\frac{1}{2}=\frac{3}{4}\}$ are empty sets. The symbol for the empty set is either ϕ or $\{\ \}$.

When sentences are encountered that are true for all replacements of natural numbers and zero, pupils should be encouraged to examine the patterns. If no counterexample is discovered and the pattern is one used over and over, then these patterns may be named.

Some of these are patterns recognized as properties of the number system:

Commutative Property for Addition
$[a+b=b+c]$
Commutative Property for Multiplication
$[a\cdot b=b\cdot a]$
Associative Property for Addition
$[a+(b+c)=(a+b)+c]$
Associative Property for Multiplication
$[a\cdot(b\cdot c)=(a\cdot b)\cdot c]$
Distributive Property of Multiplication with Respect to Addition
$[a(b+c)=a\cdot b+a\cdot c]$
Identity Property for Multiplication
$[a\cdot 1=a]$
Identity Property for Addition
$[a+0=a]$
Property of Multiplication by Zero
$[a\cdot 0=0]$.

These properties should then be used in simple two or three step proofs. No doubt this will be the pupils first exposure to proof. There is a continued emphasis on proof throughout the entire junior high school program and on into the senior high school as well. In fact, the extensive use made of deductive reasoning and proof is one of the major differences between the past program of arithmetic and algebra and the improved program of the present.

Finally, mathematical sentences should be employed as a method of expressing the structure of verbal problems and for solving them. This approach is further expanded later in this topic.

Leading questions which a teacher might ask pupils are:

1. Give a mathematical sentence whose solution set is "empty" if the replacement set consists of natural numbers and zero.
2. Give several sentences each of which has solution set $\{5\}$.

3. Give several sentences whose solution set is {0}.
4. How would the graph $x+x>20$ appear on the number line of natural numbers and zero?
5. Betty had 12 phonograph records. She bought 7 more records, but she still had fewer records than Elaine. How many phonograph records can Elaine have?
6. Dick scored 13 points during the first half of a basketball game. He scored fewer than 20 points during the entire game. How many points could he have scored during the second half?
7. What numbers can be multiplied by 3 so that each sum is less than 20?
8. Is $n+5=5+n$ a true sentence when n holds a place for a natural number or zero?
9. For what numbers, n, is the sentence $5\times n=n$ true?
10. How would you group the sentence $20\times 6+3+2\times 4=n$ for accurate solution?

Following is a group of sample lessons developing out of question 4 above. Question 4 was "How would the graph of $x+x>20$ appear on the number line of natural numbers and zero?"

Pupils should be led to see that dots can be used to make a picture of a set of numbers. Each dot pictures a point. (A point is an idea.) Each point is assigned a number so that each number matches a point and each point matches a number. Mathematicians call this matching a *one-to-one correspondence* between points and numbers. The picture looks like this:

· · · · · · · · · · · · ·
0 1 2 3 4 5 6 7 8 9 10 . . .

A line that has points matched with natural numbers is called a "natural-number line." The dots form a picture or graph of the set of natural numbers. Of course, it is impossible to make a dot for each natural number, but the three small dots to the right of the picture of a line indicate that the dots in the graph of the set of natural numbers and zero form an infinite (continuing) set.

Many subsets of the set of natural numbers and zero can be pictured by graphing.

The subset {0, 1, 2, 7} is pictured by encircling the dots.

⊙ ⊙ ⊙ · · · · ⊙ · · ·
0 1 2 3 4 5 6 7 8 9 10

The solution set $\{x|x>6\}$ forms a subset of the universe also. If U={natural numbers and zero}, the points of the solution set are pictured:

· · · · · · · ⊙ ⊙ ⊙ ⊙ ⊙ ⊙ ⊙
0 1 2 3 4 5 6 7 8 9 10 . . .

The solution set $\{x|x>6\}$ is infinite (endless) as indicated by encircling the three small dots to the right of the picture of a line.

Given the same sentence, $x>6$ and the universe {0, 1, 2, 3, . . . 10}, the solution set becomes a finite-subset of the given universe.

Statements of equality can be pictured also. The solution set $\{n|\frac{1}{2}+n=6\frac{1}{2}\}$ is a subset containing only one number, 6. Its graph is:

· · · · · · ⊙ · · · ·
0 1 2 3 4 5 6 7 8 9 10

For conditions such as $n\neq n$ which describe a null set, no dots are encircled.

Pupils should be given many varied experiences in graphing on the natural-number line the solution sets of equalities and inequalities with one variable. It is a concrete, visually apparent device for gaining understanding of our number system. These understandings will be broadened when the pupil works with subsequent number lines: integers, rationals, and reals. A firm foundation is being laid for graphing in two-dimensional space with two variables and later in three-dimensional space with three variables.

Questions 5, 6, and 7 serve to introduce several lessons in solving verbal problems about situations and numbers.

The mathematical sentence can be a powerful tool in problem solving. The sentence gets at the structure of the problem. It becomes simpler and more meaningful to work with the sentence than with the non-mathematical description of the problem.

Question 5 above was, "Betty had 12 phonograph records. She bought 7 more records, but she still had fewer records than Elaine. How many phonograph records can Elaine have?"

In question 5 we do not know the number of members in the set of the number of records Elaine could have so we use n as a placeholder for a name of the number of records she could have. Pupils must be led to recognize the type of action in the problem situation. Buying 7 records would be an additive action. Recognizing this would help us in forming a sentence to represent the situation described:

$12+7<n$. The universe for n, the number of records Elaine has, is $\{13, 14, 15, \ldots\}$.

Pupils should learn to translate the problem situation into mathematical language in the precise order that action occurs. Even though $n>19$ and $19<n$ have the same solution sets, the two sentences do not express the same problem action.

Question 6 was, "Dick scored 13 points during the first half of a basketball game. He scored fewer than 20 points during the entire game. How many points could he have scored during the second half? We do not know how many points he scored during the second half, so we let the placeholder, n, represent that number in the open sentence $13+n<20$. Again the action is additive, so the sentence reflects this. The universe for n is $\{1, 2, 3, \ldots\ldots 19\}$.

Let us consider another problem in which the action is not additive. "Sue made some potholders. She gave 12 of these potholders to her friends and has 6 left. How many did Sue make?" Since the members of one set of objects were not put with the members of another set of objects, the action is not additive. Another set is left, namely, a set of 6 potholders. The action in which the members of a subset of a given set are removed from the given set is said to be subtractive.

The terms "additive" and "subtractive" need not be mentioned explicitly. The pupil should merely recognize that in some problems, objects are put with other objects (the union of disjoint sets) and in other problems, objects are taken from a given number of objects (a subset of a given set is removed from the given set).

To be sure that the members of the solution set of a sentence are the numbers that the problem asks us to find, these numbers are used in the problem itself. The pupil should be accustomed to verifying his answers to a problem in this manner.

Situations requiring solutions involving multiplication and its inverse operation, division, are presented in a later topic on ratios and rates. Special sentences called proportions are then formed for determining the solution set.

Work sheets can be prepared containing additional additive and subtractive problems such as these:

1. A school auditorium can seat three hundred persons. Fewer than twelve seats were empty for the school play. How many persons could have attended? $300-n<12$

2. Anne bought 24 buttons. After she had sewed some of the buttons on a dress, she had more than twelve buttons left. How many buttons could Anne have sewed on the dress? $24-n>12$

3. Betty baked 25 cupcakes for 18 people at a party. Each person had at least 1 cupcake. How many people at the party could have had at least 2 cupcakes? $25-18>x$

4. Jim had some dimes. Then he spent 4 dimes for school supplies and 18 dimes for a birthday gift. Jim told Herb, "The number of dimes I spent is the same as the number of dimes I have left." How many dimes does Jim have left? $4+18=x$

5. Ed had 52 marbles. Then he gave some marbles to his brother Bill. The number of marbles Ed has left is greater than the number of marbles Ed gave to Bill. How many marbles could Ed have given to Bill? $52-x>x$

6. Mr. Thomas sold some of the 42 ducks on his farm. The number of ducks he sold was greater than the number of ducks left on his farm. How many ducks could he have sold? $x>42-x$

7. During the first six months of the year, Mr. Jones worked 43 hours overtime. By the end of that same year, he had worked 119 hours overtime. How many hours overtime did he work during the last six months of that year? $43+x=119$

8. Of the 603 employees at the Allen Co., 17 had left the company by the end of the year. At least one replacement was hired. What was the least number of employees the company could have had at the end of the year? $603-17<x$

9. The water tank for an apartment house was filled to its capacity of 12,000 gallons. A day later there was still some water in the tank, but there were fewer than 9,140 gallons left. How many gallons could have been used that day? $12{,}000-x<9{,}140$

10. The capacity of a cargo plane is 38,000 pounds. The plane left Chicago with a capacity load. Some of the cargo was unloaded in Albany, but no cargo was put on the plane. When the plane arrived in New York City, it was carrying more than 29,000 pounds of cargo. How many pounds of cargo could have been unloaded in Albany? $38{,}000-x>29{,}000$

More experience in solving verbal problems will be given to pupils in TOPIC IV (FRACTIONS ON THE NUMBER SCALE) and TOPIC V (RATIOS INCLUDING PER CENTS AS PAIRS OF NUMBERS).

Elementary proof using the basic properties of numbers should be introduced in this topic. One- and two-step proofs could be used for such sentences as:

$$2+3x=3x+2$$

$$x+y=y+x$$

$$(2x+5)+3=(3+2x)+5$$

TOPIC VIII on proof has numerous suggestions that could be used either at the seventh- or eighth-grade level.

Order of operations should be presented and pupils given practice on examples written in horizontal array, i.e., $3\times7+8-2=n$.

In conclusion, this topic has presented mathematical sentences as a means to better understanding the structure of our number system and its properties and as a means to developing more understanding and confidence in problem solving. Some of the properties of the natural numbers and zero were investigated, named, and used in simple deductive proofs.

TOPIC III

Nonmetric Geometry*

Suggested time allotment: 4-5 weeks

The main purpose of this topic is to give the pupil experiences with fundamental geometric concepts. Some of these experiences should be designed to help him see the need for precise language, particularly in definitions.

Several side gains can come out of this study. For example, many opportunities develop for the pupil to simplify his work by forming generalizations of his own. Another benefit is the reinforcement of the language he has learned in dealing with mathematical ideas.

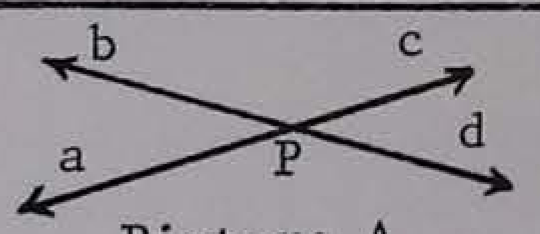
One purpose of this topic is to distinguish between geometric figures and their measures. For example, it is necessary to be aware of the distinction between the idea of an angle and the measure of the angle, the idea of a line and the length of a line.

It is suggested that the topic be started by considering sets of lines. A mark can be made on the chalkboard and the pupils can be asked whether this is a line or merely the picture of a line. Since line, point, and angle are ideas and not drawings, the "endlessness" of a line is an easy thing for pupils to understand.

Next, study sets of pairs of lines. The lines in a given pair either intersect or they do not. If the lines do not intersect, they may be parallel—or they may not be in the same plane. (It is better not to restrict thinking to a plane.) When they intersect, at least one angle is formed. How many angles? To answer this question one needs to study and understand *rays* or half lines.

Since the need for precise definition is more important than mathematical rigor at this level, there are a variety of ways to define the idea of angle. The pupil can begin to see that one of the important things here is that when one uses a word (in this case, "angle"), this word must convey the *idea* he has to his listener. So we pick the *idea* we want to associate with the word angle.

One popular definition of an angle is "two rays with a common end point." Thus, when two lines intersect, we can letter the rays a, b, c, and d. They all have a common end point. Then (a,b), (a,c), (a,d), (b,c), (b,d), and (c,d) are all angles. Since (b,c) and (c,b) are names for the same angle, all we need to do is count to know there is a total of *six* angles. This can be thought of as the number of ways four objects can be arranged taking two at a time.



Picture A

Now look at sets of triples of lines. When three lines intersect in two points, one of the lines has two special points on it and the definition of line segment arises naturally. Parallel line segments are segments of parallel lines.

The vertex of an angle also should be defined at this time.

Again there are many questions which naturally follow.

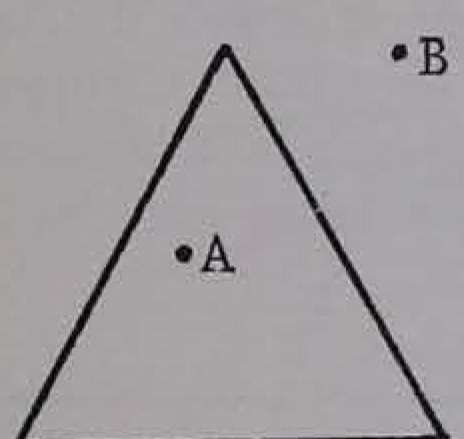
*Topic Reference: Numbers 3, 4, 30, 31, 32, 36, 37

Examples of some questions about members of the set of triples of lines are given below. Our universe is {0, 1, 2, 3, . . .}.

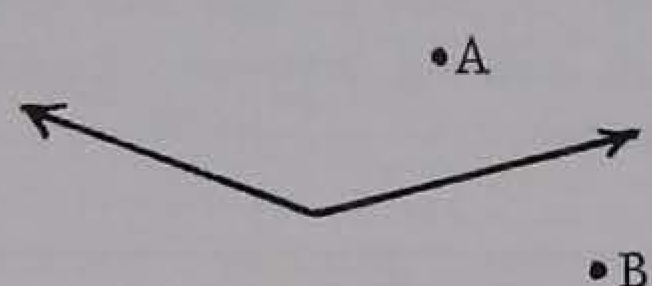
1. What is the solution set for the open sentence, "three lines may intersect in n points," that is, "In how many ways can three lines be made to intersect?"
   {0, 1, 2, 3}
2. If three lines intersect in one point, n angles are formed? {15}
3. Three lines can intersect to form n angles? {12, 15, 18}
4. Is there a member of the set of all triples of lines which has two, and not three, line segments?

It seems natural to proceed to sets of quadruples of lines, sets of quintuples, and so on.

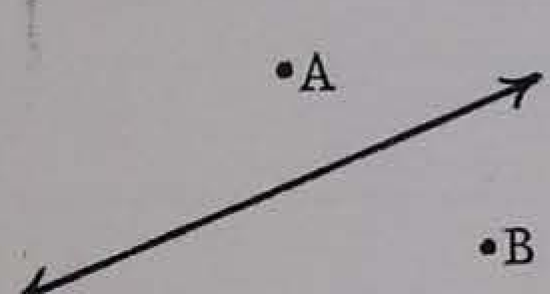
One can talk about both open polygons and closed polygons. Again a simple definition is needed. About as simple a definition as one can get is to leave "region" undefined and talk about a polygon as the "boundary of a region." Intuitively, it is easier to talk about the boundary between two regions, but the geometric idea of "betweenness" is rather sophisticated for study at this time.



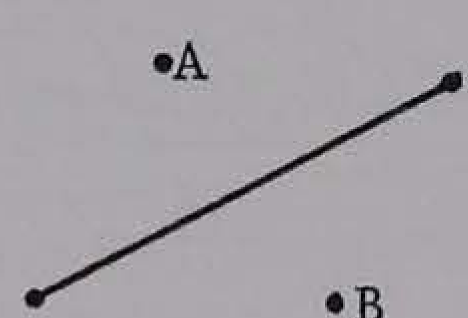
Picture B
(triangle)



Picture C
(2 rays)



Picture D
(line)



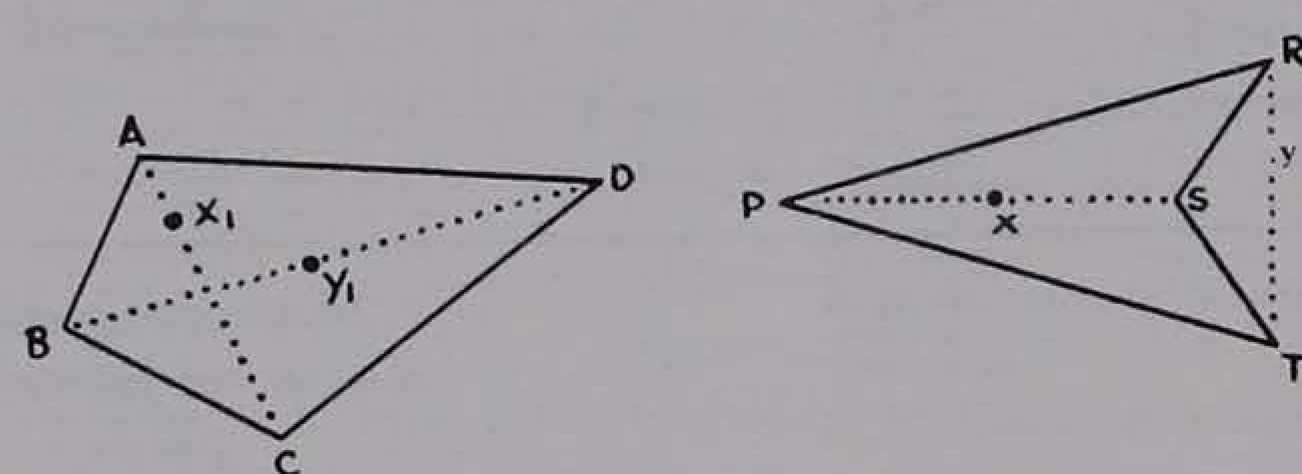
Picture E
(line segment)

In Pictures B, C, and D in order to get from certain points in one region, like point A, to points in another region, like point B, it is necessary to cross (touch) the boundary. So these are pictures of polygons. Picture C and Picture D are pictures of open poylgons.

In Picture E, it is impossible to find two regions (sets of points with members like point A and point B for which this is true), so a line segment is not a polygon. There is a "hole" in this definition if points A and B are allowed to be on the boundary in question. It is well to have this restriction come out of a discussion where pupils use the discovery approach and generalize.

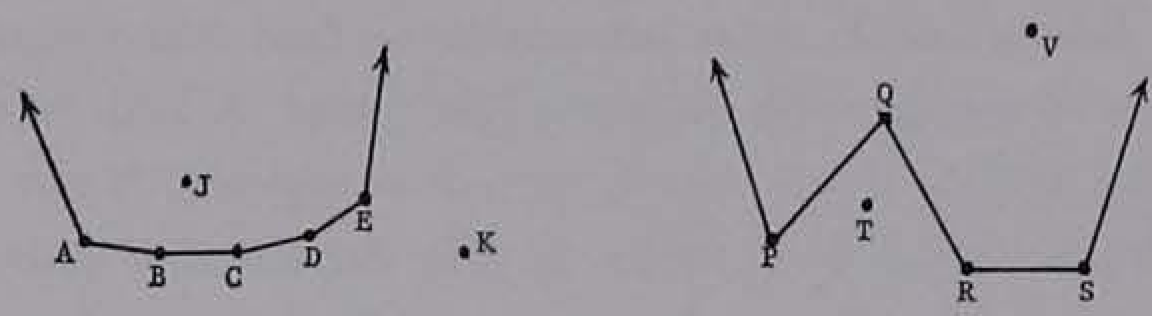
Picture B is a picture of a closed polygon. A closed polygon can now be defined as "a polygon composed of line segments." Now, the set of all quadrilaterals can be discussed as a subset of the set of all polygons. Triangles, pentagons, octagons, etc. can be handled the same way. The idea of a n-gon should not be neglected.

Diagonals can be defined as *line segments* which are not sides of the polygon, but their end points are vertices of the polygon. Then if all points but the end points of every diagonal are in the same region, the polygon is convex (see quadrilateral ABCD in Picture F). If two points can be found in the diagonals of a polygon in two distinct regions with respect to the polygon, the polygon is concave.



Picture F

Look at Picture F. Quadrilateral PRST has two diagonals. Point x is in one diagonal and point y is in the other. Any line segment (or curve) which contains both x and y on it also has a point in common with the polygon PRST. On the other hand, segment x_1y_1 has no point in common with ABCD; the same thing is true for any other placement of x_1 and y_1 in diagonals of the polygon ABCD. This enables one to avoid the concepts of "interior" and "exterior" which, like "betweenness" are far more sophisticated than they appear at first glance.



Picture G

Look at Picture G. It is not hard to decide whether open polygon ABCDE is convex or concave. One might even decide whether points J and K were inside or outside. It is more difficult to decide about whether V is inside or outside of PQRS. How about T?

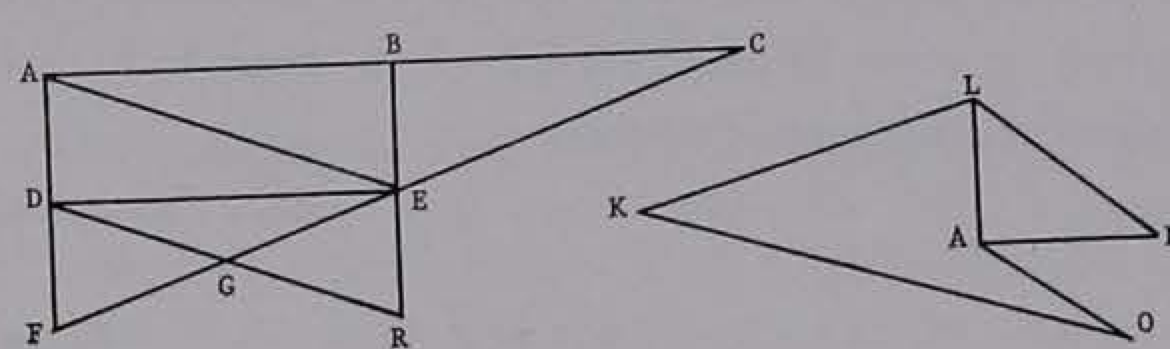
Whether PWRS is concave or convex is easily decided by choosing suitable points on diagonals QS and PR.

Good questions for pupils come up all through this topic. The following list is only intended to bring types of questions to mind.

I. Sets

U = {'polygon', 'trapezoid', 'quadrilateral', 'parallelogram', 'pentagon', 'octagon', 'triangle'}

(Note that this universe is a set of names for geometric figures—denoted by single quotes.)



Picture H

Tabulate the solution set for each statement below, tracing the figure formed by the points named. "y" is a placeholder for the elements in U above.

1. DGE is a y.
2. ABDE is a y.
3. ADEH is a y.
4. ADEC is a y.
5. KLNO is a y.
6. KLMNO is a y.

II. Tables

Many questions can be asked in tabular form. Examples: (see table below)

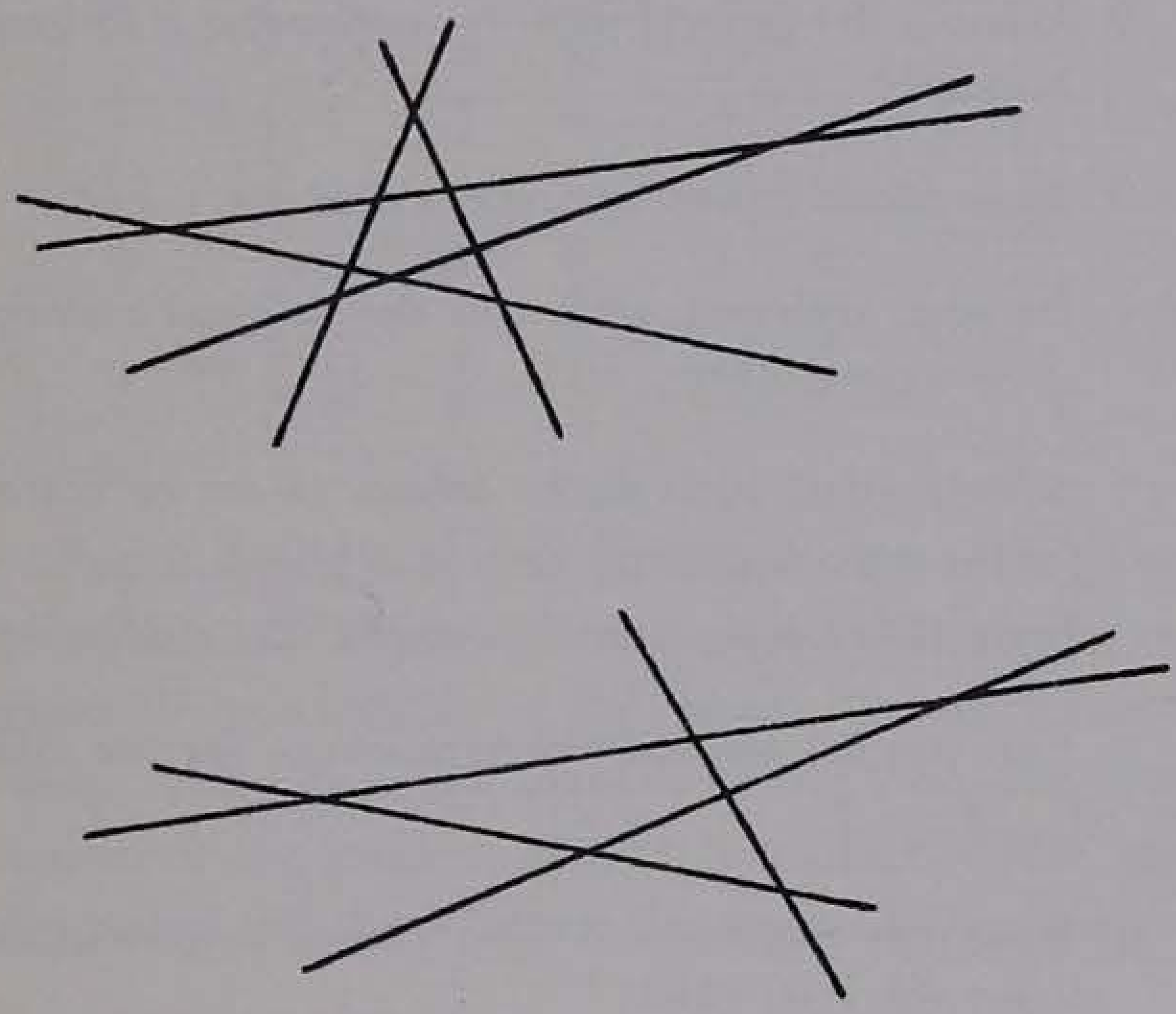
Note that here is a natural use of a placeholder where the idea it represents is that of a parameter. Actually a parameter is a variable used in a special way. The questions this brings up in the minds of boys and girls can lead to very fine thinking.

For instance, when asked about the maximum number of points in which n lines can intersect, some pupils, come up with $1+2+3+\ldots+(n-1)$. Others with a different approach come up with $\frac{n(n-1)}{2}$. The question of who is right brings up the possibility, and subsequent investigations, of whether these expressions are in fact equal. You could also have columns in this table for items such as "number of angles formed if all lines intersect in one point," or "number of angles if lines intersect in the maximum number of points." Columns for numbers of rays and line segments lend confidence in building answers. Another good table: (see next page)

Number of lines	Minimum possible number of intersection points	Maximum number of intersection points
2	0	1
3	0	3
4	0	6
5	0	10
6	0	15
⋮	⋮	⋮
n	0	$1+2+3+\ldots+n-1$ or $\frac{n(n-1)}{2}$

	Number of diagonals if convex	Number of diagonals if concave	Number of triangles formed by diagonals from one vertex
triangle			
quadrilateral			
pentagon			
hexagon			
⋮	⋮	⋮	⋮
14-gon			
⋮	⋮	⋮	⋮
n-gon			

III. Questions about specific figures. See Picture I below. (These can be done through tables or sets and solution sets.)



Picture I

How many triangles? How many quadrilaterals? How many closed polygons? Open polygons? Angles? and so on.

Since part of the purpose of the topic is to help the pupil see the *need* for precise language, he should be allowed to use his own language in describing the ideas. Some loose definitions should be made, then questions should be asked which lead the pupil into difficulty.

The wealth of individual ideas in this area could tempt a teacher to spend too much time on the topic. Four to five weeks ought to be plenty of time in spite of the fact that one can design many times that much work and pupils will continue to find it interesting.

TOPIC IV
Fractions on the Number Scale*

Suggested time allotment: 6-7 weeks

In previous topics the universe has included only the natural numbers and zero. For the universe of natural numbers and zero, pupils will recognize that a sentence such as $2n=1$ has as its solution the empty set. Given the positive rationals as the universe, the solution set becomes the fraction $\frac{1}{2}$. As with the natural numbers and zero, understanding will be enhanced by showing that each is placed in one-to-one correspondence with a point (coordinate) on the number line.

This topic is also concerned with a re-examination of operations with fractions—a more sophisticated and mature look than in previous study. Pupils should no longer be satisfied with just *how* something is done; they should learn *why*. The basic properties of numbers and operations should be emphasized whenever possible in the study of fractions. These basic properties were first introduced in TOPIC II and may be used again to develop experiences with fractions. Operations are justified by the use of such properties as commutativity and associativity for multiplication and addition, and the identity elements for multiplication and addition.

The study of prime factors and reciprocals is very important in this topic as pupils learn to quickly identify common denominators and multiples.

In the development of decimal fractions, the teacher should emphasize the decimal as a special kind of fraction notation which has certain qualities of convenience.

In teaching pupils to compute with decimal fractions, it is vital to relate the decimal notation to the common fraction notation of the same number. In this way it is relatively easy for the pupils to discover the principles used in computations with decimals.

*Topic References: Numbers 3, 30, 32, 35, 36

In the solution of verbal problems involving fractions, encourage the pupils to use the mathematical sentence to describe the action and conditions of the problem. This is the same approach as that used in TOPIC II.

Suggested Developmental Questions

The following questions may help the teacher lead pupils into a study of this topic. Discussion of the questions will follow.

1. What is the solution set for the sentence $2n=1$?
2. What is the general form for expressing a rational number?
3. How many halves are represented by $1\frac{1}{2}$?
4. In what different ways can the rational number $\frac{1}{2}$ be expressed?

   After several equivalence classes (a set of names for the same fraction; i.e., $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$) have been developed, ask the pupils the following:
5. Can you see any special significance for the class $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$?
6. How may members of other classes be developed using the ones class?
7. Which of the following is the more meaningful way to express a fraction, $\frac{13}{4}$ or $3\frac{1}{4}$?

   (Answer: one is as meaningful as the other.)
8. How may the following number pairs be reduced using the identity property for multiplication $(a \cdot 1=a)$?

   a. $\frac{2}{4}$ b. $\frac{5}{15}$ c. $\frac{12}{3}$ d. $2\frac{10}{5}$

9. What is the reciprocal of $\frac{5}{8}$?

10. Express the following numbers in more simple form.

a. $\frac{4}{8}$ b. $\frac{\frac{5}{8}}{2}$ c. $\frac{2\frac{3}{4}}{\frac{1}{2}}$ d. $\frac{6\frac{1}{2}}{2\frac{1}{4}}$

11. How are decimal fractions and common fractions related?

Discussion of Developmental Questions

Questions number 5 and 6 (above) explained:

The answer to the question in number 5 is the "ones class" $(\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots)$. Justification of this answer for the question may be developed as follows.

The pupils may recognize patterns by developing such classes as:

$$\frac{1}{2} \leftrightarrow \left\{ \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots \right\}$$
$$\frac{3}{1} \leftrightarrow \left\{ \frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \ldots \right\}$$
$$\frac{2}{3} \leftrightarrow \left\{ \frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots \right\}$$
$$1 \leftrightarrow \left\{ \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \ldots \right\}$$

Pupils should note that the "ones" class is the easiest to develop. With further investigation they should recognize the relationship of the "ones" class to the other classes. Using the first member of the class, any other member is the product of that first member and some member of the "ones" class. Examples of these are:

$$\frac{1}{2}\left(\frac{2}{2}\right) = \frac{2}{4}$$
$$\frac{1}{2}\left(\frac{5}{5}\right) = \frac{5}{10}$$
$$\frac{1}{2}\left(\frac{8}{8}\right) = \frac{8}{16}$$

Question number 6 is an outgrowth of the above examples. Pupils should use multiplication by members of the "ones" class to develop other classes. Where the pupil found it rather difficult to think of members for a class such as $\frac{7}{23}$, now there is a means of developing more members.

$$\frac{7}{23}\left(\frac{n}{n}\right) = \frac{7n}{23n} \quad \text{where } n>0$$

For question number 10, the concept of a multiplicative inverse (reciprocal) is a useful tool in operating with fractions. When a positive rational number is multiplied by its reciprocal, the product is one. Later this idea is used extensively in finding solution sets for open sentences. Here it will be used in the simplification of number pairs.

Pupils should understand that a number pair $\left[\frac{a}{b}\right]$ can represent more ideas than that of a fraction. The number pair $\frac{5}{2}$ could mean:

1. The fraction five-halves.
2. A ratio of five to two.
3. Five divided by two.

The first of these interpretations is familiar to the pupils. The second interpretation will be developed in TOPIC V.

The following comments on the third interpretation (above) of a number pair will help answer question 10.

a. $\frac{4}{8} = \frac{1}{2}$ The pupils know this from the study of equivalence classes, recognizing that both 4 and 8 are multiples of 4.

b. $\frac{\frac{5}{8}}{2}$ A few pupils may recognize the solution as being $\frac{5}{16}$. The pupil should be led to understand the most important idea is that this is a number pair naming a point on the number line. Further, it may be interpreted as $\frac{5}{8}$ divided by 2. The following questions will help pupils solve this division:

(1) What number is easiest to use as a divisor?
(2) What is the reciprocal of two?
(3) What member of the "ones" class may be used in simplifying the number pair?

The solution of the problem takes this mathematical pattern:

$$\frac{\frac{5}{8}}{2} = \frac{\frac{5}{8}}{\frac{2}{1}}\left[\frac{\frac{1}{2}}{\frac{1}{2}}\right] = \frac{\frac{5}{16}}{1} = \frac{5}{16}$$

Note that this method is also satisfactory in the solution of the problems below.

c. $$\frac{2\frac{3}{4}}{\frac{1}{2}} = \frac{\frac{11}{4}}{\frac{1}{2}}\left[\frac{\frac{2}{1}}{\frac{2}{1}}\right] = \frac{\frac{11}{2}}{1} = \frac{11}{2}$$

This development makes it clear that division by a number is equivalent to multiplication by its reciprocal. Think how mystifying this idea has been to many pupils in the past! Examine part c. again. Keep in mind that one interpretation of

$$\frac{2\frac{3}{4}}{\frac{1}{2}} \text{ is } 2\frac{3}{4} \div \frac{1}{2}$$

The reciprocal is developed by using the "ones" class to simplify the number pair. After pupils become belabored with the extra calculation involved in this process, they may be encouraged to find an easier one. They should choose to work the problem as follows, using a horizontal form:

$$\frac{2\frac{3}{4}}{\frac{1}{2}} = \frac{11}{4} \cdot \frac{2}{1} = \frac{22}{4} = \frac{11}{2}$$

Now work with fractions using the horizontal form.

In addition and subtraction of positive rational numbers, the idea of prime factors as related to common denominators must be included. The problems should be written in both vertical and horizontal form. The horizontal arrangement, of course, more nearly resembles the algebraic fractions to be studied later. For example:

(a) $$\frac{5}{6} + \frac{4}{15} = \left(\frac{5}{6} \times \frac{5}{5}\right) + \left(\frac{4}{15} \times \frac{2}{2}\right)$$
$$= \frac{25}{30} + \frac{8}{30} = \frac{33}{30}$$

(b) $$\frac{11}{10} - \frac{13}{30} - \left(\frac{11}{10} \times \frac{3}{3}\right) - \frac{13}{30}$$
$$= \frac{33}{30} - \frac{13}{30} = \frac{20}{30} = \frac{2}{3}$$

In reducing $\frac{20}{30}$ the prime factors should be examined, several "ones" classes being discovered and used to obtain the simpler equivalent fraction, $\frac{2}{3}$.

In summary, this topic has presented techniques and devices for helping the pupil gain more understanding of fractions which behave like the positive rationals. It is hoped that with the number line as a visual aid pupils will see the fractions as an extension of the universe of numbers previously used. In this more mature look at the four operations with fractions, the pupil should be guided to discover that the same properties which justify operations with the natural numbers and zero also justify the operations with this set of numbers called the fractions.

TOPIC V

Ratios (Including Per Cents) As Pair of Numbers*

Suggested time allotment: 9-10 weeks

There are various ways of comparing pairs of numbers. This topic deals with the comparison between pairs of numbers which can be expressed by the form $\frac{a}{b}$ where a and b represent positive rational numbers. We shall call such a form a *ratio*. A *ratio* is a pair of numbers expressing a rate in a physical situation in which we make a many-to-many, one-to-many, or many-to-one correspondence. Using equivalent ratios to represent equivalent rate pairs enables us to precisely describe symbolically the action in many problem situations. Rates arise in buying at a certain cost per article or in mixing materials in certain proportions or in finding areas of the interior of geometric figures.

Per cent is a special rate-pair comparison, where the second component of the ratio is 100. In the traditional study of per cent, the pupil studied three distinct types of situations, each having a specific arithmetic process designated as the correct one to use in finding the solution. In the ratio approach to per cent the pupil finds that all per cent situations can be expressed by a pair of equivalent ratios with the missing component represented by a placeholder. Research and current practice seem to indicate that the pupil who learns the ratio approach becomes much more skillful in problem solving and retains more of the concepts as well.

The pupil should have a good background of experience with ratios from his elementary schoolwork. If this is the case, then the teacher may move quickly into the consideration of per cent with emphasis placed on the use of a mathematical sentence to describe the conditions of the problem.

It would be well for the teacher to help the pupils review and later summarize the properties of rate pairs. For reference, a summary of these properties follows:

(1) If two rate pairs belong to the same set, that is, they are equivalent, then their cross products are equal.
[If $a : b \simeq c : d$, then $ad = bc$]

(2) If in a given set of rate pairs we multiply each component of a rate pair by a natural number, then we get another member of the same set of rate pairs. [$a : b \simeq a(n) : b(n)$]

(3) If we divide both terms of a rate pair by a common factor, we obtain another member of the set of rate pairs.
[$a : b \simeq \frac{a}{n} : \frac{b}{n}$]

(4) Any given rate pair (except $0 : 0$) belongs to one and only one set of rate pairs. (Thus we do not accept any rate pair if one component is 0.)

These properties should be contrasted with those of fractions. (See TOPIC IV) A fraction is defined as an ordered pair of natural numbers written (p, q) or $\frac{p}{q}$ when it is understood that $q \neq 0$. Two fractional numbers, (a, b) and (c, d), are equal if and only if $ad = bc$. The binary operations of addition and multiplication (and their inverses) can be performed giving a meaningful result. This is not the case with ratios.

For example: Mark Davis was practicing free throws in basketball. On Monday he made 15 baskets out of 20 attempts. On Tuesday he made 16 baskets out of 20 attempts. If $\frac{15}{20}$ and $\frac{16}{20}$ were added as fractions, the result $\frac{31}{20}$ would seem to indicate that Mark was successful 31 out of 20 times, which is impossible. Actually he was successful 31 times out of 40 attempts. Multiplication of ratios is likewise ridiculous.

*Topic References: Numbers 6, 7, 8, 20, 27, 35, 36

Many applications from business (commission, discount, taxes, margin, interest) should be used in this topic. Scale drawing is another application of ratio. Statistical data can be utilized in the construction and interpretation of graphs. Circle graphs give another fine application of per cent as a rate per hundred.

The first property of ratios listed above may be defined for the pupils after experimenting with special cases such as:

If $\frac{2}{3} \simeq \frac{4}{6}$, then $2 \cdot 6 = 3 \cdot 4$

And since $2 \cdot 6 = 3 \cdot 4$ we know $\frac{2}{3} \simeq \frac{4}{6}$,

leading to $\frac{a}{b} \simeq \frac{c}{d}$ if and only if ad=bc.

The following questions lead to the major concepts and ideas that must be developed with the pupils during this topic:

1. What are some examples of ratios?

2. Find solutions to the following using:

   $\frac{a}{b} \simeq \frac{c}{d} \rightarrow ad=bc.$

   (a) If Tom is paid at the rate of 25¢ per hour, how much will he earn in 5 hours? in 7 hours?

   (b) If potatoes cost 6¢ per pound, what is the cost of 10 pounds of potatoes? of 13 pounds?

   (c) If onions cost 5¢ per pound, how many pounds may be bought with 50¢? with 25¢?

   (d) When Sid shoots free throws, out of every 5 shots he makes 3. If he makes 18 shots, how many free throws did he shoot? How many free throws can he make if he shoots 100 shots?

   (e) For every dollar that Ron loaned to Gary, Gary promised to pay Ron an interest of 4¢ when he returned the money. If Gary borrows $12, how much interest must he pay to Ron? If Ron is paid 20¢ in interest, how much did he lend to Gary?

   (f) If 10 pounds of force on a lever will lift 30 pounds, how much weight will 20 pounds of force lift? How many pounds of force is required to lift 15 pounds?

3. Why is per cent a special case of ratio?

4. Per cent is identified as a ratio involving what number?

5. How may each of the following be stated as a number pair?

   (a) 5%

   (b) .05%

   (c) 175%

6. Find solutions for the following using:

   $\frac{a}{b} \simeq \frac{c}{d} \rightarrow ad=bc.$

   (a) What number is 25% of 40?

   (b) 15 is what per cent of 75?

   (c) 18 is 75% of what number?

   (d) Larry makes 60% of his shots during a game. If he attempts 20 shots, how many does he miss?

   (e) Mike lost 12% of his wrestling matches. If he lost 3 matches, how many did he win?

7. How can ratio be applied to the simple machines of physics? (Teacher should devise specific questions)

8. How are yearly interest rates on money an application of equivalent ratios? (Again teacher should devise many specific questions.)

The pupil will have no difficulty with problems like those above. He will be finding solution sets for the sentences 14x=28 and 3x=75. But ratios which will lead to sentences like $2\frac{1}{2}x=3$ require some system of solution. His easiest approach is to rewrite the sentence as $\frac{5x}{2}=\frac{3}{1}$ then the cross products work for him again to write 5x=6 and he should have no difficulty recognizing $\{\frac{6}{5}\}$ as the solution set.

Now examine the parts of question 2 stated on a previous page:

(a) Tom's rate of pay is $\frac{25¢}{1 \text{ hr.}}$

$\therefore \frac{25}{1} \simeq \frac{x}{5}$ **$1.25**

$125=1x$ {125}

$\frac{25}{1} = \frac{x}{7}$ **$1.75**

$175=1 \cdot x$ {175}

(b) The cost of potatoes is $\frac{6¢}{1 \text{ lb.}}$

$\therefore \frac{6}{1} \simeq \frac{x}{10}$ **60¢**

$1 \cdot x = 60$ {60}

$\frac{6}{1} = \frac{x}{13}$ **78¢**

$1 \cdot x = 78$ {78}

(c) The cost of onions is $\frac{5¢}{1\text{ lb.}}$

$$\therefore \frac{5}{1} \simeq \frac{50}{x}$$

5x = 50 {10} **10 lbs.**

$$\frac{5}{1} = \frac{25}{x}$$

5x = 25 {5} **5 lbs.**

(d) Sid's ratio is $\frac{3}{5}$

$$\therefore \frac{3}{5} \simeq \frac{18}{x}$$

3x=90 {30} **30 free throws**

$$\frac{3}{5} = \frac{x}{100}$$

5x=300 {60} **60**

(e) Ron's interest is $\frac{4¢}{100¢}$

$$\therefore \frac{4}{100} \simeq \frac{x}{1200}$$

100x=4800 {48} **48¢**

$$\frac{4}{100} = \frac{20}{x}$$

4x=2000 {500} **$5.00**

(f) The ratio of the lever is $\frac{10}{30}$ or $\frac{1}{3}$

$$\therefore \frac{1}{3} \simeq \frac{20}{x}$$

1 · x = 60 {60} **60 lbs.**

$$\frac{1}{3} = \frac{x}{15}$$

3x=15 {5} **5 lbs.**

Question Number 6

A major portion of the work with ratio may be with per cent. A per cent is a ratio of two positive numbers, the second of which is 100. Since per cent is a rate-per-100 number pair, the basic concept of ratio may be used.

(a) 25% may be expressed as $\frac{25}{100}$

$$\therefore \frac{25}{100} \simeq \frac{x}{40}$$

100x=1000 {10}

(b) This is a ratio of $\frac{15}{75}$

$$\therefore \frac{15}{75} \simeq \frac{x}{100}$$

75x=1500 {20} **20%**

(c) 75% may be expressed as $\frac{75}{100}$

$$\therefore \frac{75}{100} \simeq \frac{18}{x}$$

75x=1800 {28} **28 is the number**

(d)

$$\frac{60}{100} \simeq \frac{x}{20}$$

100x=1200 {12} **12 shots made**

8 shots missed

(e)

$$\frac{12}{100} \simeq \frac{3}{x}$$

12x=300 {25} **22 matches won**

Per cent of increase and per cent of decrease can be handled the same way with the original component of the rate pair being 100. For example:

Fred was paid $60 a week when he began working in a service station. After six months his wages were increased to $75 a week. What is the per cent of increase? $\frac{75}{60} \simeq \frac{x}{100}$, etc.

The emphasis of this entire topic is placed on the recognition and understanding of

$$\frac{a}{b} \simeq \frac{c}{d} \leftrightarrow ad=bc.$$

This is a concept that leads the pupil to a fundamental understanding of per cent. The solution is not the formulation of many rules but the solution of "different types" of per cent problems using the same mathematical model throughout the experience.

TOPIC VI

Metric Geometry*

Suggested time allotment: 5-6 weeks

TOPIC III entitled, "Sentences That Describe Sets of Points," introduced certain concepts of geometry to the pupil which are independent of measures (non-metric geometry). This topic again investigates such geometric configurations as points, lines, planes, and closed polygons but the emphasis here is upon their measures. The *non-metric* geometry previously studied involved the number of angles, number of lines, number of points, number of corners, and number of line segments. Here *metric* geometry will assign numbers to the size of geometric figures. Of course, more than one number may be assigned to the same geometric figure if the size of the unit is changed.

The organization of material in this topic follows this outline:

I. Creating a need for standard units of measurement.

II. Comparison of different systems of measurement and the ratios within each system.

III. Using standard units for measuring in one dimension—a line segment.

IV. Using standard units for measuring in two dimensions—areas of closed curves—by use of equivalent ratios.

The placement of this topic immediately following the topic on ratio enables the teacher to approach measurement as another application of ratio—the ratio of the measure of a figure to a previously agreed upon unit of measure. The pupils should clearly understand that any measurement of length is simply a ratio of the length of the object to the length of an arbitrary standard. (An object here could be an intangible one such as a period of time.)

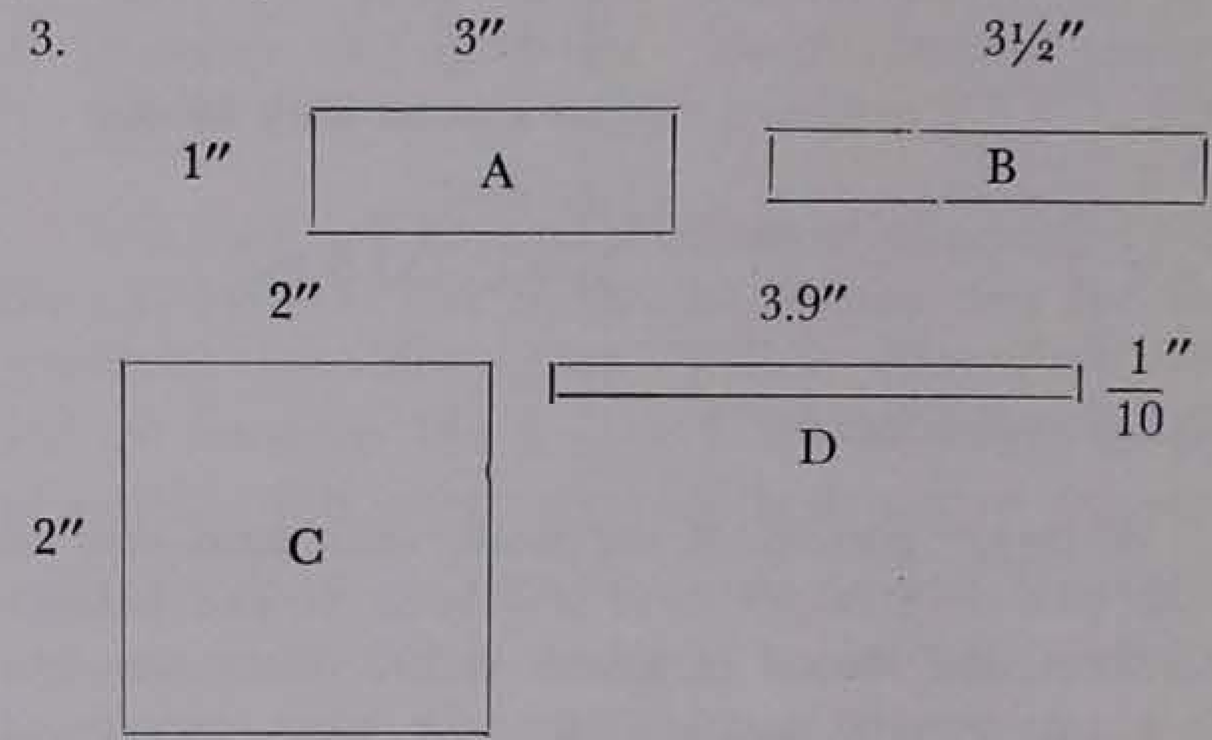
For a class moving at an average pace, this topic should be studied at the end of the seventh-grade year. This topic is open-ended and could include many measurement experiences. The weather would at this time permit some outdoor activities. By and large the mathematics class should expect to emphasize the study of the mathematical model of measurement and leave further development of the techniques of measurement and application of measurement to other courses. However, the mathematics teacher could well draw on problems of the physical world for exercise material. Proper emphasis should result in a strong carry-over into science, industrial arts, and other areas.

Questions that a teacher might ask pupils are:

1. ______________________

   How long is this line segment?

2. How wide is this room?

3. 

Quadrilaterals A, B, C, and D are all rectangles. What is the perimeter of each? A?________ B?________ C?________ D?________

The perimeters are all 4 inches.

4. Do the figures differ in any way than shape? (Discussion should bring out a need for area units in order to describe what is different about these figures.)

5. List five real-life situations where you would be interested in the size of a closed region.

6. In the following, tell whether we are interested in the length of the closed curve or the size of the closed region:

   (a) In determining how much fence is needed to enclose a yard.

*Topic References: Numbers 3, 4, 19, 30, 31, 32, 35, 36, 37

(b) In buying roofing for a home.

(c) In determining how many gallons of paint are needed to paint a house.

(d) In buying a boy's belt.

(e) In putting a new counter top on the kitchen cupboards.

(f) In buying a chrome strip for the project in situation e.

(g) In purchasing a farm.

7. Draw a figure which illustrates a pair of intersecting lines. Measure both pairs of vertical angles. Repeat this with several other pairs of lines. Do your findings suggest a general principle about pairs of vertical angles? Does this measurement constitute a proof for all pairs of vertical angles?

In getting pupils to see the need for standard units in measurement, the development can be patterned after the historical development of standard measures. Basic units for angle measure and time measure should also be included. Pupils can be asked to measure objects using several different arbitrarily chosen units.

It is best if they do not correspond to any of our "standard" units. Sticks can be cut into various lengths and the pupil asked: "How many green sticks long is this object?" or "How many blue sticks long is this object?" Through experiences such as these, the pupils can be led to recognize that measuring is simply a method of assigning a number to a characteristic of the physical world, in this case, a length. They should also see that the number assigned is dependent upon the unit that is used and that the arithmetic involved is that of ratio.

Once the need for standard units has been established in the minds of the class, different systems can be analyzed. The multiples within a system can be seen as ratios; in the English system, 1 to 12, 1 to 36, 12 to 36, 1 to 16, 1 to 5280, 1 to 1760, etc. In the metric system the ratios are the same as in the base ten numeration system: 1 to 10, 1 to 100, etc. It becomes evident that the two systems involve different ratios and are therefore cumbersome. The duodecimal system of numeration could be examined also, and its ratios compared with the English system of measure.

In examining systems of measure, pupils should be led to see that they need but one conversion ratio from one system to another. For example from English to metric, the ratio is 1 to 2.54 (one inch to 2.54 cm). All other conversion ratios follow this one.

Beginning their study with linear measurement, pupils should be provided with experience which will keep them well aware of the idea that a line segment and the *measure* of that line segment are *two distinct ideas.* The distinction should be maintained in the symbolism.

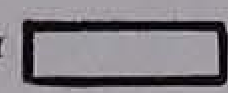
One group has suggested symbols such as these:

Idea	suggested symbol for the idea
line AB	$\overleftrightarrow{AB}$
line segment AB	$\overline{AB}$
ray AB	$\overrightarrow{AB}$
the measure of line segment AB	AB

The idea is important. Distinction should be maintained. The choice of symbols is arbitrary. Many people like symbols which, though not like those above, still carry the same understanding.

The choice of units is of course also arbitrary and for the most part if lengths are described as 3 units or 5 units, rather than using specific units such as 3 inches or 5 inches, better understanding will probably result. That is, a line segment 3 units long and a line segment 5 units long can be joined additively to form a line segment 8 units long. It makes no difference whether these units are millimeters, meters, feet, inches, miles, light years, or angstroms.

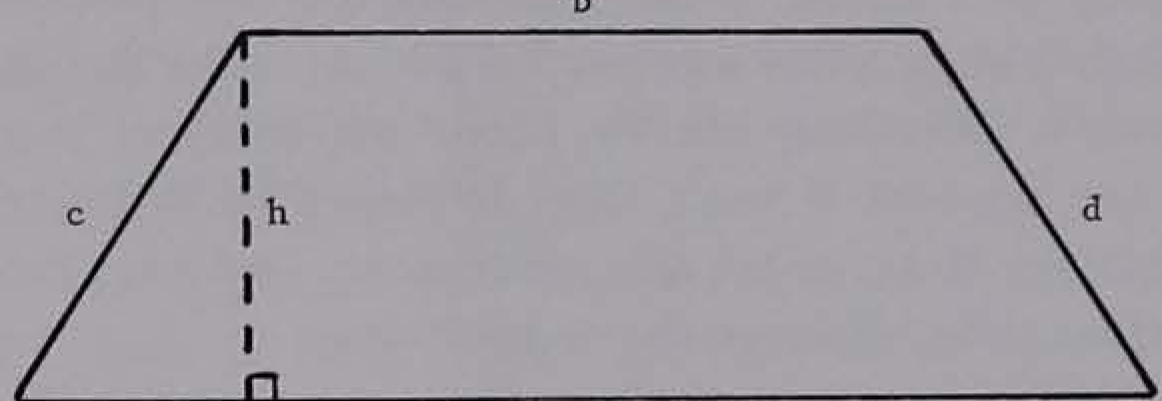
The fact that we choose many different unit segments for measuring linear lengths is familiar to the pupils; but the choice of a unit for measurement of angles is probably less familiar and the choice of names, the *degree,* is not at all suggestive of its origin. This unit for measuring angles was chosen over 4,000 years ago by the ancient Sumerians. This choice may have been based on their belief that the year was 360 days long. The Sumerians used a base of 60 for representing numbers. Thus, we divide the degree into 60 congruent angles each called a *minute.* Still more precisely we divide the minute into 60 seconds. Pupil experiences with the protractor will help them understand the principles underlying angle measurement.

Many pupils confuse area and linear units of measure. Prior to any instruction on the subject, pupils could be asked to describe the sizes of various plane figures. These should include some rectangles, parallelograms, triangles, and also some odd-shaped figures such as  and  . Questions can be asked about which is the larger  or  .

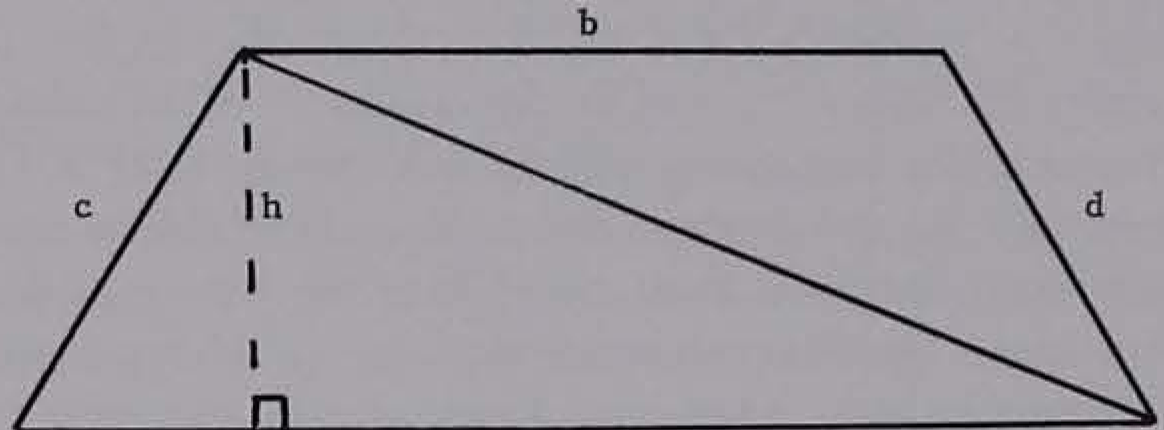
Thus, the *need* for a unit different than a linear unit is created.

Once this need is answered by the pupil, he is ready to do many of the same type of exercises in a more precise way. He should not be told that the area of a rectangle is the length times the width. Rather the formula $A=LW$ should be a natural outgrowth of a counting process and the pupils previous background with parameters (from his topic on number systems) and pairs of equivalent ratios. A rectangle 3 units wide and 4 units long can be thought of as 3 squares in one column gives us how many squares in 4 columns? Hence $\frac{3}{1} \simeq \frac{n}{4}$. Thus pupils learn that finding area is a problem in ratio, and not one of multiplying "feet times feet."

If pupils have previously studied number properties, the understanding of the properties can be reinforced in this topic. For example, as the pupil attacks the problem of the area of a trapezoid:



He may have the measurements a, b, c, d, and h given above. He can then break it down something like this:



Then he should recognize the area of the trapezoid is the sum of the areas of the two triangles.

For one triangle $A = \frac{1}{2}ah$ or $\frac{1}{2} \cdot h \cdot a$. For the other triangle $A = \frac{1}{2}bh$ or $\frac{1}{2}\ h \cdot b$.

Thus, the area of the trapezoid is $\frac{1}{2}h \cdot a + \frac{1}{2}h \cdot b$.

The pupil can now apply the distributive property and come up with $\frac{1}{2}h \cdot (a+b)$ or $\frac{h(a+b)}{2}$.

The pupil previously should have had many experiences with triangles and parallelograms and other geometric areas before he tries his hand at trapezoids. The key concept is, of course, the idea that the area of a complex figure can be computed (or in some cases only approximated) by cutting it up into more simple figures.

Measurement of circumferences and areas of circles will be taken up in the eighth grade topic on geometry (TOPIC IX), when pupils will be more ready for a universe of numbers containing irrational numbers.

Measurement of three-dimensional figures will likewise be delayed until TOPIC IX. There the arithmetic in finding volumes becomes a series of equivalent ratios as with areas.

Problem situations involving measurement of geometric figures can be found in almost any mathematics textbook. Perimeters of simple closed curves involving measurement of line segments, angle measurement, and measurement of areas of regions within closed curves. Some of the best problem situations will be brought to class by the pupils themselves.

Throughout this entire measurement topic, it should be pointed out that we are working only with physical models of our geometry. They will have discovered many interesting geometric principles which by measurement seem to be generally true; but since measurement in the physical world is subject to many errors, they must be cautious about these assumptions. Later in high school they will establish by rigorous proof some of these geometric relationships that seem to hold between geometric objects.

TOPIC VII

Sentences Whose Solution Sets Require Extensions to The Negatives of Previously Used Numbers*

Suggested time allotment: 5-6 weeks

For junior high school groups moving at an average rate through the mathematics sequence, this topic should normally be the first studied in eighth grade. Psychologically, its placement is good. Pupils come at the opening of the school year eager for something new and are intrigued with the negative numbers being added to their previously used universe of numbers—the non-negatives. Mathematically, of course, the negatives are needed to provide a solution set that is not empty for such sentences as $n+3=1$, $x+5=0$, $2x+6=4$, $\square+7<0$, $2n+8<6$, $\triangle+2<1$. Pupils have already encountered in the physical world negative temperature readings, negative signs representing changes in values of stocks, cost-of-living indexes, etc. They very naturally wonder how the familiar binary operations of addition, subtraction, multiplication, and division can be applied to these new numbers. They will want the familiar commutative, associative, and distributive properties to hold as well as the properties of 1 and 0, special numbers in our base ten system. In fact, the entire approach is that of maintaining the basic principles of previously used numbers.

An intuitive idea of negative numbers should be developed by considering them as "opposites" of previously used numbers. Opposites in direction both in the social world in which we live and in the universe of the number system should be presented. Some examples from the social world are: temperature readings, upward and downward forces, clockwise and counterclockwise motion, positive and negative charges of electricity, acceleration and deceleration, increase or decrease in cost of living or population, deviation from normal in weather. Examples from the universe of numbers are $^{+}2$ and $^{-}2$, $\frac{^{+}7}{2}$ and $\frac{^{-}7}{2}$, $^{+}1$ and $^{-}1$. These opposites should be diagramed on the number line.

* Topic References: Numbers 1, 3, 5, 7, 13, 15, 26, 29, 30, 32, 34, 36, 37.

In the pupil's early experience with negative numbers, the notation for negative four, for example, is $^{-}4$, with the negative '$^{-}$' written in a raised position to avoid confusion with the sign for subtraction. '$^{-}4$' is the name of the number which corresponds to the point 4 units to the left of 0 on the number line. Later, when pupils clearly understand the difference between the two meanings of the '$^{-}$' sign, the 'opposite of 4' will be symbolized as '-4,' the dash being written in a lowered position. In the case of subtraction '$-$' should be read "minus" to avoid confusing it with the concept of negative numbers.

It is important that the pupil gain early understanding of the absolute value of a number so that when a generalization of patterns of addition and multiplication, for example, are discovered they can be stated more precisely. Comprehending that all measures of distances can be represented by the absolute value lays a good background for using the distance formula later in coordinate geometry. The absolute value of a number can be visualized and interpreted geometrically on the number line as the distance between that number and 0. The absolute value of 0 is 0. The absolute value of any other real number is the greater of the number and its opposite. Using the symbol '$|\ |$' to stand for absolute value,

$|^{+}5|={}^{+}5$, $|\frac{^{-}7}{3}|=\frac{^{+}7}{3}$, $|^{-}\sqrt{2}|={}^{+}\sqrt{2}$ $|0|=0$ $|^{+}12|={}^{+}12$

If the operations of subtraction and division of negative numbers are discussed, they should be presented as addition of the opposite and multiplication by the inverse (or reciprocal). This is not difficult for pupils to comprehend since they know they can check the results of subtraction and division by addition and multiplication. If you wish to have pupils study subtraction and division of negative numbers, it is suggested that these operations be presented after pupils have first had much experience with the operations of multiplication and addition.

If pupils have encountered notation with negative exponents in their reading about science, the meaning of the notation could be explained; but at this level computation should be handled with positive exponents, as in this example: $10^{-2}\times10^{-3}=\frac{1}{10^2}\times\frac{1}{10^3}=\frac{1}{10^5}$

Developmental Questions

Leading questions a teacher might ask in introducing this topic are:

1. The Greek mathematician Diophantus (275 A. D.) tried to solve a problem which would be written in the open sentence, $4x+20=4$. He called the problem absurd; no number of arithmetic would satisfy the sentence. Is his answer still true today?
2. With the numbers we now have, is there a solution set (other than the null set) for the sentence $x+10=5$? For the sentence $5x+2<2$?
3. If $^{+}6$ is a trip to the right on the number line and $^{-}6$ is a trip to the left, what direction does $^{-}(^{-}6)$ suggest?
4. In a football game a team makes 3 yards on the first down, 2 on the second down, and loses 6 yards on the third down, what is the net yardage?
5. Dividing by three is the same as multiplying by what number?
6. The diameter of a molecule of hydrogen is 2.5×10^{-8}cm. This scientific notation is a name for what number?
7. In a movie film of a boat and outboard motor with forward and reverse speeds, what is the change in the boat's apparent position if the film runs forward $\frac{1}{2}$ minute and the outboard motor speed is forward at a rate of 300 feet per minute? If the film runs backward $1\frac{1}{2}$ minutes and the outboard motor speed is forward at a rate of 300 feet per minute, what is the change in the boat's apparent position?

A detailed lesson that arises out of the above question number 7 follows:

We have already established a need for negative numbers. We have observed that the properties of the non-negative rationals continue to hold in the addition of negative numbers. Let us investigate what happens in the operation multiplication. We assume that positive numbers "act" like the numbers of arithmetic, so let us set up a one-to-one correspondence between the numbers of arithmetic and the positive numbers using the symbol "$\leftrightarrow$" to stand for one-to-one correspondence:

$$1 \leftrightarrow {}^{+}1$$
$$2 \leftrightarrow {}^{+}2$$
$$3 \leftrightarrow {}^{+}3$$
$$4 \leftrightarrow {}^{+}4$$
$$5 \leftrightarrow {}^{+}5$$
$$6 \leftrightarrow {}^{+}6$$

This table indicates that to each arithmetic number there corresponds a positive number, and to each positive number there corresponds an arithmetic number. Thus, there is a one-to-one correspondence.

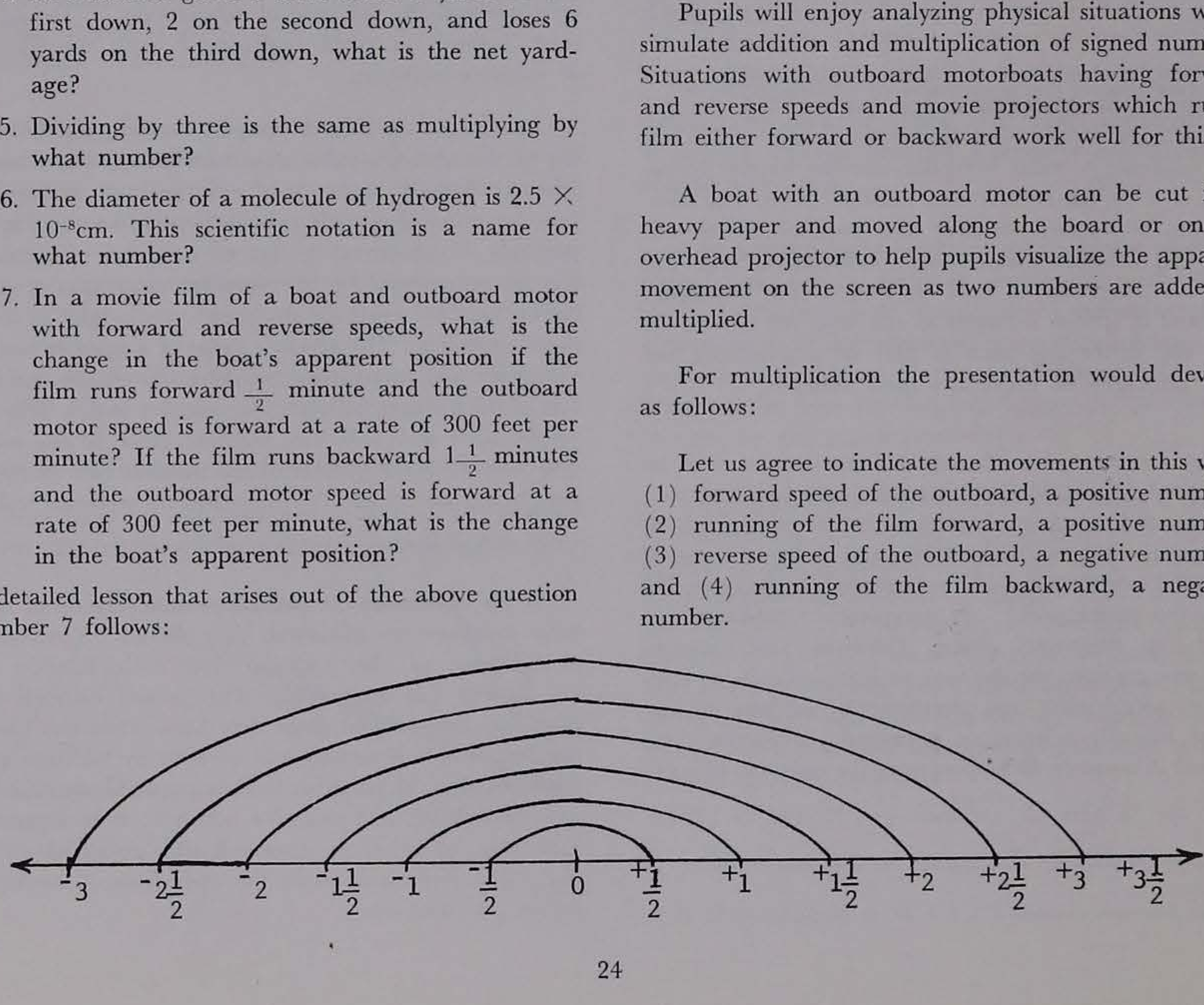
There is also a one-to-one correspondence of positive numbers and negative numbers.

Pupils will enjoy analyzing physical situations which simulate addition and multiplication of signed numbers. Situations with outboard motorboats having forward and reverse speeds and movie projectors which run a film either forward or backward work well for this.

A boat with an outboard motor can be cut from heavy paper and moved along the board or on the overhead projector to help pupils visualize the apparent movement on the screen as two numbers are added or multiplied.

For multiplication the presentation would develop as follows:

Let us agree to indicate the movements in this way: (1) forward speed of the outboard, a positive number; (2) running of the film forward, a positive number; (3) reverse speed of the outboard, a negative number; and (4) running of the film backward, a negative number.



If the outboard motor is running forward and the film showing it is running forward, the boat seems to move forward in the picture.

If the outboard motor is running forward and the film is running backward, in what direction does the boat appear to move in the picture?

If the outboard motor is running in reverse gear and the film is running forward, the boat seems to move backward in the picture.

If the outboard motor is in reverse and the film is run backward, in what direction does the boat appear to move in the picture?

Let us try some examples.

a. The outboard motor is set for a forward speed of 400 feet per minute, and the film is run forward for 2 minutes. The forward speed of the boat may be expressed as $^{+}400$. How would you indicate the running forward of the film for 2 minutes?

b. If the outboard motor were set for a reverse speed of 400 feet per minute, how would you indicate this? If the film were run backwards for 2 minutes, how would you denote this?

c. Now set the outboard speed forward at 400 feet per minute and the film forward for 2 minutes. What is the apparent change in the position of the boat as seen on the screen? This example could be written $^{+}2 \cdot (^{+}400) = {}^{+}800$. (800 feet forward)

Write a multiplication example and find the product to describe the apparent change in position of the boat in each of the following:

a. The outboard is set for 400 feet per minute forward, and the film runs backward for 2 minutes.

b. The outboard is set for a reverse speed of 400 feet per minute, and the film runs forward for 2 minutes.

c. The outboard is set for a reverse speed of 400 feet per minute, and the film runs backward for 2 minutes.

Find the products to describe the apparent change in position of the boat in the following examples:

a. The motor is set for 450 feet per minute forward, and the film runs forward for 2 minutes; 3 minutes; $3\frac{1}{2}$ minutes. $^{+}2(^{+}450) = ?$ $^{+}3(^{+}450) = ?$ $^{+}3\frac{1}{2}(^{+}450) = ?$

b. The motor is set for 500 feet per minute backward, and the film runs forward for 2 minutes; 3 minutes; 5 minutes. $^{+}2(^{-}500) = ?$ $^{+}3(^{-}500) = ?$ $^{+}5(^{-}500) = ?$

c. The motor is set for 300 feet per minute forward, and the film runs backward for 3 minutes; 4 minutes; $5\frac{1}{2}$ minutes. $^{-}3(^{+}300) = ?$ $^{-}4(^{+}300) = ?$ $^{-}5\frac{1}{2}(^{+}300) = ?$

d. The motor is set for 350 feet per minute backward, and the film runs backward for 2 minutes; 3 minutes; 3.6 minutes. $^{-}2(^{-}350) = ?$ $^{-}3(^{-}350) = ?$ $^{-}3.6(^{-}350) = ?$

Have pupils examine the completed chart to find a pattern for multiplication of directed (positive and negative) numbers.

If the outboard motor were shut off, would the boat seem to move in the picture if the film were run forward for 2 minutes? If the film were run forward for 4 minutes? Backward for 3 minutes? Does it seem reasonable to represent the speed for the outboard motor in an "off" position as a zero speed? The above situations would give:

$^{+}2 \cdot (0) = 0$

$^{+}4 \cdot (0) = 0$

$^{-}3 \cdot (0) = 0$

What is the product of zero and any integer?

A more sophisticated presentation of multiplication of signed numbers follows. It is in the form of argument based on the necessity of maintaining the "structure" of the number system—an argument that the previously encountered properties of non-negative numbers must hold.

Suppose we have the following sentence pattern $^{+}2 \cdot {}^{+}5 = n$

Let the one-to-one correspondence table on page 24 help us. First find the arithmetic number corresponding to $^{+}2$. It is 2. Then find the arithmetic number corresponding to $^{+}5$. It is 5. We know that $2 \cdot 5 = 10$, and we find that to our arithmetic number 10 there corresponds the positive number $^{+}10$. So, if our positive numbers are to behave like arithmetic numbers, $^{+}2 \cdot {}^{+}5={}^{+}10$. In the same manner, show that $^{+}3 \cdot {}^{+}4={}^{+}12$ and $^{+}2 \cdot {}^{+}3={}^{+}6$. It is evident that a positive number multiplied by a positive number gives a positive number for a product. Study the following examples and look for a pattern.

$$^{+}2 \cdot {}^{+}5={}^{+}10$$
$$^{+}2 \cdot {}^{+}4={}^{+}8$$
$$^{+}2 \cdot {}^{+}3={}^{+}6$$
$$^{+}2 \cdot {}^{+}2={}^{+}4$$

What would you write next?

$$^{+}2 \cdot {}^{+}1={}^{+}2$$
$$^{+}2 \cdot 0=0$$

What should be next to continue the pattern?

$$^{+}2 \cdot {}^{-}1={}^{-}2$$
$$^{+}2 \cdot {}^{-}2={}^{-}4$$
$$^{+}2 \cdot {}^{-}3={}^{-}6$$
$$^{+}2 \cdot {}^{-}4={}^{-}8$$
$$^{+}2 \cdot {}^{-}5={}^{-}10$$

and so on.

Now that you know, for example, that $^{+}2 \cdot {}^{-}5={}^{-}10$, you should also know from your knowledge of number properties that $^{-}5 \cdot {}^{+}2={}^{-}10$.

Thus the product of a positive number and a negative number is a negative number.

Let us build another pattern.

$$^{-}2 \cdot {}^{+}5={}^{-}10$$
$$^{-}2 \cdot {}^{+}4={}^{-}8$$
$$^{-}2 \cdot {}^{+}3={}^{-}6$$
$$^{-}2 \cdot {}^{+}2={}^{-}4$$

What would we write next?

$$^{-}2 \cdot {}^{+}1={}^{-}2$$
$$^{-}2 \cdot 0=0$$

Before reading on, decide for yourself how the pattern should continue. We would have the following:

$$^{-}2 \cdot {}^{-}1={}^{+}2$$
$$^{-}2 \cdot {}^{-}2={}^{+}4$$
$$^{-}2 \cdot {}^{-}3={}^{+}6$$
$$^{-}2 \cdot {}^{-}4={}^{+}8$$
$$^{-}2 \cdot {}^{-}5={}^{+}10$$

and so on. It appears that a negative number multiplied by a negative number is a positive number.

Suppose that we test another basic property of mathematics, the distributive property, to see if it is consistent for the product of two negatives to be positive:

Since $^{+}9+{}^{-}9=0$, it should follow that $^{-}5 \cdot (^{+}9+{}^{-}9)=0$.

(1) Using the distributive property,
$^{-}5 \cdot (^{+}9+{}^{-}9)=(^{-}5 \cdot {}^{+}9)+(^{-}5 \cdot {}^{-}9)$.
We already know that $^{-}5 \cdot {}^{+}9={}^{-}45$.

(2) Thus, statement (1) can be written
$^{-}45+(^{-}5 \cdot {}^{-}9=0$.
If this statement is to be true it is obvious that

(3) $^{-}5 \cdot {}^{-}9$ must equal $^{+}45$;
thus, $^{-}5 \cdot {}^{-}9={}^{+}45$.

In summary we see that the negative numbers, as opposites of the previously used positive numbers, "act like" the positive numbers in the operations of addition and multiplication with respect to the distributive, associative, and commutative properties, and the properties of zero and one. After addition and multiplication have been completely understood, subtraction and division should be introduced as their inverse operations—operations which are actually "adding the opposite" and "multiplication by the inverse (or reciprocal)."

TOPIC VIII

Extension of Notions of Proof and Summary of Properties*

Suggested time allotment: 8-9 weeks

In this topic mathematics is studied as a way of thinking, of logical reasoning. Pupils must learn to realize that discovering *new ideas* in mathematics may involve experimentation, observation, and inductive reasoning. Concluding statements, however, are arrived at through deductive reasoning. By deductive reasoning we prove that from certain given conditions a definite conclusion necessarily follows.

It is in this topic that pupils learn to solve equalities in the "if . . . then" formal approach to deductive reasoning. Previously, they have had only the experience of finding solution sets for open sentences by replacement methods.

Mathematical logic gives us a precise way to describe complicated situations and to analyze difficult problems. Frequently mathematical reasoning predicts the possibility or impossibility of a scientific experiment.

The special properties of equality—transitive, reflexive, and symmetric should be discussed at this point.

(1) Reflexive: $a=a$

(2) Symmetric: If $a=b$, then $b=a$

(3) Transitive: If $a=b$ and $b=c$, then $a=c$

Since pupils have studied directed numbers (TOPIC VII), they can efficiently and deductively find solution sets for equalities using these two properties.

(1) If $a=b$, then $a+c=b+c$ (Addition Property)

(2) If $a=b$, then $ac=bc$ (Multiplication Property)

We strongly urge that no formal approach to equation solving be used until *after* pupils have studied directed numbers. Equations which might be solved by subtraction or division can be better approached by "adding the opposite (or inverse)" or by "multiplying by the inverse (or reciprocal)."

Finding solution sets for equations would be based upon the definitions, properties, and operations of the real numbers and the equation properties mentioned above:

1. Prove: $r+7=12$ if and only if $r=5$

Example: (a) If $r+7=12$ then $r=5$

Given

$(r+7)+(^-7)=12+(^-7)$ — $a=b \rightarrow a+c=b+c$

$r+[7+(^-7)]=12+(^-7)$ — $(a+b)+c=a+(b+c)$

$r+0=5$ — $a+{}^-a=0$

$r=5$ — $a+0=a$

If $r+7=12$ then $r=5$

Transitive Property of Equality

(b) If $r=5$ then $r+7=12$

$r=5$ — Given

$r+7=5+7$ — $a=b \rightarrow a+c=b+c$

$r+7=12$ — $5+7=12$

$\therefore$ If $r=5$ then $r+7=12$

Transitive Property of Equality

2. Prove: $^-5x=20$ if and only if $x=^-4$

Example: (a) If $^-5x=20$ then $x=^-4$

$^-5x=20$

$\left(\frac{^-1}{5}\right)\left(^-5x\right)=\left(\frac{^-1}{5}\right)\left(20\right)$

Multiplication Property for Equality

$\left[\left(\frac{^-1}{5}\right)\left(^-5\right)\right]x=\left(\frac{^-1}{5}\right)\left(20\right)$

Associative Property of Multiplication

$1x=^-4$

Arithmetic Facts

$x=^-4$

Property of One for Multiplication

$\therefore$ If $^-5x=20$ then $x=^-4$

Transitive Property of Equality

(b) If $x=^-4$ then $^-5x=20$

$x=^-4$ — Given

$(^-5)\,x=(^-5)\,(^-4)$

Multiplication Property of Equality

$^-5x=20$

Arithmetic Fact

$\therefore$ If $x=^-4$ then $^-5x=20$

Transitive Property of Equality

*Topic References: Numbers 2, 5, 29, 31, 32, 34

3. Prove: If $6x+5=17$ then $x=2$

$6x+5=17$ Given

$[6x+5]+{}^{-}5=17+{}^{-}5$

Equation Property for Addition

$6x+[5+{}^{-}5]=17+{}^{-}5$

Associative Property for Multiplication

$6x+0=12$

Arithmetic Fact

$6x=12$

Property of Zero for Addition

$\left(\frac{1}{6}\right)\left(6x\right)=\left(\frac{1}{6}\right)12$

Equation Property for Multiplication

$\left(\frac{1}{6}\cdot 6\right)x=\left(\frac{1}{6}\right)12$

Associative Property for Addition

$1x=2$

Arithmetic Fact

$x=2$

Property of One for Multiplication

$\therefore$ If $6x+5=17$ then $x=2$

Transitive Property of Equality

What we have proven so far is: If $6x+5=17$, then $x=2$. It is easy now to prove the converse, if $x=2$, then $6x+5=17$. Have pupils do this.

4. Prove: If $3+4x=x\cdot{}^{-}4$ then $x=\frac{^{-}3}{8}$

$3+4x=x\cdot{}^{-}4$ Given

$[3+4x]+{}^{-}4x=x\cdot{}^{-}4+{}^{-}4x$

Addition Property of Equations

$3+[(4x+{}^{-}4x)]=x\cdot{}^{-}4+{}^{-}4x$

Associative Property Addition

$3+[4x+{}^{-}4x]={}^{-}4x+{}^{-}4x$

Commutative Property of Multiplication

$3+0={}^{-}4x+{}^{-}4x$

Property of the Addition of Opposites

$3={}^{-}4x+{}^{-}4x$

Property of the Additive Identity

$3=({}^{-}4+{}^{-}4)x$

Distributive Property

$3={}^{-}8x$

Arithmetic Fact

${}^{-}8x=3$

Symmetric Property of Equality

$\frac{^{-}1}{8}\cdot{}^{-}8x=\frac{^{-}1}{8}\cdot 3$

Equation Property of Multiplication

$\left(\frac{^{-}1}{8}\cdot{}^{-}8\right)x=\frac{^{-}1}{8}\cdot 3$

Associative Property of Multiplication

$1x=\frac{^{-}3}{8}$

Arithmetic Facts

$\therefore x=\frac{^{-}3}{8}$

Property of Multiplication Identity

With the emphasis upon the basic properties of the number system, it should be evident that "transposing" is not an operation and that it should no longer be taught.

The culmination of this topic is the summary of all the properties which the pupil has at his command. The pupil should be given a feeling of confidence at this point that these properties justify all the mathematical processes in his present course and the algebra work to follow. (See table on the following page.)

*Note: The special property of zero for multiplication in the real number system can be proved deductively from the properties listed in the table and then called a theorem: $\forall x,\ x\cdot 0=0$.

Pupils should be given additional experience finding solution sets for inequalities. (Formal principles in the study of inequalities are presented in TOPIC XII of the ninth-grade program.) The universe for each variable should be precisely described. Examples should include sentences having solution sets which contain one, many, or no members.

Examples:

$2x+1>3$	$\lvert z\rvert+12=6$
$4-3x<13$	$4+7x>x-6$
$2-x>6$	$2x+4>x+9$
$\lvert x-4\rvert<1$	$x^2<4$

Many verbal problems should be included in this topic and throughout the remainder of the course. Include not only those of the conventional type requiring sentences of equality for their solution, but also problems which require the use of inequalities. Some problems should result in empty solution sets and some in solution sets which contain many members. Several examples follow:

Summary of the Basic Properties of the Real Numbers (A Number Field)

L = the set of real numbers

'$\forall$' is a symbol meaning 'for each'

Commutative Properties

$\forall x \, \forall y,\ x+y=y+x$

$\forall x \, \forall y,\ xy=yx$

Associative Properties

$\forall x \, \forall y \, \forall z,\ (x+y)+z=x+(y+z)$

$\forall x \, \forall y \, \forall z,\ (xy)\,z=x(y\,z)$

Distributive Property

$\forall x \, \forall y \, \forall z,\ (x+y)z=xz+yz$

Identity Elements

For Addition $\forall x,\ x+0=x$

For Multiplication $\forall x,\ x \cdot 1=x$

Inverses

Additive

$\forall x$ there exists $({}^{-}x)$ such that $x+({}^{-}x)=0$

Multiplicative

$\forall x,\ x \neq 0$, there exists (x^{-1}) such that $x \cdot (x^{-1})=0$

Definition of Subtraction

$\forall x \, \forall y,\ x-y=x+{}^{-}y$

Definition of Division

$\forall x, \forall y,\ y \neq 0, \frac{x}{y}=x \cdot \frac{1}{y}$

Closure under Addition

If x is in L and y is in L, then x+y is in L

$\forall x$ and $\forall y$, x+y is unique

Closure under Multiplication

If x is in L and Y is in L, then xy is in L

$\forall x$ and $\forall y$, xy is unique

(1) Mary bought 15 three-cent stamps and some four-cent stamps. If she paid $1.80 for all the stamps, was she charged the correct amount?

(2) The sum of two successive positive integers is *less* than 25. Find the integers.

(3) Find two consecutive odd positive integers whose sum is less than or equal to 83.

(4) Four added to the number results in a sum that is greater than three times the original number. From what set could the original number have been chosen?

(5) John is 5 years younger than Mark. The sum of their ages is less than 39. What is the greatest possible age for John?

Thought provoking questions a teacher might ask:

1. Are the operations of subtraction and division commutative? Give examples to justify your answer.

2. Are the operations of subtraction and division associative operations? Give examples.

3. What is the reciprocal of 0?

4. Adding the opposite of what number is the inverse of adding ${}^{+}7$?

5. If $17+a=0$, what property of real numbers tells us at once that the sentence will be true for $a={}^{-}17$?

6. Make up three numerical examples for each of these properties of the real numbers. Check each sentence.
   (a) Property for adding the real number 0.
   (b) Property for multiplying by the real number ${}^{+}1$.

7. Prove: If x is a real number, then $3(2x)=6x$.

SAMPLE LESSON PLAN

A sample lesson plan which might follow from question 7 above: I am placing on the board two statements about numbers. Either one, or both, may be true or false. Do a little testing and experimentation to decide whether each is true or false. If you believe either is true, try to prove its truth by logical reasoning using the properties of real numbers.

The statements are:

1. If x is a rational number, then $1+2x=3x$.

2. If x is a rational number, then $3(2x)=6x$.

Give pupils sufficient time to investigate each before you ask them their decision.

Considering the statement $1+2x=3x$, a pupil may have believed it false because:

$$1+2(10)=3(10)$$
$$1+20=30$$
$$21\neq 30$$

Another pupil may have believed it true because:

$$1+2(1)=3(1)$$
$$1+2=3$$
$$3=3$$

But the first pupil has discovered *one* counter example, that is, one number for which the statement is not true. Even though it may be true for some numbers, *one* counter example is sufficient to make the generalization false.

Considering the statement $3(2x)=6x$, each pupil in class may have tested a different real number for x and in every case found the sentence true. Is this enough to prove it always true? How many such examples would one need? One certainly could not live long enough to test and verify each possible number replacement.

Consider another method of testing. Let us start with the phrase on the left and try to transform it into the phrase on the right, justifying each step with the properties of real numbers.

Prove: If x is a real number, then $3(x \cdot 2)=6x$.

Proof: $3(x \cdot 2)=3(2 \cdot x)$
 Commutative Property for Multiplication
$3(2 \cdot x)=(3 \cdot 2)x$
 Associative Property for Multiplication
$(3 \cdot 2)x=6x$
 Arithmetic Fact
$\therefore 3(x \cdot 2)=6x$
 Transitive Property of Equality

This proof shows that each step is a consequence of the commutative or associative properties for multiplication or the fact that $3 \cdot 2=6$. Hence, the conclusion is a generalization that one must accept if he accepts the premises.

Prove: If k is a real number, then $3k+(9k-2)=12k-2$.

Proof: $3k+(9k-2)=3k+(9k+{}^-2)$
 Definition of Subtraction
$3k+(9k+{}^-2)=(3k+9k)+{}^-2$
 Associative Property for Addition
$(3k+9k)+{}^-2=(3+9)k+{}^-2$
 Distributive Property of Multiplication over Addition
$(3+9)k+{}^-2=12k+{}^-2$
$3+9=12$
 Arithmetic Fact
$12k+{}^-2=12k-2$
 Definition of Subtraction
$\therefore 3k+(9k-2)=12k-2$
 Transitive Property of Equality

We give pupils a variety of exercises which involve proof of the type shown here. Any statement which we can prove by deductive reasoning is called a theorem.

One such theorem and its proof might be: For each non-zero real number a, there is one and only one multiplicative inverse of a.

Proof: Let us assume that a multiplicative inverse of a is b; that is, $a \cdot b=1$. If a has *another* inverse under multiplication, say x, such that $a \cdot x=1$, then we have:
$ax=1$
$b(ax)=b$
 Multiplication Property of One and Multiplication Property of Equations
$(ba)x=b$
 Associative Property of Multiplication
$(ab)x=b$
 Commutative Property of Multiplication
$1x=b$
 Definition of Reciprocal
$x=b$
 Identity Element for Multiplication

Thus, this possible second inverse x is equal to b. Hence, it follows that a has one and only one multiplicative inverse.

Sample pupil exercises might be these: Each of the following sentences is a generalization about real numbers. Some are true and some are false. Write a proof for each of the sentences which you judge to be true. Give a counter example for those you think are false.

1. If t is a real number, then $3(5t)=15t$.
2. If q is a real number, then $3+6q=9q$.
3. If r is a real number, then $3+6r=3(1+2r)$.
4. If x is a real number, then $x \cdot 1+x=2x$.
5. No matter which number you choose, if you add it to 9 and then add this sum to 1, the result will be 10 plus the chosen number.
6. If r is a real number, then $8r+7=7r+8$.
7. If x is a real number, then $x \cdot x \cdot x=3x$.
8. If x is a real number, then $3x \cdot 4=12x$.
9. If x is a real number, then $9+7x+3=7x+12$.

For each of the following sentences write the numeral in the box which will make the statement true. Tell which property the statement illustrates.

1. ${}^{+}5 \cdot \square = {}^{-}12 \cdot {}^{+}5$
2. ${}^{-}3 \cdot {}^{-}7 = \square \cdot {}^{-}3$
3. $({}^{-}3 + {}^{+}7) \cdot 5 = {}^{-}3 \cdot {}^{+}5 + \square \cdot {}^{+}5$
4. ${}^{-}6 \cdot ({}^{+}8 \cdot \square) = ({}^{-}6 \cdot {}^{+}8) \cdot {}^{+}15$
5. ${}^{+}9 + {}^{-}3 + {}^{+}5 = {}^{+}9 + (\square + {}^{+}5)$
6. ${}^{-}2 \cdot {}^{-}7 + {}^{-}2 \cdot \square = {}^{-}2 \cdot ({}^{-}7 + {}^{+}17)$
7. ${}^{+}9 \cdot \square = 0$
8. ${}^{-}3 \cdot \square = {}^{-}3$
9. ${}^{-}5 \cdot \square = {}^{-}5$
10. ${}^{+}5 \cdot {}^{-}7 \cdot \square = 0$
11. $0 = \square \cdot 0$
12. $\square + 0 = 0$
13. $4 \cdot (7 + \square) = 4 \cdot 7 + 4 \cdot {}^{-}6$
14. ${}^{-}25 = \square + 0$
15. ${}^{-}25 = \square \cdot 0$

Complete the following proofs:

1. Prove: $8(4+3)=4 \cdot 8+3 \cdot 8$

Statements	Reasons
(1) $8(4+3)=8 \cdot 4+8 \cdot 3$	(1) Distributive Property (multiplication over addition)
(2) $8 \cdot 4+8 \cdot 3=4 \cdot 8+3 \cdot 8$	(2) ....
(3) $\therefore 8(4+3)=4 \cdot 8+3 \cdot 8$	(3) Transitive Property of Equality

2. Prove: $7 \cdot 8+5 \cdot 9=9 \cdot 5+8 \cdot 7$

Statements	Reasons
(1) $7 \cdot 8+5 \cdot 9=8 \cdot 7+9 \cdot 5$	(1) ....
(2) $8 \cdot 7+9 \cdot 5=9 \cdot 5+8 \cdot 7$	(2) Commutative Property of Addition
(3) $\therefore 7 \cdot 8+5 \cdot 9=9 \cdot 5+8 \cdot 7$	(3) Transitive Property of Equality

3. Prove: If a and x are real numbers, then $a(7+x)=a \cdot x+7 \cdot a$.

Statements	Reasons
(1) $a(7+x)=a \cdot 7+a \cdot x$	(1) ....
(2)	(2) Commutative Property of Addition
(3) $a \cdot x+a \cdot 7=a \cdot x+7 \cdot a$	(3) ....
(4) $\therefore a(7+x)=a \cdot x+7 \cdot a$	(4) ....

4. Prove: $7(17+3)=3\cdot 7+17\cdot 7$

Statements	Reasons
(1) $7(17+3)=$	(1) Distributive Property
(2)	(2) Commutative Property of Multiplication
(3)	(3) Commutative Property of Addition
(4) $\therefore$	(4) Transitive Property of Equality

5. Prove: $23+2(5+11)=23+10+22$

Statements	Reasons
(1) $23+2(5+11)=23+2\cdot 5+2\cdot 11$	(1) ....
(2) $23+2\cdot 5+2\cdot 11=23+10+22$	(2) Multiplication Facts
(3) $\therefore$	(3) ....

6. Prove: $19\cdot 7+19\cdot 3=19\cdot 10$

Statements	Reasons
(1) $19\cdot 7+19\cdot 3=$	(1) Distributive Property
(2) $19(7+3)=19\cdot 10$	(2) Addition Fact
(3)	(3) ....

7. Prove: If a, b, and c are real numbers, then $a\cdot b+a\cdot c=c\cdot a+b\cdot a$.

Statements	Reasons
(1)	(1) ....
(2)	(2) ....
(3)	(3) ....

8. Prove: If □, △, and * are natural numbers, then □+(△+*)=(△+□)+*.

Statements	Reasons
(1)	(1) ....
(2)	(2) ....
(3)	(3) ....

9. Prove: If △ and □ are natural numbers, then *(△+□)=□ · *+△ · *.

Statements	Reasons
(1) *(△+□)=* · △+* · □	(1) ....
(2) * · △+* · □=△ · *+□ · *	(2) ....
(3) △ · *+□ · *=□ · *+△ · *	(3) ....
(4) *(△+□)=□ · *+△ · *	(4) ....

10. Prove: $50 \cdot 27+13 \cdot 50=50(27+13)$

Statements	Reasons
(1) $50 \cdot 27+13 \cdot 50=50 \cdot 27+50 \cdot 13$	(1) ....
(2) $50 \cdot 27+50 \cdot 13=50(27+13)$	(2) ....
(3) $50 \cdot 27+13 \cdot 50=50(27+13)$	(3) ....

11. Prove: If a, b, and c are natural numbers, then $(a \cdot b) \cdot c=c \cdot (a \cdot b)$

Statements	Reasons
(1) $(a \cdot b) \cdot c=a \cdot (b \cdot c)$	(1) ....
(2) $a \cdot (b \cdot c)=a \cdot (c \cdot b)$	(2) ....
(3) $a \cdot (c \cdot b)=(a \cdot c) \cdot b$	(3) ....
(4) $(a \cdot c) \cdot b=(c \cdot a) \cdot b$	(4) ....
(5) $(c \cdot a) \cdot b=c \cdot (a \cdot b)$	(5) ....
(6) $(a \cdot b) \cdot c=c \cdot (a \cdot b)$	(6) ....

12. Prove: If □, △, and * are natural numbers, then □ · △+*=*+△ · □

Statements	Reasons
(1) □·△+*=	(1) Commutative Property of Addition
(2)	(2) ....
(3)	(3) ....

If the statement is *true,* state which property of real numbers has been used.

If the statement is *false,* rewrite it to make it true.

1. $a(b \cdot c)=a \cdot b+a \cdot c$ ....
2. $x-y=y-x$ ....
3. $c+d=c \cdot d$ ....
4. $xy+xz=x \cdot z+x \cdot y$ ....
5. $x+y=y+x$ ....
6. $x+(y+z)=x \cdot y+x \cdot z$ ....
7. $100+19=119$ ....
8. $12 \cdot 9=108$ ....

The following are suggested questions about inverses and identity elements.

1. A member of the set of natural numbers has the property that when it is multiplied by any number of the set, the product equals that member by which it was multiplied.

   For example: □ · a=a.

   (a) What number has this property?....................

   (b) Give an example to illustrate this.................

   (c) The answer to (a) is called the........................ element for multiplication.

   (d) What number has the above property in the set of fractions?...

   ....

   (e) Give an example to illustrate this..................

   ....

2. A member of the set which includes natural numbers, zero, and the fractions has the property that when it is added to any member of the set, the sum equals that member to which it was added.

   Foı example: a+□=a.

   (a) What number has this property?....................

   (b) Give an example to illustrate this...................

   (c) The answer to (a) is called the...................... element for addition.

3. If a and b are symbols which reprsent natural numbers and $a \cdot b=0$, what number does a represent when $b=3$?

   (a) $b>7$ and $a \cdot b=0$ a= ....

   (b) $a=\frac{1}{2}$ and $a \cdot b=0$ b= ....

If the universe for a, b, and c is the set of fractions, which of these are valid conclusions?

1. If $\frac{a}{b}=c$, then:
   (a) $b\neq0$
   (b) If $b>1$, $a>c$
   (c) $\frac{a}{c}=b$
   (d) If $c=0$, $a=0$

2. If $\frac{a}{b}>1$, then:
   (a) $\frac{a}{b}\times c>c$
   (b) $a>b$
   (c) $b<a$
   (d) $a<b$

3. If $\frac{a}{b}=\frac{c}{d}$, then:
   (a) $a>b$
   (b) $a\times b=c\times d$
   (c) $a\times d=b\times c$
   (d) $a=c$

4. If $a>b$, then:
   (a) $b<a$
   (b) $a+1>b$
   (c) $b+100<a$
   (d) $\frac{a}{c}<\frac{b}{c}$

5. If $a+b=a+c$, then:
   (a) $b>c$
   (b) $a<b$
   (c) $b=c$
   (d) $a=b$

6. If $a+b=c$, then:
   (a) $b=c$
   (b) $b<c$
   (c) $c-b=a$
   (d) $c>a$
   (e) $c+b=a$

7. If $\frac{a}{b}<1$, then:
   (a) $\frac{a}{b}+\frac{5}{6}<1$
   (b) $a>b$
   (c) $a=b$
   (d) $\frac{a}{b}\cdot c=\frac{a\cdot c}{b}$

8. If $a+b<1$, then:
   (a) $a \cdot b$ are both <1
   (b) a and b are both fractions
   (c) $\frac{1}{2}\cdot a+\frac{1}{2}\cdot b>1$
   (d) $a<1$
   (e) $a>b$

9. If $a-b=1$, then:
   (a) $a=b$
   (b) $a>b$
   (c) $a\neq0$
   (d) $a \cdot b=a^2$

10. If $a\times b=c$ and a, b, and c are natural numbers, then:
    (a) $a>c$
    (b) $a+c=c$
    (c) $c>b$
    (d) $2a+2b=4c$
    (e) $6c=3a\times2b$

11. If $a+b=c$ and a and b are natural numbers, then:
    (a) c is a natural number
    (b) c is an even number
    (c) $c>b$
    (d) $\frac{a}{c}<1$

Justify each step with a basic property of the real numbers:

1. $(y-x)+(y+x)=[y+(^-x)]+(y+x)$
   $=y+(^-x)+(x+y)$
   $=y+[(^-x)+x]+y$
   $=y+[x+(^-x)]+y$
   $=y+0+y$
   $=(y+0)+y$
   $=y+y$
   $=1\cdot y+1\cdot y$
   $=(1+1)y$
   $=2y$

2. $\frac{xr}{yr} = \left(\frac{x}{y}\right)\left(\frac{r}{r}\right)$

$= \frac{x}{y} \cdot \left[r \cdot \left(\frac{1}{r}\right)\right]$

$= \frac{x}{y} \cdot 1$

$= \frac{x}{y}$

3. $xy + (^{-}x)y = [x + (^{-}x)]y$

$= y[x + (^{-}x)]$

$= y\ (0)$

$= 0$

Although this is presented here as a block of work, teachers will probably find it most successful to space the work on proof throughout the remainder of the year. Only through sustained and repeated practice will pupils improve their ease and power in proof. This is one of the cyclical growths in mathematical ability for the pupil. The mastering of the use of the basic properties in proof is highly important because it is here that a pupil sees mathematics as an organized structure of properties rather than as a group of isolated mechanical processes. The attention given to rigor in proof at any stage of the child's development can, however, be over-emphasized; these ideas are acquired slowly over a long period of meaningful experiences.

TOPIC IX

Sets of Points: Measurement and Construction*

Suggested time allotment: 7-8 weeks

The content of this topic is geometry, the study of measurement and the construction of figures in space. The ideas, however, are not to be developed with the rigor of a high school course. Rather the teacher should call upon his pupils for original and intuitive thinking; stimulate them with geometric models and figures to focus attention upon *concepts* related to figures in space.

Do not limit discussions to the plane. The ideas of geometry are better understood if originally developed in space. By intersecting different solids with planes, various lines and curves are formed. Points of intersection of surfaces will lead to a review of the concepts of point and line. Mental imagery and visualization is extremely important here.

Draw various polygons and have the pupils construct a figure exactly like the one drawn, knowing the measures of specified sides and angles. The discussion which follows can lead to the postulates and theorems of congruency. Congruence can be considered as the existence of a certain kind of one-to-one correspondence between two or more geometric figures.

By intersecting a solid figure, ideas of similarity may also be examined. Establish how similar polygons may be constructed. Have the pupils write generalizations that may be established for all similar polygons.

In the study of area, it is worthwhile to begin by considering a *unit square* as the basic unit of measure. Area then expresses the number of square units required to "cover" a specific surface. This number of square units can be determined by counting the number required in a row across the area and by multiplying this number times the number of rows. Notice the following example:

5 units per row; 7 rows
$5 \times 7 = 35$
area $= 35$ square units

This problem can also be approached as a type of ratio problem:

$$\frac{1 \text{ row}}{5 \text{ units}} \sim \frac{7 \text{ rows}}{n \text{ units}} \longrightarrow \frac{1}{5} \sim \frac{7}{n}$$

$n = 35$
area $= 35$ square units

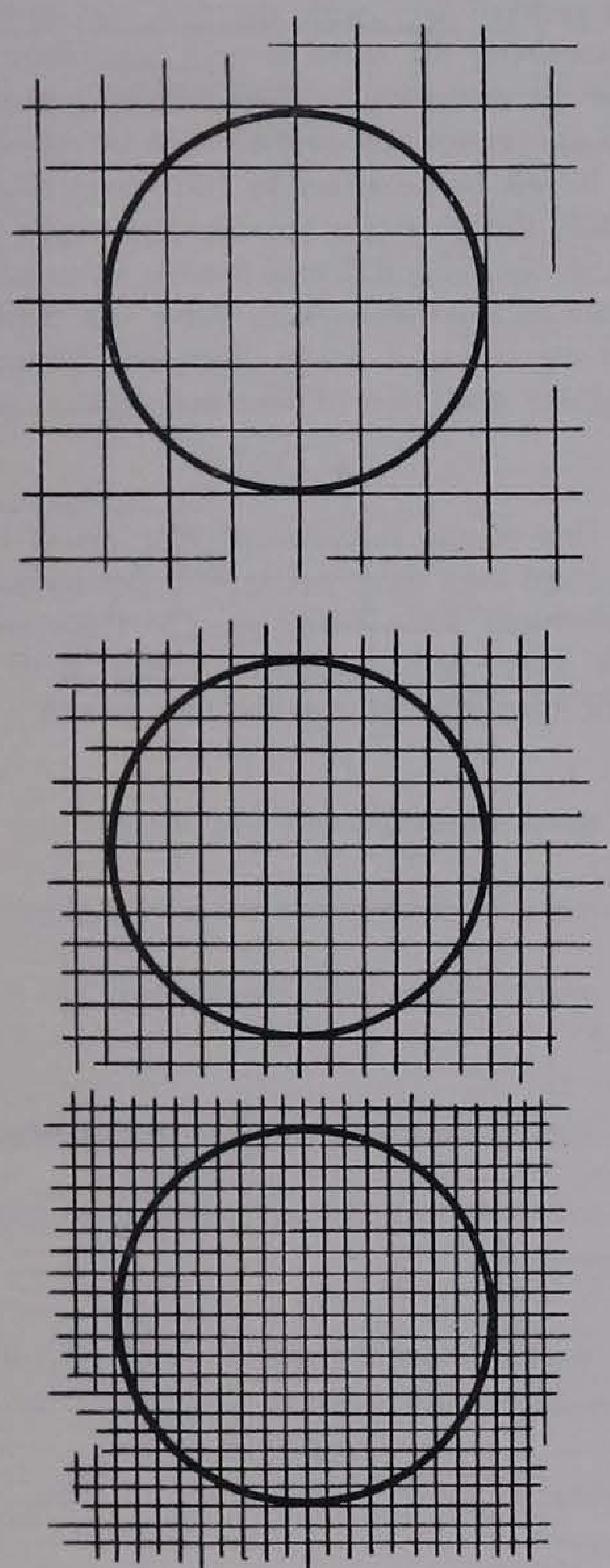
It is important that pupils understand that there is a *unit* of area—the square — before proceeding to develop the short-cut method or common formula $A = b \cdot h$.

Following the study of the area of the rectangle, a fundamental approach to the area of the parallelogram is through the use of equal ratios. The further development of areas for other polygons may be explored after pupils have a command of the formula for finding the area of a triangle. Mathematical sentences may be written describing areas of various polygons. Following consideration of the development of the area of a regular polygon with "n" sides, the area of a circle will naturally follow.

Through the use of the idea of ratio, pupils can be led to discover the relation which exists between the circumference of a circle and its diameter or radius. Pupils should be given the opportunity to experiment freely, collecting data by measuring many circular objects. Then they should be encouraged to "discover" the relationship involving approximations.

*Topic References: Numbers 1, 11, 15, 19, 32

An interesting approach to the study of the area of a circle is to overlay several transparent grids made up of different-sized squares on a circle.



By approximating the number of square units covering the circle (count one-fourth of the circle and multiply by 4), the pupils can be encouraged to intuitively arrive at the general formula for the area of a circle.

Problem situations in the study of geometry should not be limited to the plane. Use various solids and find the surface areas of the solids. Given the height, base, and lengths of the sides of a pyramid, find the area of the triangular faces. Problems like this commonly require the Pythagorean Theorem. Perhaps the best intuitive development of the Rule of Pythagoras involves its development through the study of the areas of squares on the sides of the right triangle. This is a very familiar approach and will not be expanded here.

A new word may be introduced while finding the surface areas of various solids. The word is "parallelepiped," describing any parallelogram prism (a prism with all faces parallelograms). A special parallelepiped, the *cube*, is the basic unit for the measurement of volume. Relate the area formulas already developed to the formulas for finding volume. As much as possible, let pupils write the sentences which will describe the volume of various solids such as: (1) parallelepipeds, (2) pyramids, (3) cylinders, (4) spheres.

Through the entire topic, concepts of measurement must be developed. The use of different systems of units of measure in problem-solving situations should be employed. This can readily lead to the study of the metric system of measurement, placing emphasis on using it as a *system* rather than upon changing units to English-measure equivalents.

Suggested Developmental Questions

Following are some key questions which may be studied during the development of this topic.

1. How many planes are required to limit a solid in space?
2. How may various polygons be formed in space by a plane intersecting various solid figures?
3. What measures must be known before a triangle may be constructed?
4. What types of areas may be described by three intersecting lines? four intersecting lines? five intersecting lines? etc.

5. Given a constant value for the length of one side of a triangle, what mathematical conditions may be stated for the other two sides, if they are allowed to vary?

6. What is the most concise mathematical sentence that can be written to describe the surface area of a cube? a rectangular prism?

7. In how many ways can we divide a line segment into two equal segments?

8. Questions related to an isosceles triangle:
   a. How is the vertex related to the base?
   b. What is the relationship of the altitude to the vertex angle? to the base?

9. What possible combination of given line segments always form a right triangle?

10. What happens when the Pythagorean Theorem is applied to an isosoceles right triangle?

11. How may the Pythagorean Theorem be used to determine whether a triangle is (a) acute, (b) obtuse, or (c) right?

Discussion of Developmental Questions

The following is largely a discussion of logic in terms of mathematical theory. It is an argument intended to stimulate the pupils' thoughts toward point, line, plane, and solid.

The location of a point in space is a strictly mental activity. Even the placement of a dot does describe a point; for if the dot is to be visible, it must take on the properties of three dimensions. The pupil must be able to use his imagination to *think* about this point in space. Its location must be of the pupil's own making.

What happens if a second point is brought into the discussion? Provided that the second point is distinct (not of the same placement as the first point), there should be relations between the two points. The pupil should be led to investigate these relations. The thought of line will enter the discussion. Try to decide what a line is. Work toward the idea that a solid line is not a necessary condition and that a line may well mean a set of ordered points. Now there are not just two points, but a great number of points lying within a defined pattern which includes the two original points and forms a line.

Introduce the idea of a third distinct point. Do not let this third point be colinear (an element of the first ordered set) with the first two points. The system involving the three distinct noncolinear points will limit the discussion to ideas involving planes. The word "plane" is not new to the pupils by this time, but possibly it will be necessary to help them discuss and clarify their thoughts. Try to stimulate pupils to think in terms of the many different pattern variations which may occur in this new system. Give the pupil confidence in the realm of points, lines, and planes before beginning the discussion of four noncoplanar points in space.

The idea of the existence of four points in space with no more than three points in a determined plane may give pupils some difficulty. The following set of questions may help them to orient their thinking. These questions refer to the four points.

1. How many distinct points are there?

2. How many lines are determined by these points?

3. How many planes are determined by the four points?

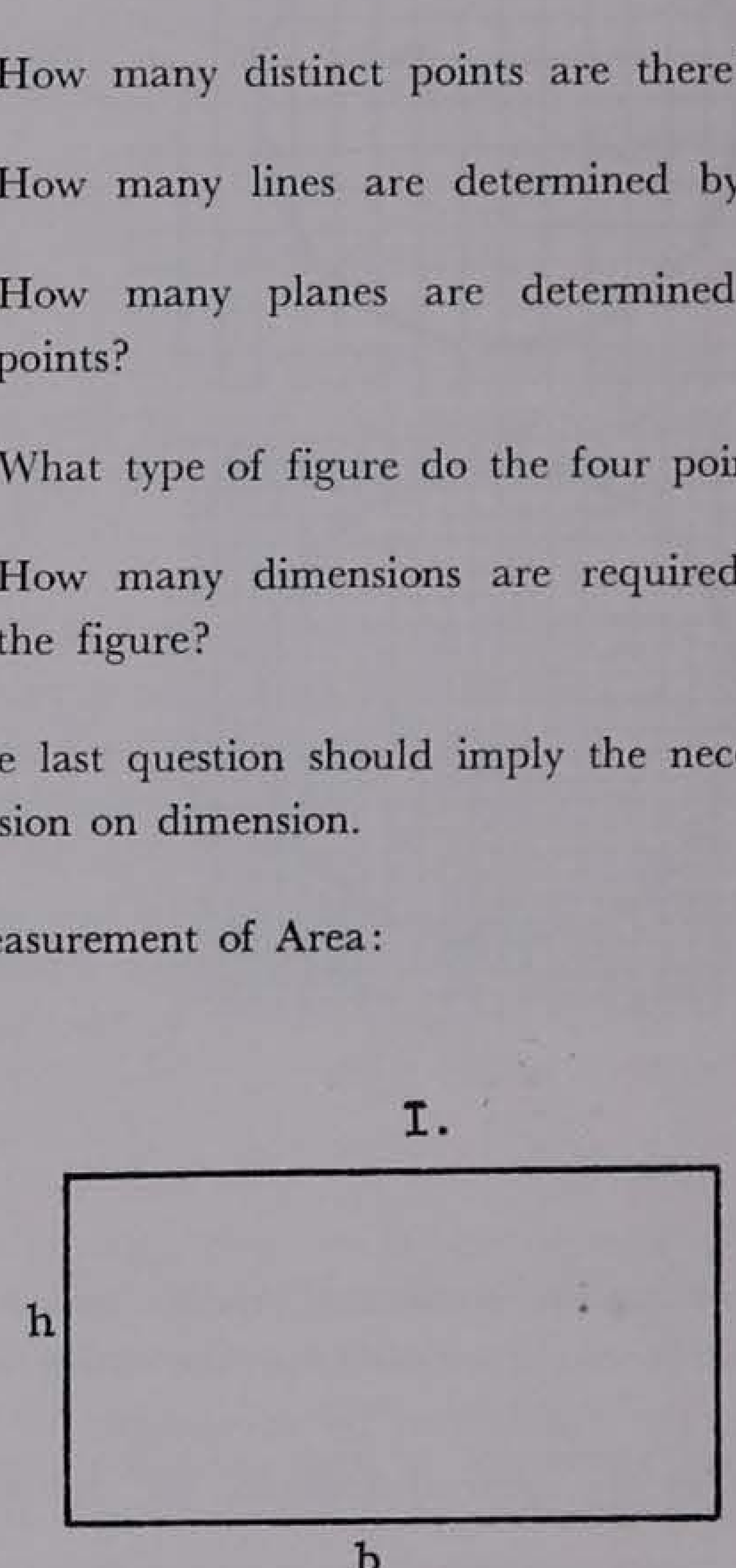
4. What type of figure do the four points suggest?

5. How many dimensions are required to determine the figure?

The last question should imply the necessity of a discussion on dimension.

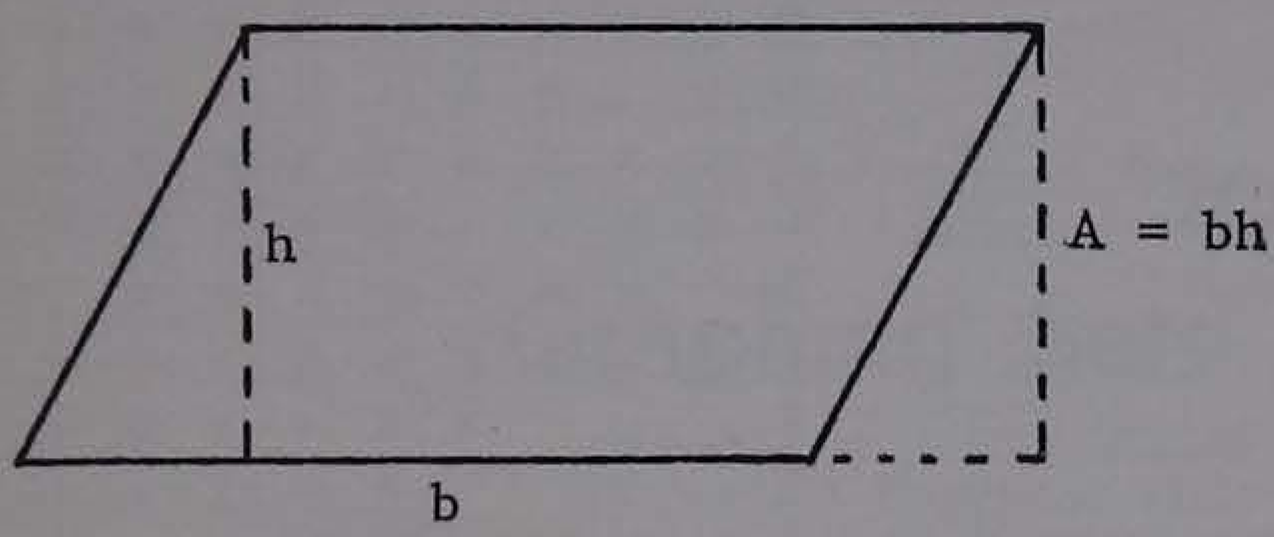
Measurement of Area:

I.

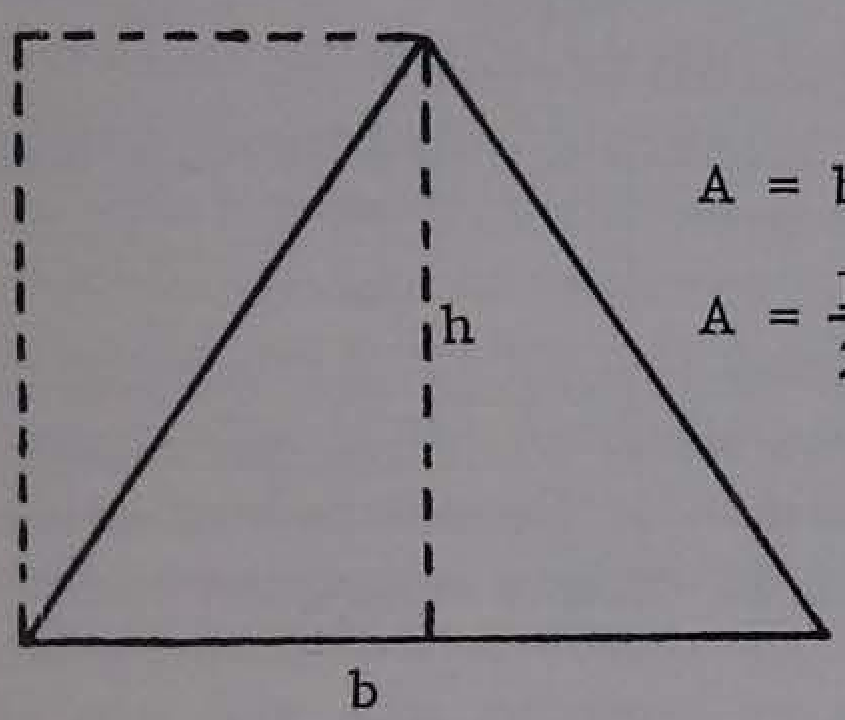


$A = bh$

II.



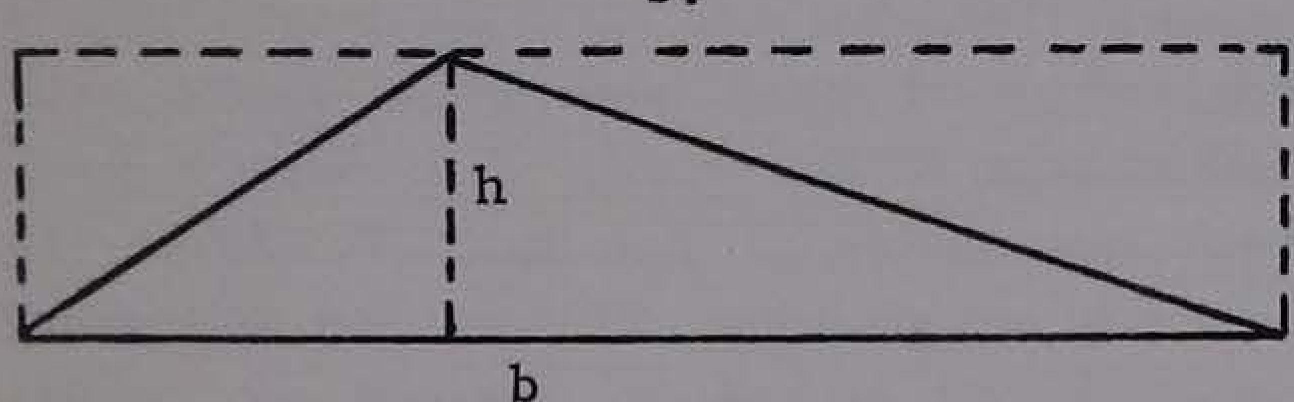
III. a.



A = bh rectangle

$A = \frac{1}{2}bh$ triangle

b.



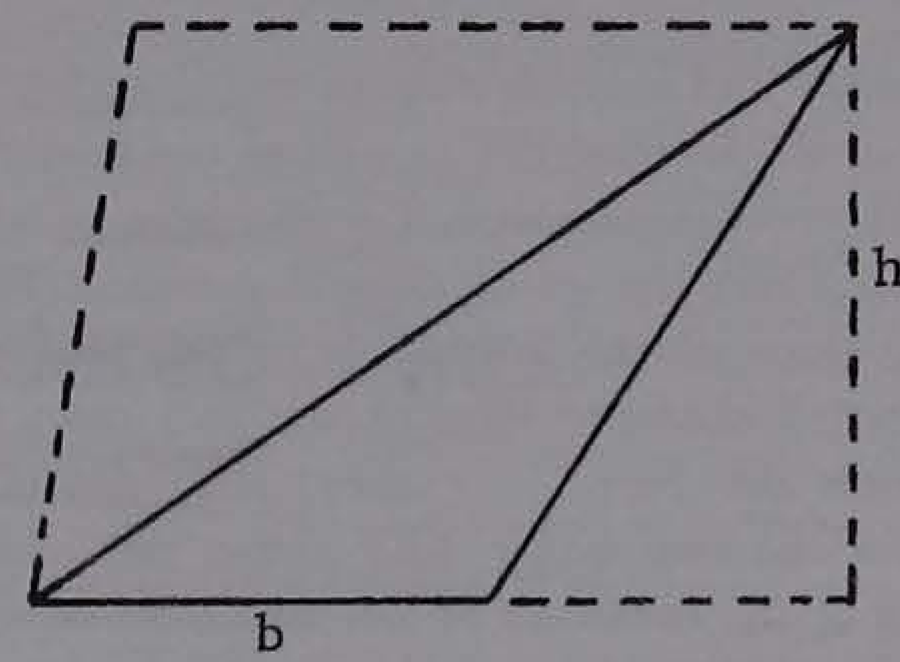
A = bh rectangle

$A = \frac{1}{2}bh$ triangle

or

$A = \frac{bh}{2}$

c.



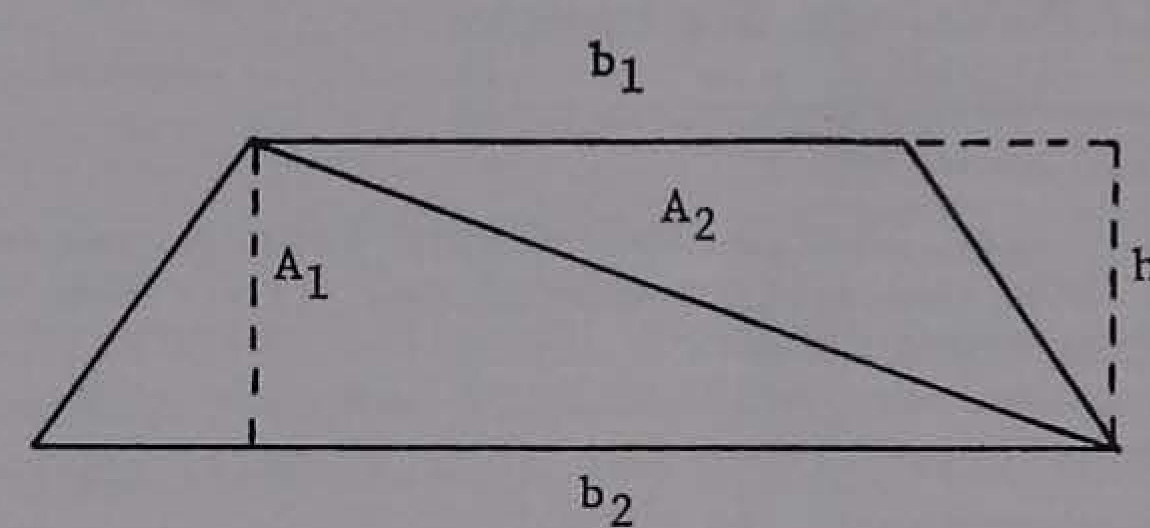
A = bh parallelogram

$A = \frac{bh}{2}$ triangle

$A = \frac{1}{2}bh$

It may now be observed that any polygon of higher order than three sides may be divided into triangles. With this as a basic premise, the formulas for the areas of any polygon may be developed.

Example:



$A_1 + A_2 = A_t$ (Area of the Trapezoid)

$A_1 = \frac{1}{2} b_2 h$ and $A_2 = \frac{1}{2} b_1 h$

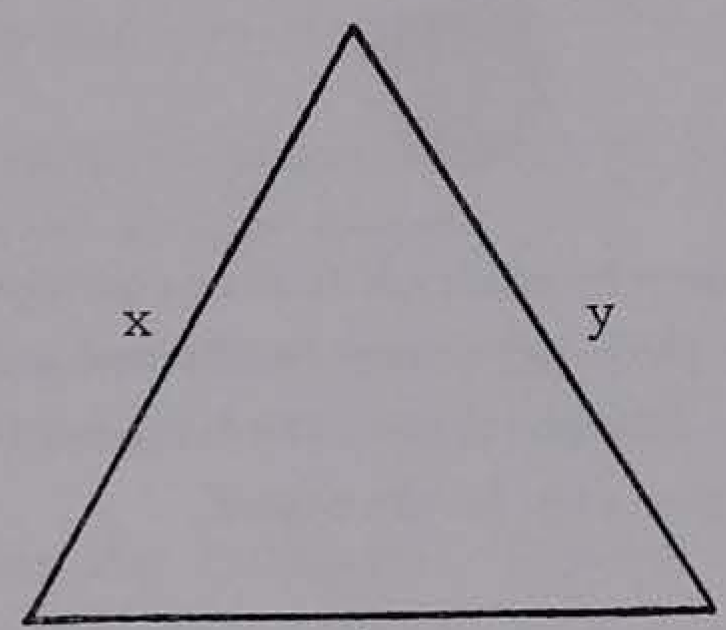
then: $\frac{1}{2} b_2 h + \frac{1}{2} b_1 h = A_t$

or $\frac{1}{2} h (b_2 + b_1) = A_t$

or $A_t = \frac{b_1 + b_2}{2} h$

Discussion of Question 7

The study and investigation of this question involves the ideas of the inequalities as well as the first introduction to the coordinate plane. Let the base of a triangle be six units using the set of natural numbers. How may the other sides of the triangle vary as to length? It may be noted that the sum of the two sides must be greater than six. Now, how great may the difference be between the two sides? This shows the relationship of the sum of the measures of any two sides of a triangle as related to the measure of the third side. Draw a triangle with a base of the given measure. Let one of the variable sides be x and the other side be y. Each time an acceptable pair of values is given, plot them on a lattice.



For the sake of brevity let the values of x and y be of a finite universe. For example:

Let $S=\{1, 2, 3, \ldots, 10\}$
and let $U=S\times S$

Furthermore, after you have studied Topic X, you may plot these values as an ordered pair, (x,y). Note the patterns which develop.

y 10	.	.	.	.	x	x	x	x	x	x
9	.	.	.	x	x	x	x	x	x	x
8	.	.	x	x	x	x	x	x	x	x
7	.	x	x	x	x	x	x	x	x	x
6	x	x	x	x	x	x	x	x	x	x
5	.	x	x	x	x	x	x	x	x	x
4	.	.	x	x	x	x	x	x	x	.
3	.	.	.	x	x	x	x	x	.	.
2	.	.	.	.	x	x	x	.	.	.
1	.	.	.	.	.	x	.	.	.	.
	1	2	3	4	5	6	7	8	9	10 x

Can you answer the following questions about the triangles represented on the above lattice?

1. Where are the isosceles triangles?
2. Where are the right triangles?
3. Where are the obtuse triangles?
4. Where are the acute triangles?
5. Where is the equilateral triangle?
6. Using your own imagination, what more can be done with this configuration?

Summary

As teachers we must recognize that the role of geometry in the high school curriculum has changed. It is no longer practical, or desirable for that matter, for schools to delay the study of basic geometric concepts, measurement, and constructions until tenth grade. These topics must be expanded upon and take on greater significance in the junior-high-school years. This will allow pupils in high school geometry to go more deeply into the study of the nature of proof, logical systems, and analytic geometry. This topic gives the teacher some leading ideas into what may be done in geometry at the eighth grade level.

TOPIC X
Graphing Sets of Ordered Pairs*

Suggested Time Allotment: 7-8 Weeks

This is the pupil's first hard look at the concept of the coordinate plane. So far he has dealt only with linear coordinate systems. Now his understanding of the idea of a point will be expanded. He will also study the use of lines in the coordinate plane as sets of ordered conditions which lead readily into the study of linear programing. The notation and vocabulary of this topic is extremely important, as it is the foundation of advanced study in mathematics. Teachers face an interesting task in leading their pupils to a fundamental understanding of the coordinate plane and its role in the study of mathematics. Teachers might well plan to place considerable emphasis on this study.

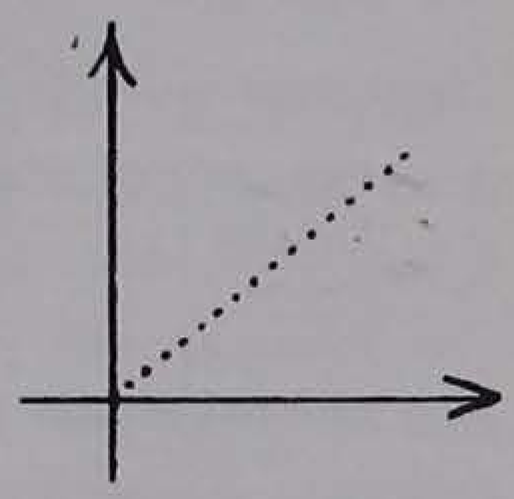
It should include exercises and games directed toward helping the pupil locate points on a lattice. Use as a universe, $U=\{1, 2, 3, \ldots, 10\}$ and develop $U \times U$ in such a way that a finite part of the first quadrant is developed. The games can lead to competency in locating points. It is desirable to use a finite set for these early experiences in graphing.

Find solution sets in the finite universe from conditions written in the form (x,y). Intersection and union of subsets of the universe may be examined for: (1) domain, (2) range, and (3) ordered pairs. After intersections and unions of small finite sets have been studied, some generalizations may be established using Venn diagrams.

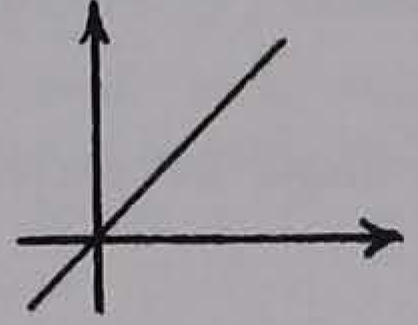
Investigation of the properties of equivalence classes (reflexive, symmetric, and transitive) as related to the coordinate plane can also be dealt with at this time.

Using mathematical sentences, describe possible conditions for point, line, and step graphs in graphing discrete and continuous data. Examine these possibilities, first using the positive integers and then using the set of positive rationals.

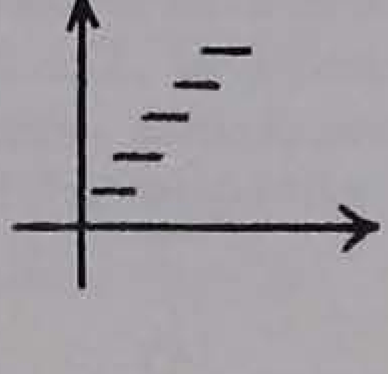
Examples of discrete data would be the graphing of the purchase price of numbers of football tickets, phonograph records, cars, desks, etc. at a given unit price. Since discrete data can be processed in terms of whole units only, the graph must be a point graph; it is not reasonable to draw the line joining the integral points on the graph. Example:



In contrast, line graphs are used to show continuous data situations as, for example, the raising and lowering of temperature, the flow of water through a pipe per unit of time, and so on. Here a continuous line may be drawn.



A common example of a step graph can be seen by having the pupils make a graph of the cost of mailing first class postage at 5¢ per ounce or fraction thereof.



*Topic References: Numbers 2, 3, 5, 7, 8, 9, 29, 30, 32, 34, 36

Many interesting variations of these types of graphs will be discovered by imaginative pupils.

Sentences which describe sets of points in the coordinate plane have two variables. There is the possibility of horizontal and vertical movement. Discover what possible relationships may be developed in terms of (the change in "y" as related to the change in "x").

Suggested Developmental Questions

The following are some questions which may be used in the study of this topic.

1. How may a specific point be located in a line? in a plane? in space?

2. How many dimensions in space are required to relate a point to another point in a plane? in a line?

3. What possible games may be developed that will make use of the finite coordinate system?

4. Given a finite U×U (coordinate system or lattice) consider:

   a. What conditions may occur for y, given a constant x?

   b. What are the possibilities for the intersection of two ordered sets in the first quadrant for a finite U?

   c. What are the possibilities for the union of two ordered sets in the first quadrant for a finite U?

5. What sets of ordered pairs describe a reflexive set? a symmetric set? a transitive set?

6. How may "intersection" be explained using Venn diagrams?

7. How may "union" be explained using Venn diagrams?

8. What does a graph for the sale of postage stamps look like?

9. What is meant by the term "slope" as used in mathematics?

Discovery teaching has as an element of its philosophy the idea of activity on the part of the pupils. They must take active part in a discussion before being able to discover patterns and generalizations. The problem, then, is to keep the pupils active. Games which allow every member of the class to participate are very effective.

In using discovery teaching procedures, the teacher must encourage the pupils to guess and use intuitive hunches. Many times the pupil will make a good guess, other times the guess will not be valid. When a pupil offers a guess, it must be examined for its possible value. The pupil has committed himself to active learning when he guesses or plays a hunch. Once the pupil has committed himself, the teacher must make use of his guess, and lead him to investigate the reasoning which stimulated his interpretation of the question.

This kind of learning may be used in developing a game using a finite U×U in the first quadrant of the coordinate system. The pupil has had some acquaintance with the location of points in the plane, but this knowledge should not be called upon directly.

Lattice Games

Use a pair of dice, one green and the other red. This will enable us to readily distinguish an ordered pair. The object is to roll the dice and read the numbers which appear. Since each die will have a number, the idea of an ordered pair is quickly developed. Visualize a certain area of the blackboard containing a 6x6 lattice. Roll the die and indicate to the pupils that the resultant *pair* of numbers designates a point which can be located in the plane of the blackboard. To locate this first point is truly a guess. When a pupil takes the chalk and places a dot on the board, the teacher may discuss the degree of accuracy of the guess. Now, place the dot where you envision it to belong and label it with the appropriate numerals. Repeat the rolling of the die until either the lattice is completed or the pupils are able to identify the total configuration. The pupils should now have the ability to locate points in the plane, given an origin and a unit of measure. Note that the only numbers dealt with are the natural numbers and that zero has not been included. In other words:

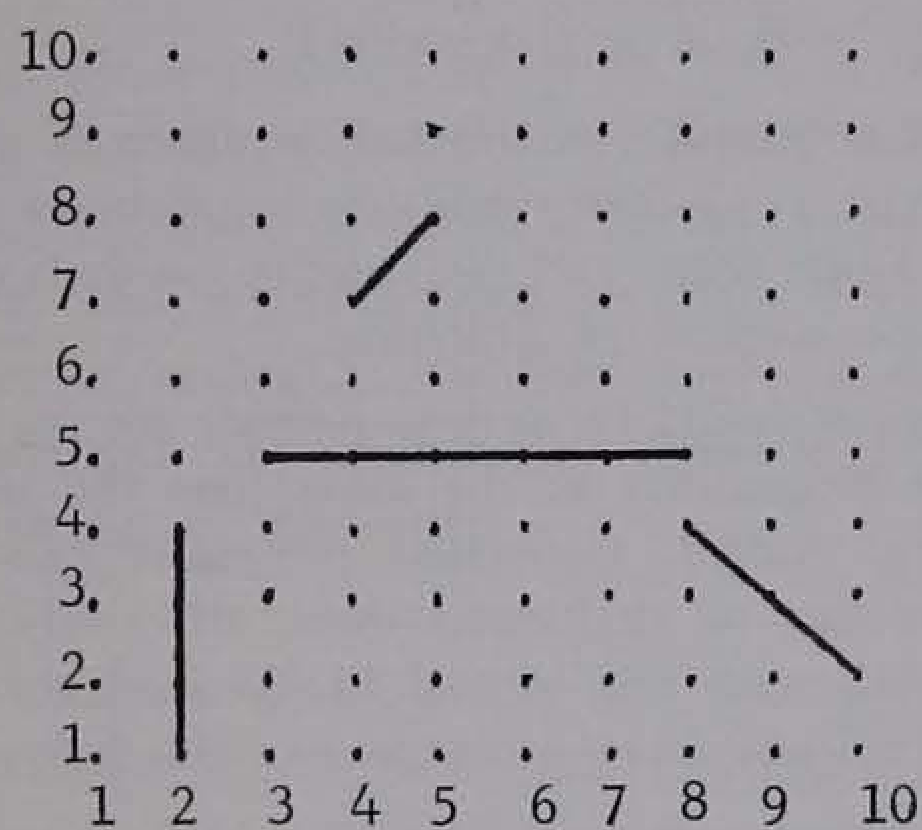
$$U \text{ (Green)} = \{1, 2, 3, 4, 5, 6\}$$
$$U \text{ (Red)} = \{1, 2, 3, 4, 5, 6\}$$

Any other games used should continue to use a finite universal set. This limits the possible combinations of ordered pairs to a reasonable number.

One such game is "Submarine," which may be played by teams or individuals. For those who have not played "Submarine," the rules are as follows:

1. There are at least four different types of ships involved: (1) submarine, (2) battleship, (3) cruiser, and (4) carrier.

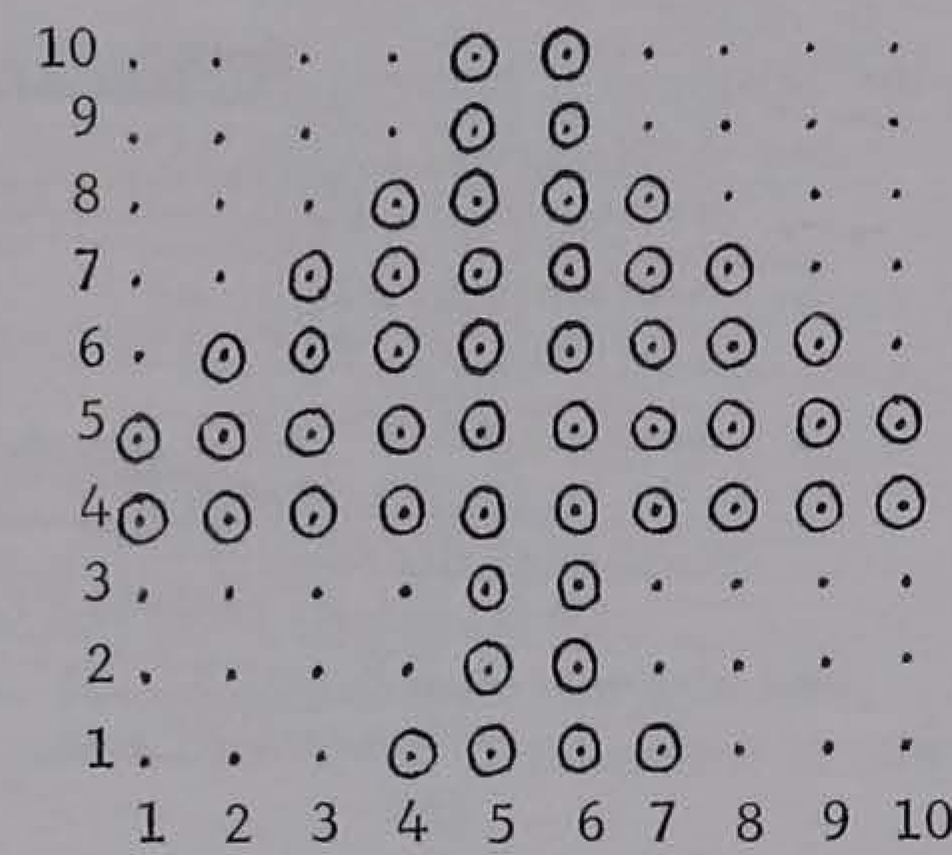
2. Each of these ships occupies a different set of ordered pairs. Each ship is made up of adjacent points.
   a. submarine—any set of two ordered pairs
   b. cruiser—any set of three ordered pairs
   c. battleship—any set of four ordered pairs
   d. carrier—any set of six ordered pairs
3. The game may be played as follows:
   a. Establish some UxU, preferably at least a 10x10 lattice.
   b. Mark the individual ordered sets on the lattice.



   c. Random guesses of ordered pairs are made by the opponents to try to locate and "sink" the ships. When a "hit" has been made, the player must say only which ship has been hit. It is up to the opponent to make further random guesses until the various ships are sunk.
   d. Each player has his own lattice, and guesses are made alternately.
   e. If this game is played by groups, limit the number of groups to no more than six. Use a rotation method for the guessing.

There is another game called "Tree" which possibly has greater value for the development of sets of ordered pairs as solution sets. Again, this is played on a 10x10 lattice and only integers are used. The only materials needed are: a deck of 20 cards, two sets of 1 through 10, {1, 2, 3, . . . , 10}, and a 10x10 lattice with some established configuration, in this case a tree. An ordinary bridge deck of cards can be used. This game is limited to two players. The rules for the game are as follows:

1. The deck of 20 cards is shuffled and dealt alternately by one of the two players. The deal changes after each hand.
2. The non-dealer always leads the first card. This card may be any card he desires to lead that he feels may not be "treed." The second player trys to win the trick by playing some card which will "tree" the card which is led. This illustration of a "tree" lattice may help to clarify the rules.



3. The lead card will always be the horizontal element of the ordered pair and the second card will be the vertical one.
4. It is not *required* that the second card "tree" the first card.
5. The cards are shuffled and redealt after each hand.
6. Each time a player "trees" the other player, he is the winner of that trick, and he must then lead a card.
7. If a player cannot "tree" the lead card, or does not desire to "tree" the lead card, he still must play one of his cards.
8. The first player to win 20 or more tricks at the completion of a hand wins the game.

Procedure of the game: After the cards have been dealt, suppose the lead card is a "2." Any member of the set {4, 5, 6} will result in a number pair which is a member of the "tree." Suppose the next lead card is a "7." Any member of the set {1, 4, 5, 6, 7, 8} will result in a number pair which is a member of the "tree."

The "tree" is only one possible configuration which may be used in playing this game. Other games may be developed using the same 10x10 lattice.

Remember that games such as the above are for the purpose of acquainting pupils with the idea of ordered pairs. After this has been established, use the previously developed configurations to solve problems such as the following:

1. Tabulate this set:
   $\{y \mid (4,y) \epsilon T\}$
   The problem is read: The set of y's determined by (4,y) such that (4,y) *is an element* of the tree. The symbol $\notin$ (not an element of) could be used also. Answer: $\{1, 4, 5, 6, 7, 8\}$

2. State the relationship between the following sets.
   a. $A=\{y \mid (6,y) \epsilon T\}$
      $B=\{y \mid (1,y) \epsilon T\}$
   b. $A=\{x \mid (x,4) \epsilon T\}$
      $B=\{x \mid (x,9) \epsilon T\}$
   a. Answer: $B \subset A$ or $B \subset A$ – $A \supset B$
      B is a subset of A
      or
      B is properly contained in A
   b. Answer: $A \supset B$ and $B \subset A$
      A is contained in B
      and
      B is a subset of A

3. Perform the indicated binary operations.
   a. $\{x \mid (x, 3) \epsilon T\} \cap \{x \mid (x, 7) \epsilon T\}$
      where "$\cap$" is the binary operation symbol which means *intersection*. Answer: ϕ, where 'ϕ' is the symbol for null set or empty set.
   b. $\{y \mid (7, y) \epsilon T\} \cup \{y \mid (10, y) \epsilon T\}$
      where "$\cup$" is the binary operation symbol which means *union*. Answer: $\{2, 3, 4, 5, 9, 10\}$

4. The above problems could also be demonstrated using Venn diagrams.

Problems like the above should be used for configurations developed by the pupils. This topic demonstrates the advantages of using a finite set in the development of mathematical understandings. Pupils can gain very powerful insights to aid in the future study of mathematics working with a small, familiar set of numbers.

After pupils have arrived at an intuitive understanding of the coordinate plane, and have become familiar with graphing techniques through the use of games, it is important that they learn to graphically represent linear sentences. It is best to begin with conditions of this type:

Graph each set:

1. $\{(x, y) \mid y=x\}$
2. $\{(x, y) \mid y=x+3\}$
3. $\{(x, y) \mid y=3x\}$
4. $\{(x, y) \mid y=2x-1\}$
5. $\{(x, y) \mid y=6\}$
6. $\{(x, y) \mid x-y=4\}$

Through a carefully programed sequence of problems of this type, pupils can readily be led to make generalizations about slope, and can develop on their own the point-slope method of graphing.

Teachers should be alert to provide for the graphic study of inequalities at the same time the equalities are being studied. Important principles and understandings can be developed more effectively in this way. Pupils with this sort of background get considerable challenge from problems like the following:

Graph each set:

1. $\{(x, y) \mid x+y>8\}$
2. $\{(x, y) \mid y \leq 2x+3\}$
3. $\{(x, y) \mid x+y=8\}$
   and
   $y<6$
4. $\{(x, y) \mid y=4x\}$
   and
   $7+x=10$
5. $\{(x, y) y=0\}$
   and
   $x-y>4$
6. $\{(x, y) \mid y \leq x\}$

TOPIC XI

Patterns That Arise In Counting*

Suggested time allotment: 5-6 weeks

This topic is intended to be an "end of the year" topic for the eighth grade. The material holds much interest for pupils, provides a great variety of experiences, and is open-ended to allow for extreme flexibility in timing as the final topic of study for the school year. The teacher may expect to spend from 1 to 5 weeks on this topic.

The contents are unique in that each activity acts as an introduction to new and challenging areas of mathematics. This gives the teacher great latitude in deciding which to pursue further and in selecting interesting enrichment materials for individual pupils. The materials on measures of central tendency (mean, median, mode) are considered to be basic and should be studied to develop a background for other topics.

Pupils have been developing a "seeing eye" for patterns all through the seventh- and eighth-grade program. Therefore, they should have a background which will enable them to appreciate these "patterns that arise in counting."

The study of median, mode, and mean should receive the conventional treatment. Frequency tables and histogram bar graphs can be made of random events such as 10 tosses with 5 coins, 25 tosses with 10 thumbtacks, or the frequency of particular letters used in any 100 letters on a randomly chosen page. Finding the arithmetic mean by using positive and negative deviations from an assumed mean is recommended. Another investigation might be to find the frequency of integers on a given problem page of a textbook.

*Topic References: Numbers 1, 9, 13, 18, 28

Suggested Developmental Questions

The following are questions a teacher might ask to arouse interest in the development of this topic.

1. Which letter of our alphabet has the greatest frequency of use? Choose a page at random and test your answer.
2. Does the mean of the squares of the numbers in the set $\{1, 2, 3, 4, 5\}$ equal the square of the mean of the numbers? Could this be true for other sets of numbers?
3. A newspaper reports the "average" income per family in Loop City is $6,085. Which measure of central tendency do you think has been used, the median or the mean? Which do you think would be more typical?
4. Experiment with two sets of six numbers each, using whatever numbers you wish. Find the mean of each set, then find the mean of the set of the sums of the corresponding numbers in the first two sets. Does this mean equal the sum of the first two means?

Discussion of Developmental Questions

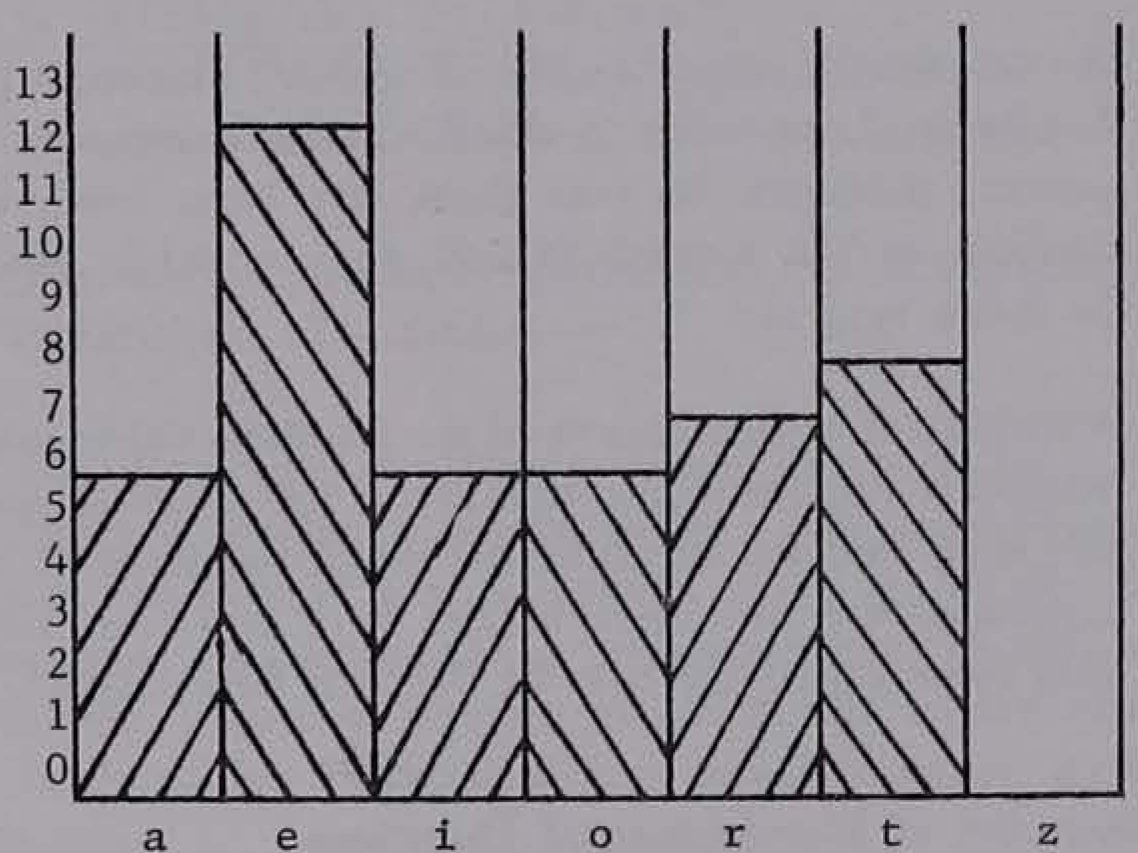
A sample lesson developing out of question 1 is explained below.

"Which letter of our alphabet has the greatest frequency of use? Choose a page at random and test your answer." Have each pupil select a page from a book which he has available in class. Then have the pupils block out any 100 words and make a frequency table for the letters a, e, i, o, r, t, and z. One pupil's frequency tabulation might appear as the following.

Frequency Distribution of Letters of the Alphabet on a Page Chosen at Random

Letter	Tally
a	~~////~~ //
e	~~////~~ ~~////~~ ///
i	~~////~~ //
o	~~////~~ //
r	~~////~~ ///
t	~~////~~ ////
z	0

Have pupils prepare a bar graph of these frequencies (called a "histogram") as below:



Histogram of Frequency of Letters of the Alphabet on a Page Chosen at Random

Have pupils compare their longest vertical bar with that of the histogram above. Compare the lengths of the other bars.

The teacher might also suggest to his pupils that facts from such histograms can be of help in decoding letter codes.

Another random sampling situation pupils might enjoy follows. Have pupils toss 5 pennies 100 times and tally a frequency table as above.

Number of Heads	Frequency
0	2
1	18
2	36
3	24
4	14
5	6

From their frequency table pupils can make a histogram as before. Also a cumulative frequency distribution table could help them answer many questions about the incidence of heads.

Number of Heads	Frequency	Cumulative Frequency	Cumulative Relative Frequency
5	6	100	1.00
4	14	94	.94
3	24	80	.80
2	36	56	.56
1	18	20	.20
0	2	2	.02

Some questions which might be asked about this table are:

1. How many times did four or fewer heads occur?
2. What per cent of the tosses showed three or fewer heads?
3. What per cent showed three or four heads?
4. What per cent showed at least four heads?
5. What per cent showed five heads?

Frequently the study of special properties of numbers and special sets of numbers is omitted in the conventional mathematics curriculum. The remainder of this topic is an investigation of these properties. There will be a listing of ideas and references with the hope that this will be more helpful to the teacher. These topics are not listed in sequence or priority, so that a teacher may use any or all of them with his classes.

Triangular numbers. Many number patterns can be represented geometrically. One such pattern is that of simple right-triangular numbers.

The first number represented is one, the next is three, etc. What is the pattern used to arrive at the picture of a right-triangular number by using the previous one? Draw the fifth and sixth right-triangular numbers. Without drawing it, what number would be the seventh right-triangular number? Could it be written as the sum of a set of positive whole numbers? Describe the eighth right-triangular number as the sum of a set of positive whole numbers.

Square numbers. Another number pattern that can be represented geometrically is that of square numbers.

Number one is the first of the set, four next, etc. Do you see the pattern involved in forming the successive square numbers (no exponents allowed)? Then draw a picture of the fifth of the square numbers. Without drawing the sixth, decide what it would be. Does your pattern involve sums? Does the second of the square numbers equal $1+3$? Does the third equal $1+3+5$? Write a similar sum for the fourth; the fifth.

Are all the addends odd numbers? Will the n^{th} square number equal the sum of the first n odd numbers?

Another way to describe square numbers with which you are familiar is in exponent form. Consider: $\{1^2, 2^2, 3^2, \ldots, n^2\}$. Will the sum of the first n odd numbers always equal n^2?

Pupils should be encouraged to investigate the possibility of other kinds of numbers that may be represented geometrically.

Reversal of digits of a number. Quite often in algebra books a section is devoted to problems dealing with two-digit numbers. Consider the example: A certain two-digit number has the sum of its digits equal to 12 and the ten's digit is three times the unit's digit. Invariably, some pupils will correctly reason that formal algebra is not essential to the solution of problems of this type. Here is an area of investigation which is concerned with the reversal of digits in any number.

Begin by listing some two-digit numbers. Look for the pattern when the digits are reversed. Pupils should learn quickly to recognize a pattern.

Number	Number with Digits Reversed	Difference Between Digits	Difference Between Numbers
72	27	5	45
96	69	3	27
43	34	1	9
18	81	7	63
22	22	0	0
70	07	7	63

How is the difference between the numbers related to the difference between the digits?

Now consider the case of the three-digit numbers. Can we establish a pattern of differences at this level?

Number	Number with Digits Reversed	Difference of the Two Numbers	Difference Between 1st and 3rd Digit
216	612	396	4
418	814	396	4
801	108	693	7
321	123	198	2
919	919	0	0
200	002	198	2
620	026	594	6

Pupils should discover that the differences are multiples of 99; and they are precisely multiples of 99 times the difference between the first and third digit of the starting number.

Listed below are some four-digit numbers. No conclusions have been reached. This can be a challenge for the pupils to discover a pattern which may exist with four-, five-, and six-digit numbers.

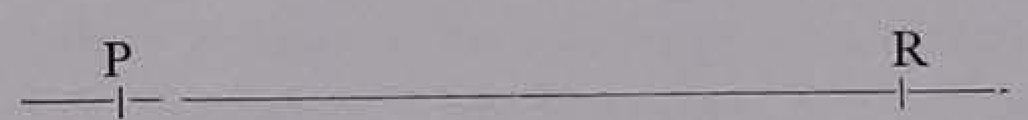
	Number	Number Reversed	Difference of the Two Numbers
I.	6254	4526	1728
	4586	6854	2268
	7365	5637	1728
	3145	5413	2268
	7859	9587	1728
II.	1286	6821	5535
	3938	8393	4455
	4179	9714	5535
	6821	1286	5535
	2717	7172	4455

III.	6154	4516	1638
	7845	5487	2358
	9267	7629	1638
	3731	1373	2358
	1953	3591	1638

Other Topics for Investigation

1. Pythagorean Triples
2. How Money Earns Money
3. Magic Squares
4. Patterns Involving π
5. Properties of Odd and Even Integers
6. Sums of Number Sequences

Vectors as Directed Line Segments



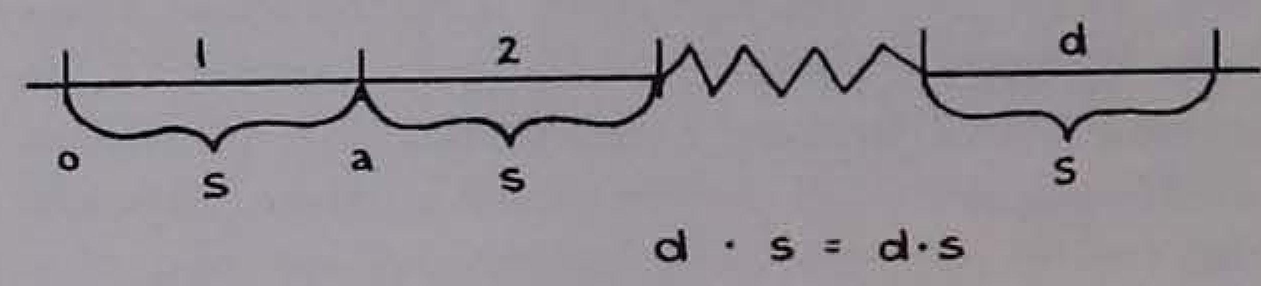
Given a line passing through points P and R ($\overleftrightarrow{PR}$; line PR), the distance between the two points may be noted as a directed segment. PR would then be that distance which would place P on R inversely RP would be that distance which would place R on P. This provides a fairly fundamental definition of a vector: A vector is a directed line segment.

Examine the sums of vectors (resultant vector) in one dimension.

MN + NR = MR
MR + RN = MN
MN + NM = O
RM + MN = RN

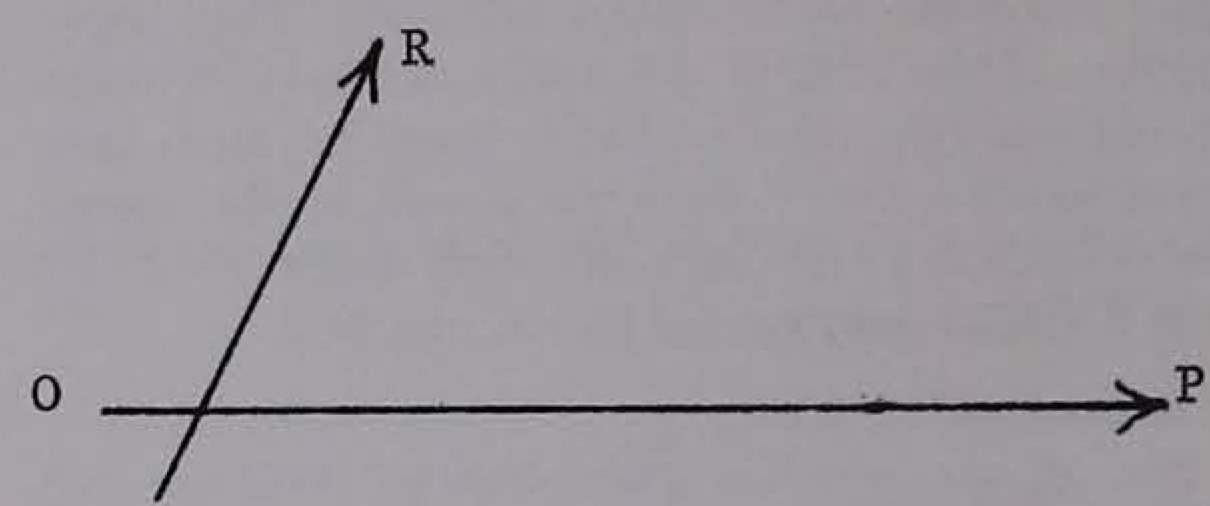
The above statements can be considered to be of a general nature. Other problems may be developed using a number line.

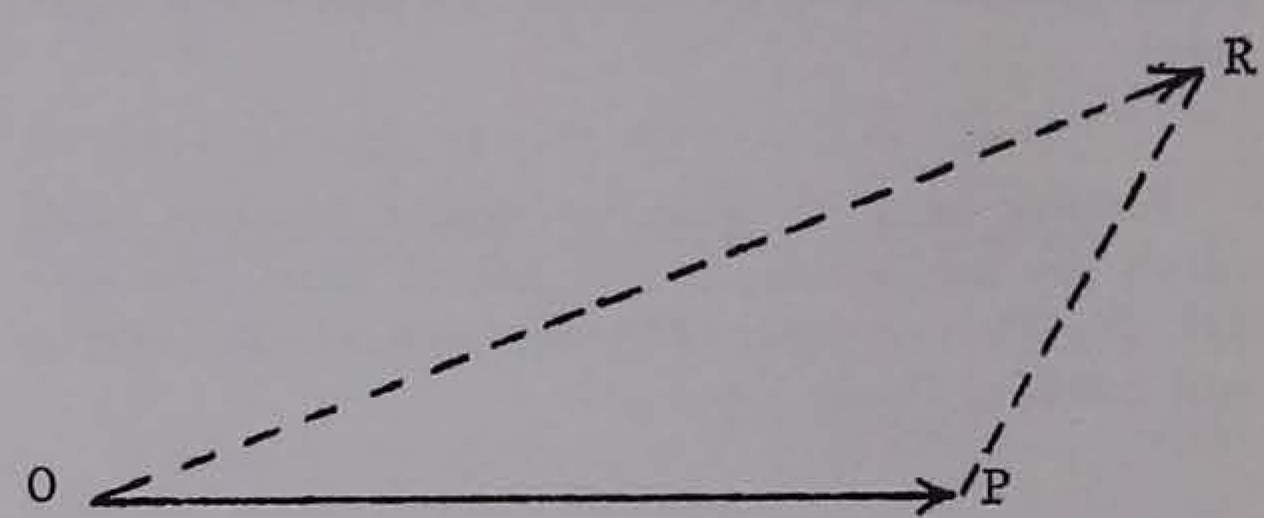
Examine now the resultant vector for multiplication.



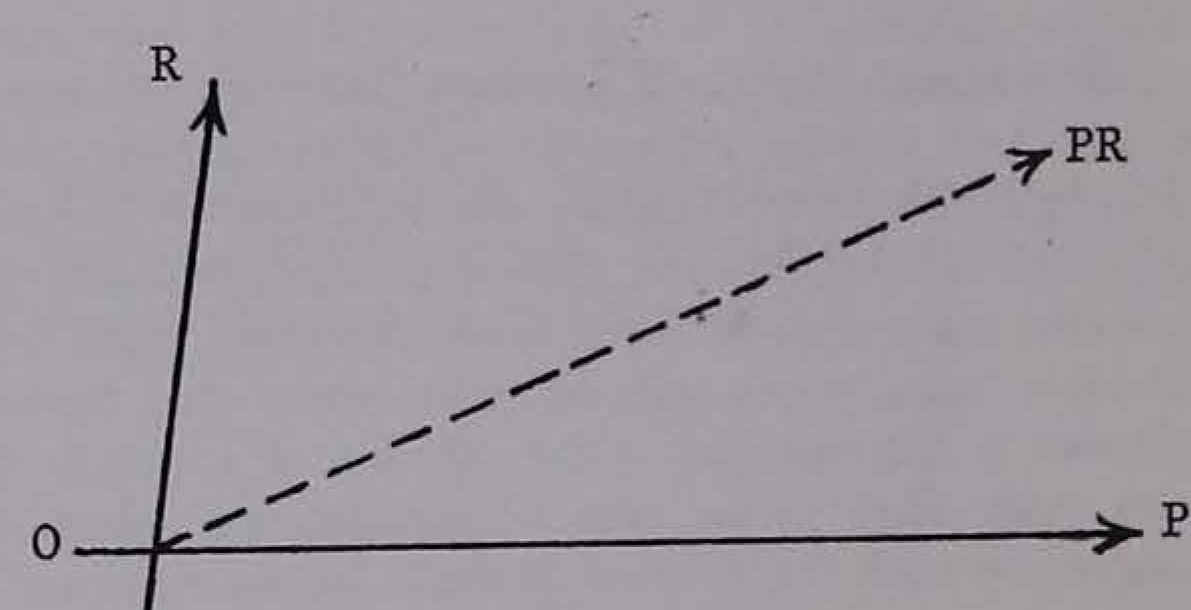
What this actually stipulates is: There are "d" number of segments added together, each of length "s." Here again, problems may be developed using the rational number line. It cannot be overemphasized that it is better to first establish the general case, before dealing with the application.

What about the resultant vector in relation to two dimensions? Examine the addition first. Locate the resultant vector for going first the distance $\overrightarrow{OP}$ and then the distance $\overrightarrow{OR}$.

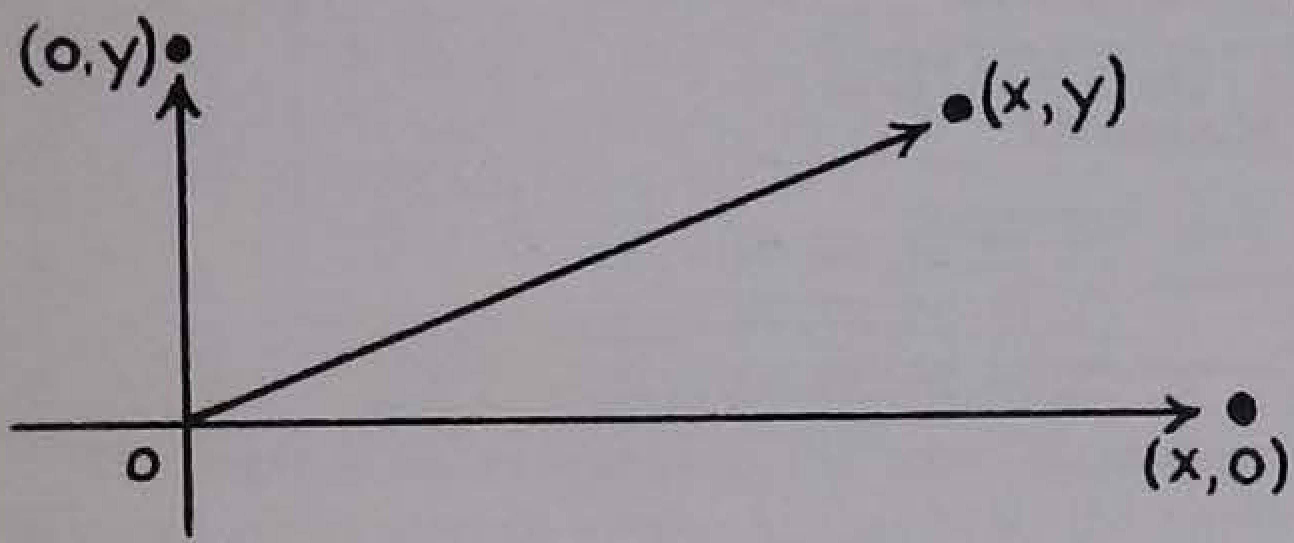




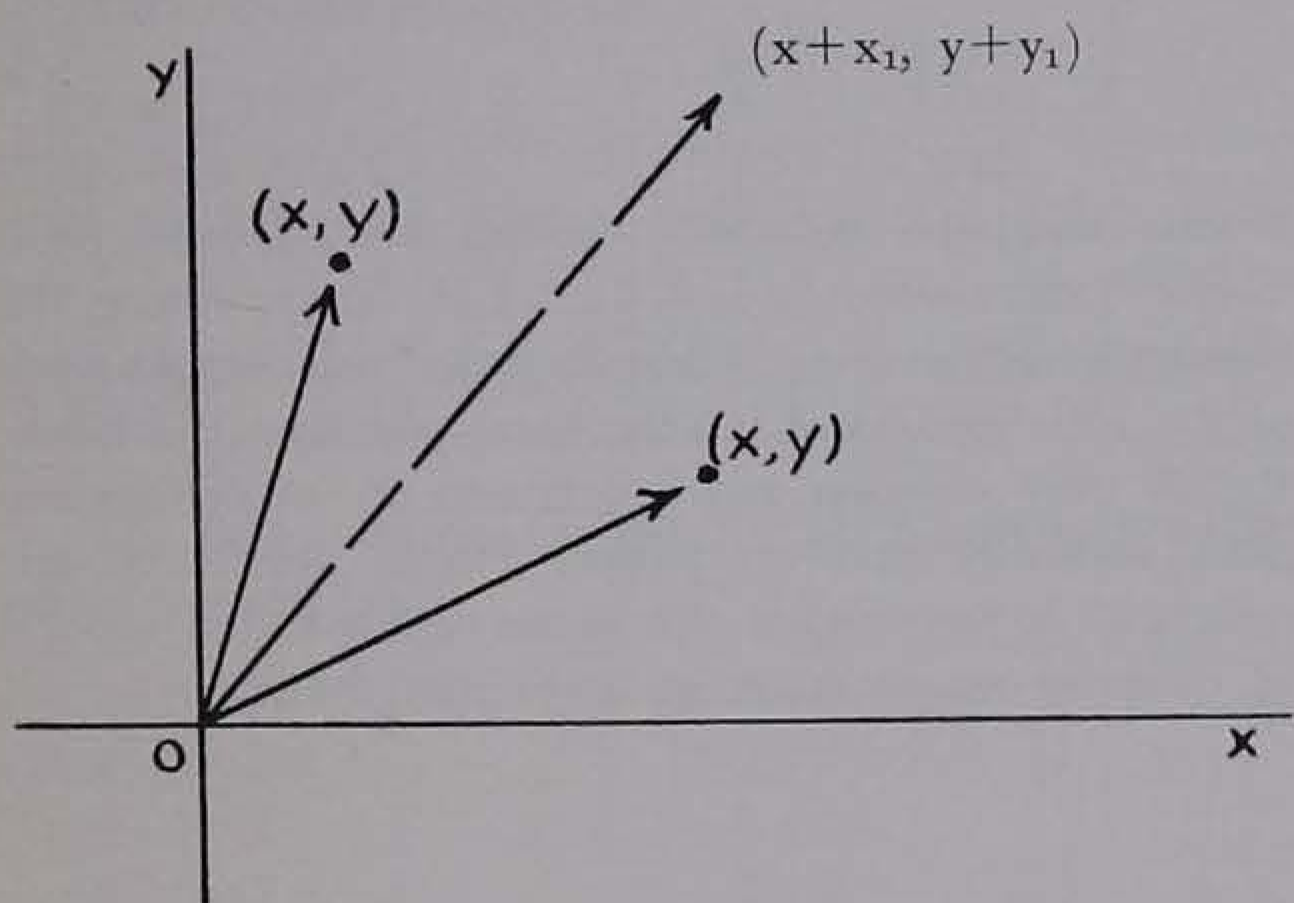
The resultant would be the addition of the two directed segments.



Now using the coordinate axis, the operation may be defined for addition.



This, however, is a specific case. What about the resultant for any (x, y) and (x_1, y_1)?

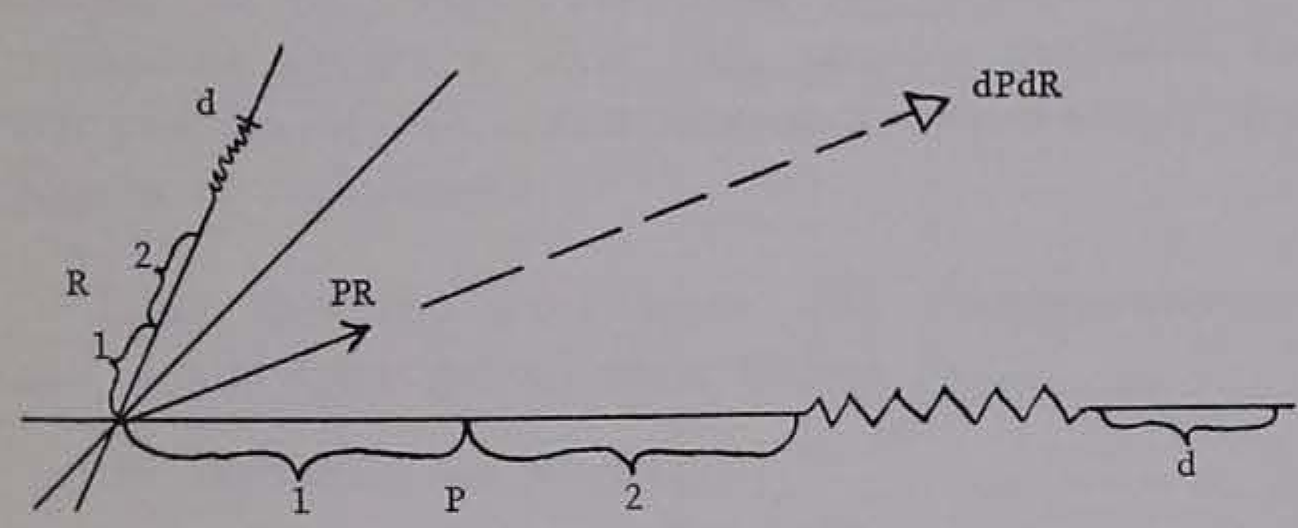


$(x,y) + (x_1, y_1) = (x+x_1, y+y_1)$

What about the resultant vector for 3 vectors? 4 vectors? n vectors?

How may multiplication be defined in relation to two dimensions?

Given PR, what does d·PR equal?



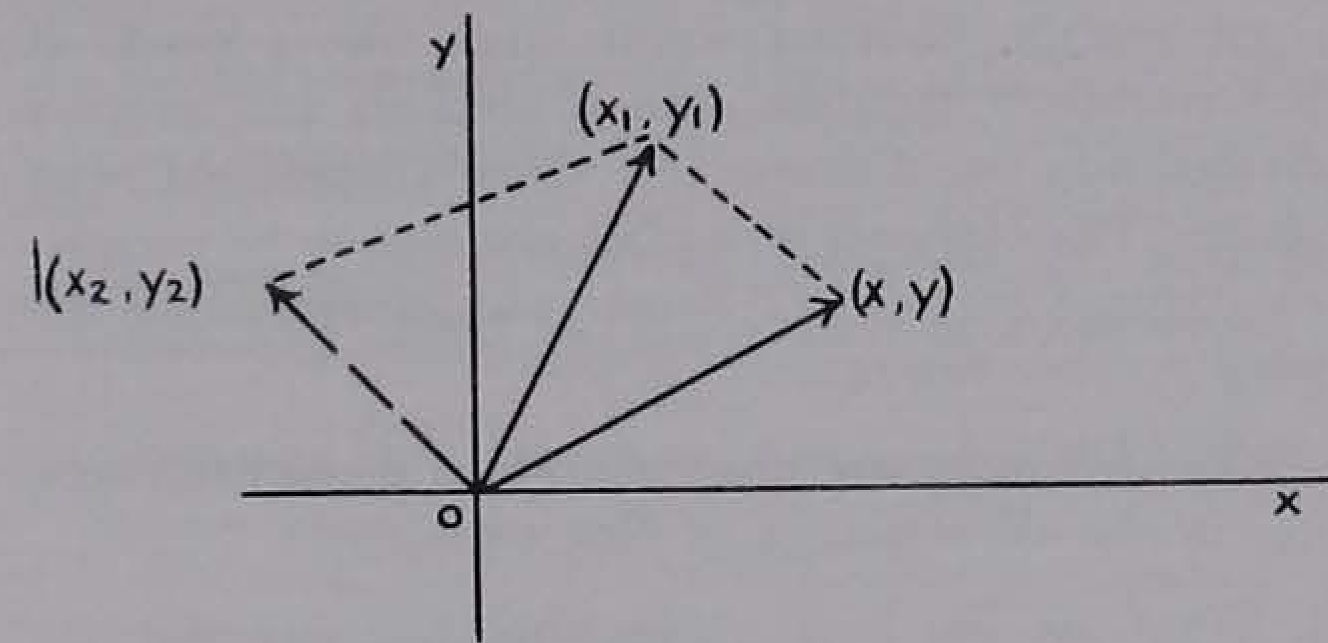
d · PR = d·P d·R

Problems may be developed using the coordinate plane for a given (x,y).

The sum vector has been developed in detail. Now let's study the difference vector. This may be developed in terms of an unknown quantity.

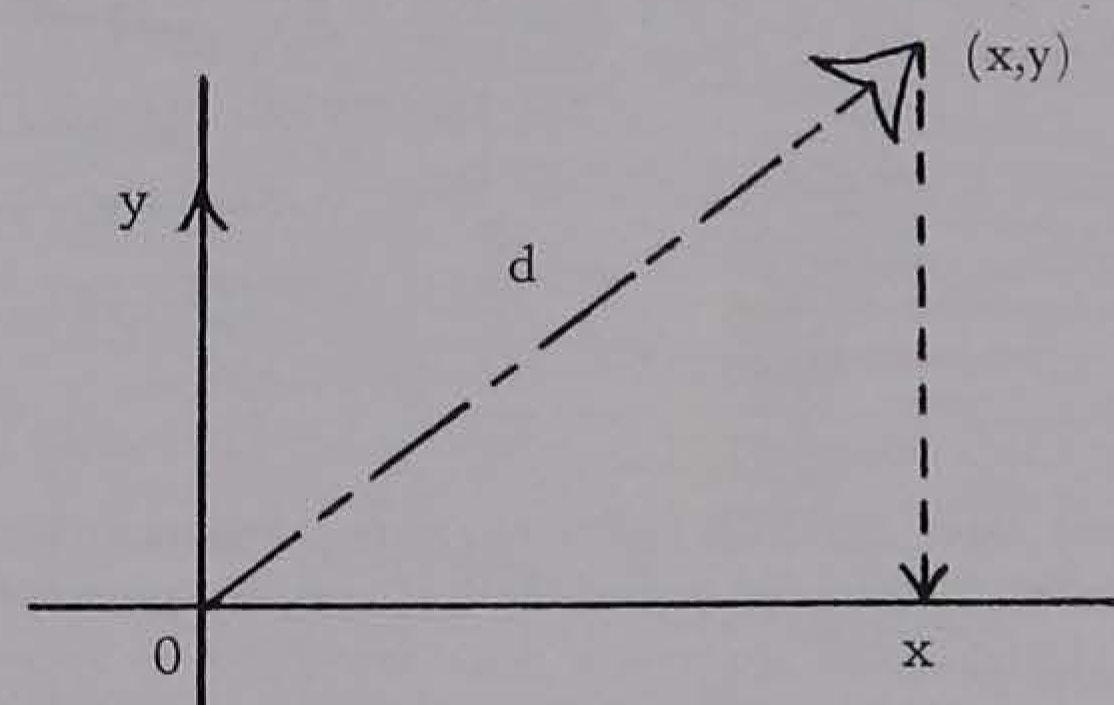
4 + □ = 6	a + □ = b
7 + □ = 2	(x,y) + □ = (x_1,y_1)

Remember that x_1y_1 is the resultant vector.



$\therefore (x,y) + (x_2,y_2) = (x_1,y_1)$

It must be remembered that (x,y) does not give the length or magnitude of the vector. What will yield the magnitude of the vector? This is another application of the Pythagorean Theorem. To find the length of (x,y) use x as one leg and y as the other leg of a right triangle.



$$x^2 + y^2 = d^2$$

This leads to the formula for the distance between any two points in the coordinate plane.

The work to this point only scratches the surface of the elementary work which may be done with vectors. Investigate many physical applications based upon the concepts which have been developed that merit investigation. Keep in mind that there must be an adequate amount of problem-solving situations to assure understanding the ideas set forth in this topic.

The congruency of modular systems. Congruency may be thought of in different forms. In modular arithmetic, sometimes called clock arithmetic, the idea of classes of equivalent, or congruent numbers is developed. Thus, for modulo 12, the numbers of the set {1, 13, 25, . . ., 1 + 12n} are congruent. It may be said that two vectors are congruent if they have the same magnitude and direction. Congruency is a special kind of equivalence.

A modulo partition works much like a clock. If it is modulo 5, then the natural numbers and zero are divided into five different sets which are referred to as classes. The elements of a class are said to be congruent because each is a different way of expressing the same number concept.

The following are questions which a teacher might use in the development of this topic.

1. What are the basic classes that can represent the natural numbers and zero using modulo 12?
2. In the partition modulo 5, what relation exists between the elements of the "ones" class and the elements of the "fours" class?
3. Given the class: 3, 10, 17, 24, . . . This class is a subset of what partition? Develop the other classes of the partition.

Comments on Question 1 stated above:

Look at the clock on the wall to develop the classes of modulo 12.

Modulo 12:

0 {0, 12, 24, 36, 48, . . .}
1 {1, 13, 25, 37, 49, . . .}
2 {2, 14, 26, 38, 50, . . .}
.
.
.
8 {8, 20, 32, 44, 56, . . .}
.
.
.
11 {11, 23, 35, 47, 59, . . .}

Notice that the natural numbers are divided into twelve basic classes: 0, 1, 2, . . . , 11. In examining the elements of the set in class 8, it is evident that each of the numbers stand for the same point on the clock. Pupils may develop new partitions in other modulo both mentally and on paper. Pupils should be encouraged to investigate the more sophisticated aspects of modular numbers on an individual basis.

TOPIC XII
Equivalent Sentences*

Suggested time allotment: 4-5 weeks

This topic is an extension of the topic on sets and sentences in the seventh grade. At that time, the pupils were faced with many experiences such as the following:

$U=\{1, 2, 3, \ldots, 10\}$

Tabulate the solution set.

1. $x+3=7$	4. $12-11=x$
2. $3x<27$	5. $7+x>15$
3. $5 \cdot 8>x$	6. $2x+5=12$

At that time, they should have thought of x as a variable in sentences such as the foregoing.

The first question, $(x+3=7)$, for example, is a sentence pattern for:

$1+3=7$	$6+3=7$
$2+3=7$	$7+3=7$
$3+3=7$	$8+3=7$
$4+3=7$	$9+3=7$
$5+3=7$	$10+3=7$

Nine of these ten statements are false, and $4+3=7$ is the *only* true statement. Four is the only number that will replace x and make a true statement. This was defined by saying $\{4\}$ is the solution set for the sentence $x+3=7$.

The same types of questions were then answered with with sets for universes which were continually increased in size. Since they had no experience with equivalent sentences, their only possible approach to the solution of the above sentences was through the process of replacement.

Later they did some work with number systems and should have gained some feeling for:

the commutative property:
$a+b=b+a, \quad a \cdot b=b \cdot a$

the associative property:
$(a+b)+c=a+(b+c), \quad (ab)c=a\ (bc)$

the distributive property:
$a(b+c)=ab+ac$

They also should know the properties of equality (though they might not know them by these terms):

Reflexive $a=a$ (sometimes spoken of as the "rubber stamp identity")

Symmetric if $a=b$, then $b=a$

Transitive if $a=b$ and $b=c$, then $a=c$.
If the pupils have had no experience with these, this could be a good place to begin the unit on equivalent sentences.

Most good teaching moves from the familiar to the new. A good first lesson for this topic might look something like this:

$U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Tabulate the solution set.

1. $2x+3=7$

2. $4(x+2)>36$

⋮

These exercises would be much like the ones studied before, except for the inclusion of several short sequences of sentences with the same solution set. For example:

$$\text{Sequence A}\begin{cases}4x+8>25\\8+4x>25\\7<x+3\\4(x+2)>25\\5x-1>21\\\frac{33}{x}<7\end{cases}\qquad\text{Sequence B}\begin{cases}2x-3=15\\5x-2=43\\x+5=14\\5+x=14\\2(x+6)=30\\30=2x+6\end{cases}$$

$$\text{Sequence C}\begin{cases}4+2x<18\\2+x<9\end{cases}\qquad\text{Sequence D}\begin{cases}x=3\\3x=9\\x+4=7\end{cases}$$

*Topic References: Numbers 3, 5, 9, 15, 23, 26, 29, 30, 34, 36

After the pupils have the solution sets tabulated, the discussion should be focused on the fact that certain subsets of the exercises have identical solution sets. Then *equivalent sentences* can be defined and the next set of exercise experiences can reverse the process. For example:

$U=\{1, 2, 3, \ldots, 20\}$

Each of the subsets of U in the exercises below could be the solution set for many sentences. For each exercise, write at least five sentences which have the given set as a solution set.

1. $\{5\}$	....	4. ϕ	....
	....		....
	....		....
	....		....
	....		....
2. $\{8\}$		5. $\{1, 2, 3, 4, 5\}$	
	....		....
	....		....
	....		....
	....		....
	....		....
3. $\{17, 18, 19, 20\}$		6. $\{U\}$	
	....		....
	....		....
	....		....
	....		....
	....		....

These exercises are more difficult than one may realize, and it is easy to assign too many of them.

The tendency of the brighter pupils here is to write extremely long and complicated sentences. This is fine but it does not lead them to making the discoveries toward which you are working. The slower pupils are capable of doing very good work on exercises of this kind.

Some pupils may come up with sequences like this:

For example:

1: $x=5$, $2x=10$, $3x=15$, $x+1=6$, $x+2=7$

or for example

3: $x>16$, $2x>33$, $3x>50$, $x+1>17$, $2x+1>33$

These of course will lead directly to some of the ideas about equivalence you want the pupils to discover. Many real good generalizations should come from the discussion of these exercises.

A set of exercises like this next set will help to focus on the power of the simpler sentences.

$U=\{1, 2, 3, \ldots, 100\}$[1]

For each exercise write five sentences which are equivalent to it (have the same solution set).

1. $x=3$
2. $5x+7=57$
3. $5x>28$
4. $\frac{x}{7}>67$
5. $\frac{85}{x-3}<85$

The discussion should bring out that exercise number 1 is very easy because it is in simple form. Exercise number 5 is more difficult except for those who may write $x>4$ as the first sentence. From $x>4$ other sentences can be written very easily.

In exercise number 2 it can well be pointed out that $x=10$ is not an answer but rather just another sentence which happens to have $\{10\}$ as its solution.[2]

After experiences of this kind, the question of how to analyze a sentence and write another which is equivalent to it is understood quite naturally.

Certainly $2(x+3)=26$ and $2x+6=26$ are equivalent because of an application of the distributive property. $x+5=9$ and $5+x=9$ are equivalent because of an application of the commutative property. $x+3=20$ certainly is equivalent to $x=17$ (they have the same solution set), but how can we decide this without first knowing the solution set? Other examples such as $5x+17=83$ and $5x=65$ leave the solution set hidden a little better while the same question is asked. This builds a need for learning to use one of two properties depending on the materials being used in the program.

Some programs build the *cancellation laws* of addition and multiplication:

$\forall$ a, b, c $\quad a+b=a+c \leftrightarrow b=c$

$\forall$ a, b, c, $a\neq0$ $\quad ab=ac$ and $a\neq0 \rightarrow b=c$

$b=c \rightarrow ab=ac$

[1]Any universe can be used as long as there is a specific one. The choice should depend somewhat on the abilities of the pupils in your class.

[2]Pupils should not acquire a conditioned response—step by step—pattern for grinding out in a mechanical way a series of sentences ending in the form x=some number. Instead this should become a thinking process with only what is necessary being written in step form.

The nature and idea of mathematical proof enters this topic for two reasons:

1. It is a natural place for the study of proof—and proof itself is a worthwhile mathematical objective.
2. The "steps" of solving equations and the understanding of same can be demonstrated adequately in proof, leaving the pupils free to write the solution set as soon as he discovers it in solving equations.

Here is an example of the way in which pupils might use proof.

Prove that $2x+8=36$ and $x=14$, where x is a rational number, are equivalent sentences.[3]

The proof might go something like this:

$2x+8=36$	given
$2x+8=28+8$	addition of rational numbers
$2x=28$	cancellation law (of addition)
$2x=2\cdot 14$	multiplication of rational numbers
$x=14$	cancellation law (of multiplication)

or like this:

$2x+8=36$	given
$2(x+4)=36$	distributive property
$2(x+4)=2\cdot 18$	multiplication of rational numbers
$x+4=18$	cancellation law (of multiplication)
$x+4=14+4$	addition of rational numbers
$x=14$	cancellation law (of addition)

These proofs constitute the writing of a succession of equivalent sentences. Each is equivalent to the preceding sentence, because of a particular property of the number system or of equivalence.

Later as they study sentences in two variables, the same ideas will again come into focus.

Pupils can be asked to graph the solution set in $R\times R$ (R=set of rational numbers) for $x+y=7$ and for the sentence $2x+2y=14$. They will recognize that these two sentences have the same solution set. The same properties which enable one to write equivalent sentences in one variable are again brought to bear.

There are many possible motivations for writing equivalent sentences in two variables.

If pupils have a good understanding of slope and intercept, it is much easier to graph $y=\frac{2x}{3}+4$ than it is to graph $3y-2x=12$, and it becomes meaningful to study the relationship between the two sentences. On the other hand, some pupils are more adept at looking at $3y-2x=12$ and recognizing the x and y intercepts as (0, 4) and (−6, 0). In this case, given equations like $y=\frac{2x}{3}+4$ to graph they will again need to understand the meaning of equivalence.

They could be proven equivalent like this:

$3y-2x=12$	given
$3y+3(\frac{-2x}{3})=3\cdot 4$	multiplication of real numbers
$3(y+\frac{-2x}{3})=3\cdot 4$	distributive property
$y+\frac{-2x}{3}=4$	cancellation law (of multiplication)
$y+\frac{-2x}{3}=0+4$	property of zero $(0+a=a)$
$y+\frac{-2x}{3}=(\frac{2x}{3}+\frac{-2x}{3})+4$	property of additive inverse $(a+{}^{-}a)=0$
$y+\frac{-2x}{3}=\frac{2x}{3}+(\frac{-2x}{3}+4)$	associative property
$y+\frac{-2x}{3}=\frac{2x}{3}+(4+\frac{-2x}{3})$	commutative property
$y+\frac{-2x}{3}=(\frac{2x}{3}+4)+\frac{-2x}{3}$	associative property
$y=\frac{2x}{3}+4$	cancellation law (of addition)

It should then be emphasized that when pupils are called upon to use the "equivalent sentences" skills to manipulate or solve equations in one or two variables; they need not, indeed, they *should* not, be asked to follow a set pattern. Conversely, they should be strongly encouraged to use all the insight and intuition which has been developed in the study of equivalent sentences.

[3]Pupils should recognize that if they can show (1) that 14 will replace x in both sentences to make true statements and (2) that no number other than 14 will replace x and make a true statement in either sentence this would constitute a proof. This is, after all, how equivalent sentences are defined and pupils should not lose sight of that definition. They should understand that they are being asked for a specific type of proof in this case, namely one involving equivalent sentences.

When faced with a set of exercises involving complicated-looking sentences to solve, pupils should be encouraged to write the solution set as soon as it is recognized and go to the next problem. They should, of course, recognize the relationship between this work and the previous work on proof. The quality of pupil thinking periodically can be checked by asking for a proof by equivalent sentences that a particular solution is correct.

By avoiding the step-by-step procedure, pupils can be encouraged to "take bigger steps" in their reasoning. In the end, pupils will be able to work stereotyped problems much more rapidly, and at the same time be free to think when faced with a problem which does not happen to fit any previously studied pattern.

TOPIC XIII

Equivalent Sentences In More Than One Variable*

Suggested time allotment: 6-7 weeks

The purposes of this topic are to develop more efficient procedures in graphing linear sentences (equalities and inequalities), and to study the solution of simultaneous linear sentences by both graphic and algebraic means.

Early in the topic the pupils should become proficient in the slope-intercept method of graphing. A series of well-directed exercises can make this method very meaningful. Using previously learned techniques, sentences can be written in the familiar $y = mx + b$ form. Then the graphing of two or more sentences on the same set of axes can lead to a discussion of intersecting sets. Several problems of this type can lead the pupils to see a need for developing rapid, efficient graphing techniques for finding solution sets. (At this stage it is wise to select the problems carefully so that the sentences will have integral solution sets.) The power of the modern approach to graphing linear conditions is readily evident in the study of linear programing. This study of problem solving by graphic analysis can easily lead the pupil to the limitations of graphic solutions. Then the pupil will be ready for the development of logical algebraic methods of solution.

Some teachers may wish to provide opportunity for better pupils to study beyond the level of pairs of equations in two variables, including, perhaps a brief study of simple determinants.

Recommendations of various study groups have stressed the inclusion of considerable work with inequalities since the graphing of inequalities is such a vital part of solving problems in game theory, linear programing, etc.

*Topic References: Numbers 3, 5, 9, 15, 23, 26, 29, 30, 34, 36

Through the study of inequalities, pupils gain broad insights into basic ideas of the coordinate plane, mathematical limits, and infinity. It also prepares them well for work in coordinate geometry and the study of space. Any linear equation divides $L \times L$ into three subsets.[1] See FIGURE 1. The plane is divided into the set of points for $y<2x+3$ and $y>2x+3$. One of the reasons for the importance of $y=2x+3$ is the fact that it is the boundary of each of the other two sets. Comparisons in the physical world utilize inequalities more frequently than equalities.

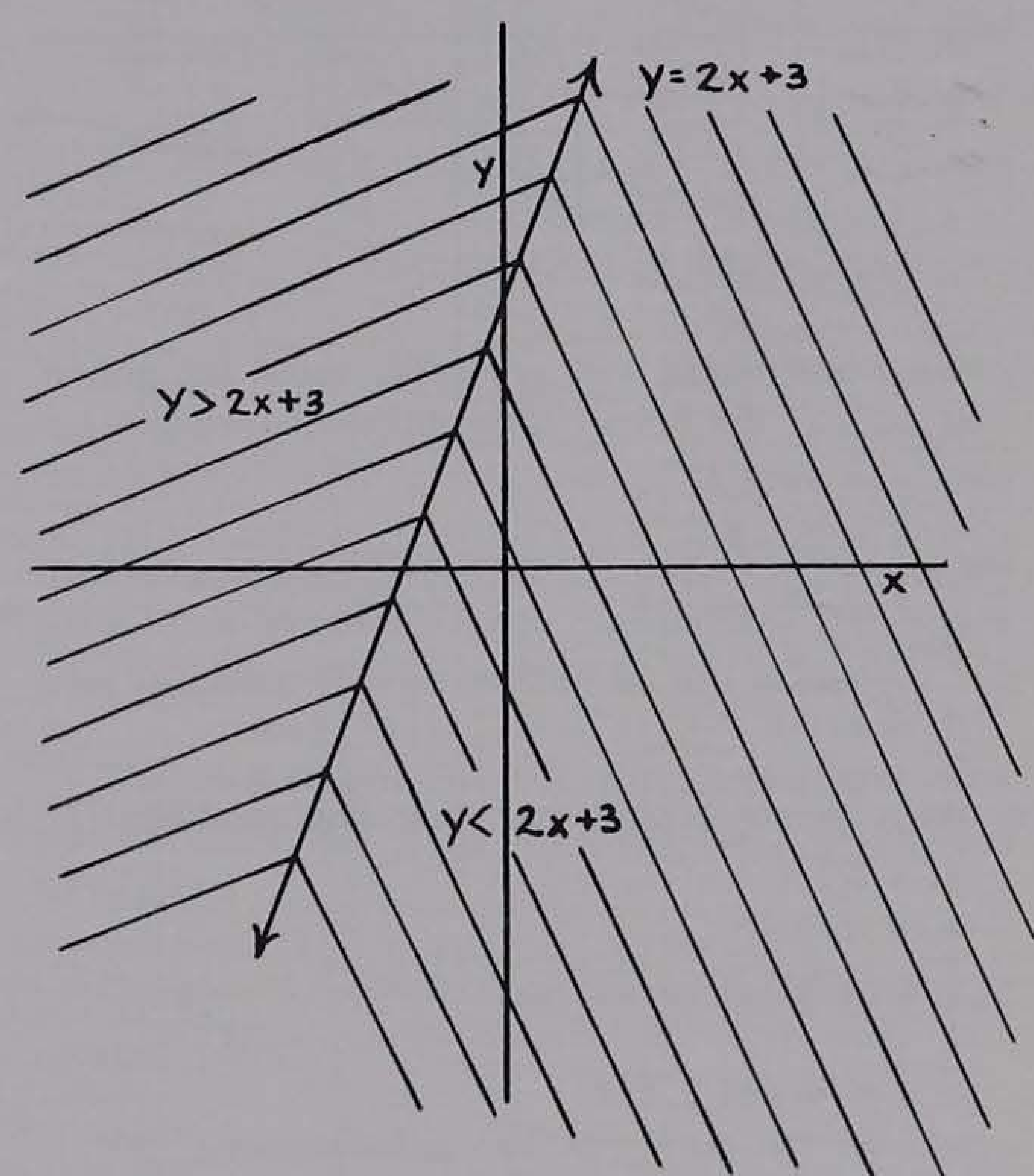


FIGURE 1

[1] L=the set of real numbers

Suggested Developmental Questions

Here are some questions which a teacher might ask his class to help develop this topic:

1. What pairs of numbers, (x, y), satisfy the following sentences?
   a. $x = 3y + 4$
   b. $2x + y > 5$
   c. $10x - 5y = 30$
   d. $7x + 8y < 13$

2. What are some members of the solution set of:
   a. $3x - 7y \_ 6 = 0$
   b. $5x - 3y + 8 > 0$
   c. $y - 8x < 5$

3. What points belong to:
   a. $\{(x, y) \mid y = 3x + 6\}$
   b. $\{(x, y) \mid y < -2x + 8\}$
   c. $\{(x, y) \mid y > x - 2\}$

4. What point in the plane is common to these sets of points?
   a. $\{(x, y) \mid y = 2x + 5\}$
   b. $\{(x, y) \mid y = 4x + 5\}$
   c. $\{(x, y) \mid 2y + 3x = 10\}$
   d. $\{(x, y) \mid 15 - 5x = 3y\}$
   e. $\{(x, y) \mid x = 2y + 5\}$

5. What observation can you make about the graphs of each of the following conditions for equality?
   a. $y = 3x + 6$
   b. $y = 3x - 2$
   c. $y = 3x$
   d. $3x + 8 = y$
   e. $\frac{1}{3} y = x + 3$

6. What numbers satisfy the following conditions?
   a. $x > 5$
   b. $x < 10$
   c. $y > 8$
   d. $y < {}^{-}7$
   e. $x > 5$ and $x < 8$

7. What points belong to each of the following sets?
   a. $\{(x, y) \mid y < 2\}$
   b. $\{(x, y) \mid x < {}^{-}6\}$
   c. $\{(x, y) \mid x > 3 \text{ and } x < 6\}$
   d. $\{(x, y) \mid y < 3 \text{ and } y > 5\}$

8. What pairs of numbers belong to each of the following sets?
   a. $\{(x, y) \mid y > 5 \text{ and } x < 3\}$
   b. $\{(x, y) \mid x + y < 8 \text{ and } y < 5x\}$
   c. $\{(x, y) \mid x + y = 12 \text{ and } x - y = 15\}$
   d. $\{(x, y) \mid 3x - 5y = 17 \text{ and } 4x + 8y = 29\}$

Discussion of Developmental Questions

It is possible that questions 1 and 2 may be the pupils' first contact with the idea of an ordered number pair. Therefore, they should be introduced to the idea of Cartesian set. They should see that order here is important, and should also recognize the idea that many sets of ordered pairs can satisfy a condition with two or more variables. The pupils should also see very early in the topic that they can use manipulative skills from the previous topic to help find ordered pairs. Here again the use of inequalities is of prime importance. It is also possible that some intuitive work can be done here with three variables.

As work is done with the problems in 1 and 2, the pupils should develop a method of determining ordered pairs that satisfy a condition. For example, when finding pairs for example a of problem 1, should we first assign values to x or y?

In example a of problem 2, the pupil should see that the following table and procedure will aid him in finding ordered pairs.

$3x - y - 6 = 0$
$3x - 6 = y$
$y = 3x - 6$

x	0	1	2	3	4	5
4	y					

Before finishing this part of the topic, pupils should have a clear, definite method of determining ordered pairs that will satisfy a given condition.

For some conditions, it may be wise to limit the universe under consideration. This could be done as a finite number of natural numbers, or it could be only done by considering the set of natural numbers. There are many interesting problems which can be developed using a limited universe.

The purpose of the problems in question 3 is to lead the pupils to relate ordered number pairs to the coordinate plane and thereby learn to draw the graph of a condition in two variables. Only a limited sampling of problems is presented here. Many others should be used to give adequate practice in the graphing of conditions. Note that the problems listed are in the form of $y = mx + b$. Experiences should be provided so pupils will learn to transform any given condition to an equivalent condition of the above form.

Pupils should draw graphs of many inequalities. They should see that by graphing an equality the plane is divided into three sections ($=$, $>$, $<$). Part of this work should be done with a limited universe and a correspondingly limited Cartesian set.

The purpose of questions 4 and 5 is to provide the pupils with opportunities to learn to draw graphs by use of slope-intercept form. Most pupils should readily discover the convenience of graphing by slope-intercept method. If a set of graphs are placed on the same axis, the pupils can be led to discover the relation that exists when lines contain the same point or when they are all parallel.

A great deal of time should be devoted to this. Positive, negative, and fractional slopes should all be investigated, beginning with the construction of many graphs. Discovery should play an important part in the development of slope-intercept concepts.

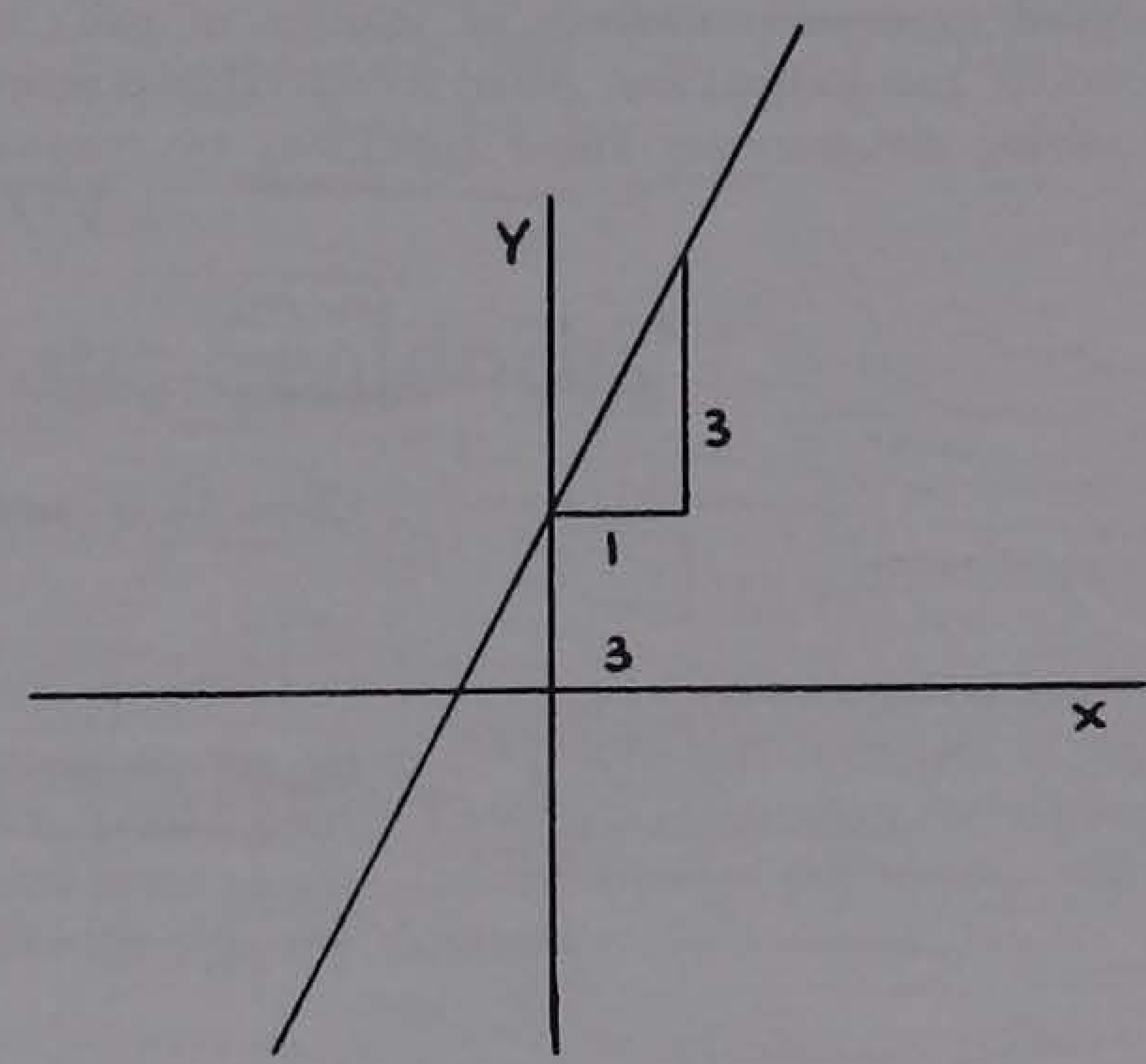
For example, some pupils may see the pattern in the ordered pairs of a problem by placing it in tabular form:

$y = 2x + 1$

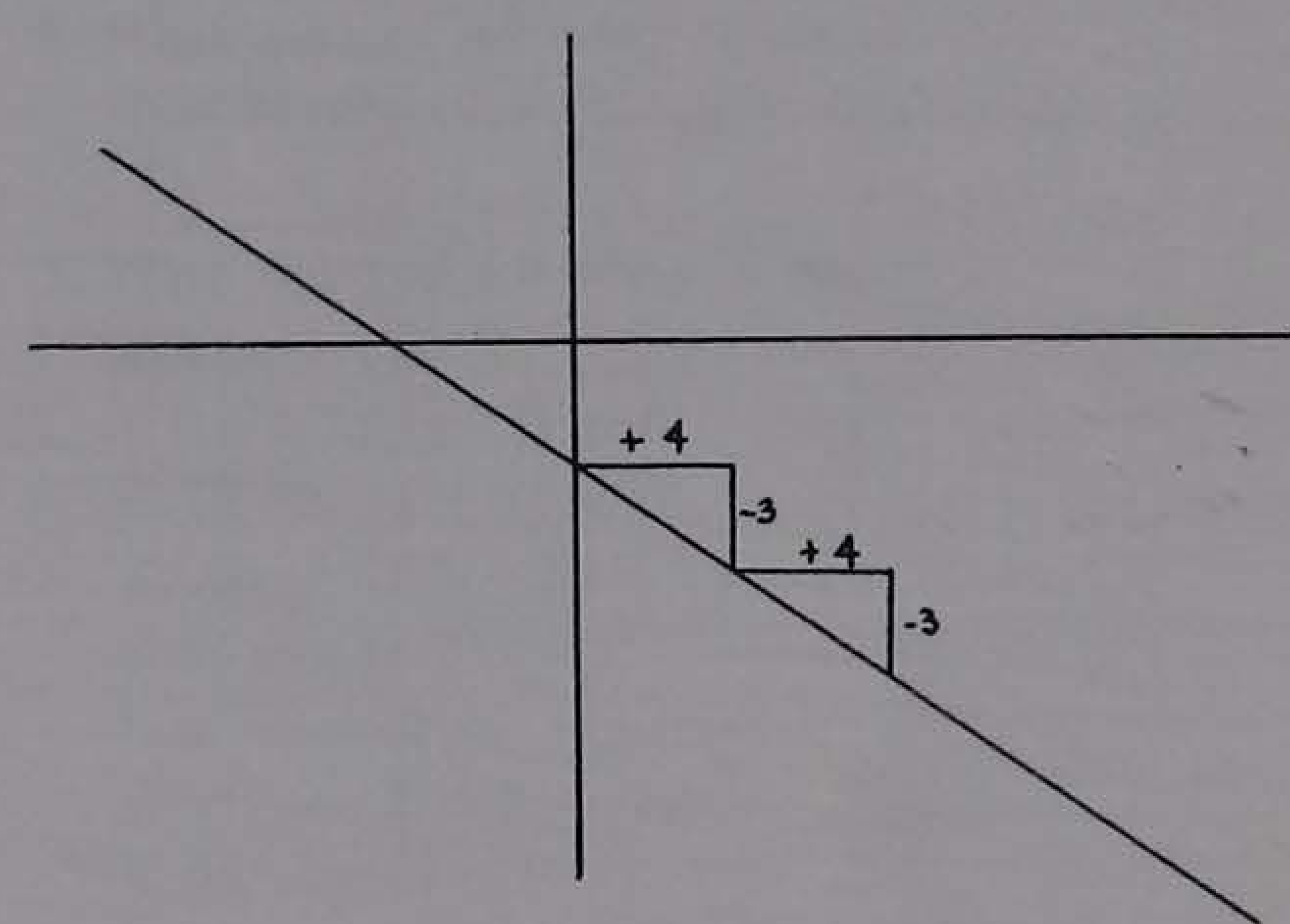
x	0	1	2	3	•	•	•
y	1	3	5	•	•	•	•

Others may see the slope concept more readily on a graph:

$y = 3x + 2$



$y = \frac{3}{4}x - 3$



The purpose of questions 6 and 7 is to provide practice in graphing simple inequalities and conditions involving two inequalities in two variables.

The work on question 6 is quite simple; more complicated conditions using the same ideas should also be used.

Question 7 provides practice in the graphing of ordered pairs.

The graphing of c and d, involving the intersection of two sets, should be of interest to pupils. Problems providing for the intersection of more than two sets of points may also be investigated.

Problems of the type presented in question 8 are designed to develop methods of solution of pairs of equations and inequalities. After considerable practice in solving simultaneous linear conditions by graphic means, pupils should be led to see the inadequacy in this kind of solution. With this background pupils will see the need for finding more sophisticated methods of solving pairs of equations.

TOPIC XIV

Solution of Quadratic Sentences*

Suggested time allotment: 9-10 weeks

The primary purpose of this topic is to formalize the solution of the quadratic equation. However, to accomplish this, two other important abilities must be attained. Early in the topic, the pupils must see that by the use of the distributive property we can more easily solve the quadratic sentence. With this as a motivation, it becomes quite essential for them to master use of the distributive property. They should understand: (1) the factoring of a general trinomial, (2) the factoring of a perfect-square trinomial, and (3) the factoring of the difference of two perfect squares. The properties of exponents will be used in this.

Having attained the above skills, it is then possible to formally solve the quadratic sentence $ax^2 + bx + c = 0$ where $ax^2 + bx + c$ can be expressed in factored form. To assure real understanding, early solution should be done as formal proof.

When the pupil has successfully solved quadratic sentences of the form $x^2 - a^2 = 0$, he should be asked to solve sentences of the form $x^2 - c = 0$ where c is not a perfect square. This will lead into the use of radical expressions. All work with fractional and negative exponents can be done at this time as a natural outgrowth of the study of quadratics.

Neither pupils nor teachers should lose sight of the main purpose of this topic. Factoring is not done without purpose. It is motivated as a necessary skill for the solution of quadratic equations. Likewise, work with radicals is done to aid in the solution of the quadratic equation.

Suggested Developmental Questions

Following is an outline of questions the teacher might ask in developing this topic. Each question could be asked at several different times during the study of the topic using different universes of possible replacements. Later each question is expanded and some of the kinds of answers the teacher will be looking for are discussed.

1. What can you tell about a and b, given that $ab = 12; ab = 5; ab = 1; ab = 0$?

2. What can you tell about x, given: $x^2 = 16; x^2 = 7; x^2 = 1; x^2 = 0; x^2 = -1$?

3. What can you tell about x given:
   a. $x(x + 1) = 0$
   b. $(x + 2)(x - 2) = 0$
   c. $(x - 2)(x + 1) < 0$
   d. $(2x + 3)(x - 2) = 0$
   e. $x^2 + x = 0$
   f. $x^2 - 4 = 0$
   g. $x^2 - x - 2 = 0$
   h. $2x^2 - x - 6 < 0$

4. What can you tell about x, given:
   a. $(x + 1)(x + 1) = 0$
   b. $(x - 3)(x - 3) > 0$
   c. $(x + 5)(x + 5) = 0$
   d. $(2x + 3)(2x + 3) = 0$
   e. $x^2 + 2x + 1 = 0$
   f. $x^2 - 6x + 9 > 0$
   g. $x^2 + 25 + 10x = 0$
   h. $4x^2 + 12x + 9 = 0$

5. What can you tell about x, given:
   a. $(x + 3)(x - 3) = 0$
   b. $(x - 5)(x + 5) = 0$
   c. $(2x + 3)(2x - 3) = 0$

*Topic References: Numbers 3, 4, 5, 7, 8, 21, 23, 24, 30, 32, 34, 36

d. $x^2 - 49 < 0$

e. $x^2 - 25 > 0$

f. $4x^2 - 9 = 0$

g. $x^2 - 7 = 0$

h. $x^2 - 150 = 0$

i. $x^2 - 8 = 0$

j. $x^2 - 50 = 0$

6. What can you tell about x, given:

a. $(x + 2)^2 - 9 = 0$

b. $(x^2 + 6x + 9) - 25 = 0$

c. $x^2 + 4x + 5 = 0$

d. $x^2 + 6x = 11$

e. $(x^2 + 8x + 16) - 24 = 0$

f. $(x^2 + 8x) + 16 = 21$

g. $(x^2 + 4x) - 12 = 0$

h. $(2x^2 + 5x) - 8 = 0$

i. $(x + 2)^2 + 4 = 0$

j. $ax^2 + bx + c = 0$

Discussion of Developmental Questions

The purpose of exercises of the type found in questions 1 and 2 is to establish conditions for the product of two factors to be 0. Areas which the teacher may wish to pursue further include the definition of exponents, ideas of prime, composites, complex numbers, irrational numbers, and existence of multiplicative inverse. Solution sets in S and S × S where S is the set $\{0, 1, 2, . . .\}$

$U = S \times S$

$\{(a, b) \mid ab = 12\} = \{(1, 12), (12, 1), (6, 2), (2, 6), (3, 4), (4, 3)\}$

$\{(a, b) \mid ab = 5\} = \{(1, 5), (5, 1)\}$

$\{(a, b) \mid ab = 1\} = \{(1, 1)\}$

$\{(a, b) \mid ab = 0\} = \{(0, b), (a, 0)\}$

$U = S$

$\{x \mid x^2 = 16\} = \{4\}$

$\{x \mid x^2 = 7\} = \phi = \{\ \}$

$\{x \mid x^2 = 1\} = \{1\}$

$\{x \mid x^2 = 0\} = \{0\}$

$\{x \mid x^2 = -1\} = \phi = \{\ \}$

Solution sets in I and I × I where I is the set of integers

$U = I \times I$

$\{(a, b) \mid ab = 5\} = \{(1, 5), (-1, -5), (5, 1), (-5, -1)\}$

$\{(a, b) \mid ab = 12\} = \{(1, 12), (-1, -12), (12, 1), (1, 12), (6, 2), (2, 6), (-6, -2), (-2, -6), (3, 4), (4, 3), (-3, -4), (-4, -3)\}$

$\{(a, b) \mid ab = 1\} = \{(1, 1), (-1, -1)\}$

$\{(a, b) \mid ab = 0\} = \{(0, b), (a, 0)\}$

$U = I$

$\{x \mid x^2 = 16\} = \{4, -4\}$ $\{x \mid x^2 = 0\} = \{0\}$

$\{x \mid x^2 = 7\} = \phi = \{\ \}$ $\{x \mid x^2 = -1\} = \phi = \{\ \}$

$\{x \mid x^2 = 1\} = \{1, -1\}$

Solution sets in R and R × R where R is the set of rationals.

To tabulate a set is to list *all* the members of the set and to use the braces { }, to show that we have a set. In the set, N × N, it is easy to tabulate the solution set for ab = 5. Simply write, $\{(1, 5), (5, 1)\}$. The braces imply that these, and only these, replacements make a true statement.

In R × R the solution set for ab = 5 is endless.

$$\left\{ \left(\frac{2}{3},\ 7\frac{1}{2}\right), \left(\frac{1}{5},\ 25\right), \left(\frac{1}{3}, 15\right), \left(15,\ \frac{1}{3}\right), \left(\frac{1}{2},\ 10\right), \left(1, 5\right), \left(\frac{7}{13},\ \frac{65}{7}\right), \ldots\ldots \right\}$$

It is suggested that these solution sets be shown graphically rather than attempting to tabulate them.

Other very good questions for discussion are:

1. Why with U = S or I does $\{a, b) \mid ab = 12\}$ have more members than $\{(a, b) \mid ab = 5\}$? More examples of this type should bring out the factors of composite numbers into primes.

2. What is the relation of each ordered pair which satisfies the conditions ab = 1 and $x^2 = 1$ when U = R × R (multiplicative inverse).

3. What would we need for a solution of $x^2 = 7$, and

what would we need for a solution of $x^2 = -1$? (Irrational and complex, respectively)

Pupils should be well aware that for all x and y: if $xy = 0$, then $x = 0$ or $y = 0$ or both are equal to 0.

In the development of questions 3, 4, and 5, practice has been provided to develop skill in expressing algebraic expressions in factored form. The teacher should be sure pupils realize the need for simplification of irrational expressions.

Questions 3, 4, and 5 asked for solution sets to sentences. The sentences in 3 are listed below along with their solution in the set of rational numbers.

$U = R$

a. $x(x + 1) = 0$ — $\{0, -1\}$
b. $(x + 2)(x - 2) = 0$ — $\{-2, 2\}$
c. $(x - 2)(x + 1) < 0$ — $\{x \mid -1 < x < 2\}$ or

−2 −1 0 1 2 3

d. $(2x + 3)(x - 2) = 0$ — $\{2, -\frac{3}{2}\}$
e. $x^2 + x = 0$ — $\{0, -1\}$
f. $x^2 - 4 = 0$ — $\{-2, 2\}$
g. $x^2 - x - 2 = 0$ — $\{-2, 2\}$
h. $2x^2 - x - 6 < 0$ — $\{x \mid -\frac{3}{2} < x < 2$
or − 3 − 2 − 1 0 1 2 3

Observing the ease of solving the first set of problems, the pupils should recognize the advantage of expressing algebraic sums as algebraic products. Teachers are generally familiar with several methods of teaching factoring. Pupils can usually develop sufficient skill in factoring in a matter of three to five days.

Looking now at the problems for Questions 4 and 5, $(U = R)$.

Question 4

a. $x^2 + 2z + 1 = 0$ $(x + 1)(x + 1) = 0$ $\{-1\}$
b. $x^2 - 6x + 9 > 0$ $(x - 3)(x - 3) > 0$ $\{x \mid x \neq 3\}$
c. $x^2 + 25 + 10x =$ $(x + 5)(x + 5) = 0$ $\{^-5\}$
d. $4x^2 + 12x + 9 = 0$ $(2x + 3)(2x + 3) = 0$ $\{-\frac{3}{2}\}$

Question 5

a. $x^2 - 9 = 0$ $(x + 3)(x - 3) = 0$ $\{3, -3\}$
b. $x^2 > 25$ $(x + 5)(x - 5) > 0$
$\{x \mid x > 5 \text{ or } x < ^-5\}$
c. $4x^2 - 9 = 0$ $(2x + 3)(2x - 3) = 0$ $\{-\frac{3}{2}, \frac{3}{2}\}$

These examples show the forms of the perfect square trinomial and difference of two squares.

Additional Questions

1. How does the solution set of $x^2 - s^2 = 0$ relate to s^2?
2. The idea of a square root should be explored.
3. Can we fill a gap? $\{x \mid x^2 - 25 = 0\}$
4. Observe the solution of $\{x \mid x^2 - 25 = 0\} = \{5, -5\} = \{\sqrt{25}, -\sqrt{25}\}$
$\{x \mid x^2 - a^2 = 0\} = \{a, -a\} = \{\sqrt{a^2}, -\sqrt{a^2}\}$
5. Define: $\sqrt[n]{x^m}$
6. How can we obtain a solution?
$\{x \mid x^2 - 7 = 0\} = \{\sqrt{7}, -\sqrt{7}\} = \{7^{\frac{1}{2}}, -7^{\frac{1}{2}}\}$
7. Can we find solutions to these problems?
$\{x \mid x^2 - 5 = 0\} = \{\sqrt{5}, -\sqrt{5}\} = \{5^{\frac{1}{2}}, -5^{\frac{1}{2}}\}$
$\{x \mid x^2 - 8 = 0\} = \{\sqrt{8}, -\sqrt{8}\} = \{8^{\frac{1}{2}}, -8^{\frac{1}{2}}\}$
$\{x \mid x^2 - 200 = 0\} = \{\sqrt{200}, -\sqrt{200}\} = \{200^{\frac{1}{2}}, -200^{\frac{1}{2}}\}$
$\{x \mid x^2 - 163 = 0\} = \{\sqrt{163}, -\sqrt{163}\} = \{163^{\frac{1}{2}}, -163^{\frac{1}{2}}\}$
$\{x \mid x^2 - c = 0, c > 0\} = \{\sqrt{c}, -\sqrt{c}\} = \{c^{\frac{1}{2}}, -c^{\frac{1}{2}}\}$

Some of the above responses can be improved if we can demonstrate that if $yx > 0$ and $y > 0$, then $\sqrt{xy} = \sqrt{x}\sqrt{y}$.

The properties of exponents should be developed at this time. Once more the teacher should encourage intuitive study and investigation. Based upon the logical application of previous knowledge, the following properties can be easily produced.

1. $x^m \cdot x^n = x^{m+n}$
2. $\frac{x^m}{x^n} = x^{m-n}$; if $x \neq 0$, then $1 = \frac{x^n}{x^n} = x^{n-n} = x^0$
3. $(x^m)^n = x^{mn}$
4. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

Prove:

$\sqrt{xy} = \sqrt{x}\,\sqrt{y}$

$\sqrt{xy} = (xy)^{\frac{1}{2}}$

$(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}\, y^{\frac{1}{2}}$

$x^{\frac{1}{2}}\, y^{\frac{1}{2}} = \sqrt{x} \cdot \sqrt{y}$

$\therefore \sqrt{xy} = \sqrt{x}\,\sqrt{y}$

Prove:

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}} \quad y \neq 0$

$\sqrt{\frac{x}{y}} = \left[\frac{x}{y}\right]^{\frac{1}{2}}$

$\left[\frac{x}{y}\right]^{\frac{1}{2}} = \frac{(x)}{(y)}$

$\frac{(x)^{\frac{1}{2}}}{(y)^{\frac{1}{2}}} = \frac{\sqrt{x}}{\sqrt{y}}$

$\therefore \sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

Having proved that $\sqrt{xy} = \sqrt{x}\,\sqrt{y}$ and $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$, work with simplification of radicals should be completed. Pupils should also review methods of extracting square roots. Equations of the type $\sqrt{x}+2=18$ should be studied for solution.

Areas of exploration which are now open:

1. To what extent have we now developed our number system?
2. Do all the properties of the rational numbers apply to this expanded system?
3. Do we see a need for ways of expressing more numbers?

Expansion of Question 6

These problems are of a type to develop a method of solving any quadratic equation which has real-number coefficients. (Equations with complex-number solutions should not be studied at this time.)

When asked for values of x which will satisfy the condition $(x + 2)^2 - 9 = 0$, the pupil is most likely to square the binomial, combine terms, and factor. An alternate method is proposed here. The problem should be treated as a form of $x^2 - a^2 = 0$. Many solutions of problems of this type will strengthen the concepts of the perfect-square trinomial and the difference of two squares. The sequence of problems should lead to an understanding of completing the square in a quadratic sentence.

Pupils should not be required to write every step in the solution of such problems. Proof of the solution of some quadratic equations should be assigned as exercises. It is recommended that this vehicle be used to assure understanding of the step-by-step process of solution.

Here is an example of the use of proof:

Prove: $3x^2 - 8x + 5 = 0 \rightarrow x = 1$ or $x = \frac{5}{3}$

1. $3x^2 - 8x + 5 = 0$, only if $3xx - 3x - 5x + 5 = 0$
   $-3x - 5x = -8x$
   previously proved
2. $3xx - 3x - 5x + 5 = 0$, only if $3x(x - 1) - 5(x - 1) = 0$
   Distributive Property
3. $3x(x - 1) - 5(x - 1) = 0$, only if $(3x - 5)(x - 1) = 0$
   Distributive Property
4. $(3x - 5)(x - 1) = 0$, only if $3x - 5 = 0$, or $x - 1 = 0$
   if $a = 0$ or $b = 0$, then $ab = 0$
5. $3x - 5 = 0$, only if $3x = 5$
   $a = b \leftrightarrow a + c = b + c$
6. $3x = 5$, only if $x = \frac{5}{3}$
   $a = b \leftrightarrow a \cdot c = b \cdot c,\ c \neq 0$
7. $x - 1 = 0$, only if $x = 1$
   $a = b \leftrightarrow a + c = b + c$

We have shown that if any roots of the equation $3x^2 - 8x + 5 = 0$ exist, then these roots are 1 and $\frac{5}{3}$. It is still necessary for the proof of the original statement to show that:

$3(1)^2 - 8(1) + 5 = 0$ and $3(\frac{5}{3})^2 - 8(\frac{5}{3}) + 5 = 0$

Problems such as: $2x^2 + 5x - 8 = 0$ should lead the pupil to the solution of $ax^2 + bx + c = 0$.

$$2x^2 + 5x - 8 = 0$$

$$x^2 + \frac{5}{2}x + \frac{25}{16} - 4 - \frac{25}{16} = 0$$

$$\left[x + \frac{5}{4}\right]^2 - \frac{89}{16} = 0$$

$$\left[x + \frac{5}{4} + \frac{\sqrt{89}}{4}\right]\left[x + \frac{5}{4} - \frac{\sqrt{89}}{4}\right] = 0$$

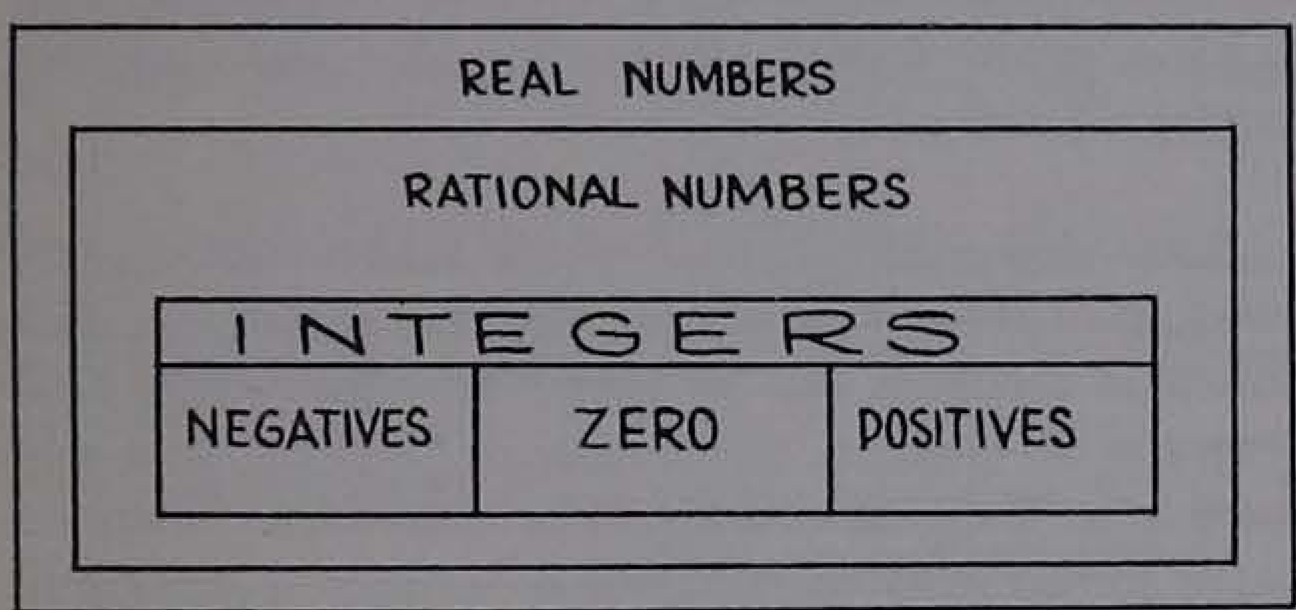
$$x = \frac{-5 = \sqrt{89}}{4}$$

The development of the quadratic formula should follow directly, applying the process of completing the square to the general quadratics equation.

Area of further exploration: How does the value $b^2 - 4ac$ affect the solution set of the quadratic sentence?

Summary

The development of the material in this topic involves the further expansion of the number system to the set of real numbers. Thus it becomes possible to investigate many problems based upon a knowledge of the evolution of our number system. The following representation should be helpful in this study.



The closure of operation within any given system should be again reviewed. The properties which hold within a given number system should be reviewed.

Pupils should be led to recognize that we are still limited in attempts to solve equations of the type $x^2 = -1$. The suggestions that a larger number system may be needed is quite apparent.

Verbal problem situations should be studied at length, giving pupils the opportunity to write the quadratic sentence which expresses the problem situation. This should include problems involving both equalities and inequalities.

TOPIC XV

Introduction to Relations and Functions*

Suggested time allotment: 10-11 weeks

This topic will be devoted to a formal study of the characteristics of some of the sets of ordered pairs with which pupils have had prior experience. It will include modern definitions of relations and functions and the accompanying vocabulary.

The topic may represent as much as one semester's work, depending on the attention given to the study of the general concept of function—such things as graphing, inverse of functions, composition of functions —and the numbers of applications of functions which are investigated.

Two types of functions should be studied in considerable detail—linear and quadratic. The slope of linear functions is an idea which has been introduced earlier. His idea should now be extended to establish the conditions for parallel and perpendicular functions. In TOPIC XIII the solution of systems of linear equations was introduced. Those techniques may be reviewed at this time in connection with determining the conditions for the intersection of linear functions to be a unique point, the empty set, or an infinite set. Some attention may be given to a geometric interpretation of the solution of linear equations in two variables by elimination through addition. If time permits, systems of three linear equations and perhaps determinants may be introduced. The usual applications—mixture, age, rate, and numbers problems—may be repeated here and serious thought should be given to including linear programing.

The main purpose of TOPIC XIV was to give pupils some familiarity with solving quadratic equations before beginning a serious study of quadratic functions. Graphing quadratic functions is an open-ended topic which is properly included here. Finding zeros of quadratic functions by graphing and factoring should precede the introduction of the more refined methods of completing the square and the quadratic formula. With these tools at hand pupils should attempt to solve many applied problem situations.

Some interesting games which lead to an understanding of the properties of reflexive, symmetric, and transitive relations can be found in many of the new programs in modern mathematics. (TOPIC V is devoted completely to a study of relations and functions which should provide excellent conceptual understanding.)

Precise definitions should follow the above activities. The following need to be carefully defined: domain, range and field of relations, reflexive, symmetric, and transitive relations.

Suggested Developmental Questions

Some questions which should be considered during the topic are:

1. Identify the relations $\{(1, 1), (2, 2), (4, 4), (5, 5), (9, 9)\}$
   $\{(1, 1), (3, 3), (5, 1), (7, -5), (-1, -5)\}$
   $\{(1, 2), (3, 3), (2, 5), (1, 5), (2, 3), (1, 3)\}$
2. What kind of relations are "equal," "greater than," "less than?"
3. What are some relations that are only reflexive? Just transitive? Just symmetric?
4. Are there some relations which have but two of these properties?
5. What do the graphs of these functions have in common? $\{(x, y) \mid y = 4x + 1\}$, $\{(y, x) \mid 4y - 1 = x\}$, $\{[(x, f(x)] \mid f(x) = 4x + 3\}$

*Topic References: Numbers 1, 2, 3, 4, 5, 7, 8, 9, 15, 24, 31, 33, 34, 37

6. What do the graphs of these functions have in common?

   $\{(x, y) \mid y + 3x + 2\}$ $\{(m, n) \mid 2n = 2m + 4\}$, $\{(a, b) \mid b = {}^-2a + 2\}$

7. What do you notice about the graphs of the pairs of functions defined by $f(x) = 3x + 2$ and $f(x) = \frac{-1x}{3} + 4$, $f(a) = a - 3$ and $f(a) = {}^-a - 1$?

8. What conditions determine when two linear functions intersect in a single point, many points, no points?

9. Graph the following pairs of equations on the same set of axes:

   $y = 3x + 1$ and $y = \frac{-1x}{3} + 2$, $y = {}^-5x + 3$ and $y = \frac{1}{5}x - 1$

   $y = \frac{2}{3}x + 4$ and $y = \frac{3}{2}x + 3$. What do you notice?

10. Consider the equations $3x + 2y - 5 = 0$ and $2x - y + 3 = 0$

    Graph 1. $\{(x, y) \mid 2(3x + 2y - 5) + 3(2x - y + 3) = 0\}$ and $\{(x, y) \mid 3(3x + 2y - 5) + 3(2x - y + 3) = 0\}$

    2. $\{(x, y) \mid {}^-2(3x + 2y - 5) + 1(2x - y + 3) = 0\}$ and $\{(x,y) \mid 4(3x + 2y - 5) + {}^-3(2x - y + 3) = 0\}$

    3. $\{(x, y) \mid {}^-2(3x + 2y - 5) + 3(2x - y + 3) = 0\}$ and $\{(x, y) \mid (3x + 2y - 5) + 2(2x - y + 5) = 0\}$

11. Is the graph of the following equation a horizontal or vertical line?

    $4(3x - 5y - 2) + 6({}^-2x + 3y + 1) = 0$

12. Using $R \times R$ as the universe, draw graphs of the following functions:

    1. $y = x^2$
    2. $y = x^2 + 1$
    3. $y = x^2 - 2$

13. Using $R \times R$ as the universe, draw graphs of the following functions:

    1. $y = 2x^2$
    2. $y = {}^-3x^2$
    3. $y = \frac{1}{2}x^2$
    4. $y = \frac{1}{10}x^2$

14. Using $R \times R$ as the universe, draw graphs of the following functions:

    1. $y = x^2 + x$
    2. $y = x^2 - x$
    3. $y = x^2 + 2x$
    4. $y = x^2 = 2x$

15. Using $R \times R$ as the universe, draw graphs of the following functions:

    1. $y = (x - 2)^2 + 1$
    2. $y = (x + 3)^2 - 3$
    3. $y = {}^-2(x + 1)^2 - 1$
    4. $y = x^2 + 8x + 16 - 1$
    5. $y = 3(x^2 - 6x + 9) + 7$
    6. $y = 2x^2 - 7x + 9$

16. What values of x will make $y = 0$ in the following functions?

    1. $y = x^2 + 7x + 6$
    2. $y = x^2 = 9x - 10$
    3. $y = 2x^2 + 6x$
    4. $y = 3x^2 + 5x - 8$
    5. $y = {}^-2x^2 + 5x + 3$
    6. $y + ax^2 + bx + c$

17. A radar station in Turkey observes the following:

    Six seconds after a Russian rocket is fired it is 1,736 feet high. Eight seconds after firing it is 2,204 feet high. Nine seconds after firing it is 2,411 feet high.

    Assuming no booster rockets:

    1. How high will it climb?
    2. How soon will it reach its maximum height?
    3. When will it strike the earth (ground level 0)?
    4. What elevation was it fired from?

Here is a sample unit of work based on question 7.

The ideas of question 7 should be preceded by a precise definition of function. An intuitive definition can be given which depends on the graph of a relation. A relation **A** is a function if and only if no two points belong to the same vertical line. Another definition is—a relation **A** is a function if and only if no two ordered pairs of **A** have the same first component (coordinate, abscissa).

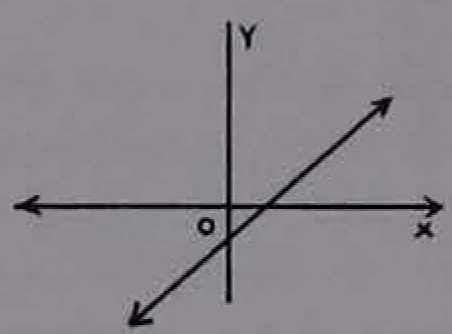
At this point, many exercises should be given where the pupils must identify which sets of ordered pairs are functions. Examples are:

a. $\{(2, 3), (4, 5), (3, {}^{-}2), (4, {}^{-}2), (6, \frac{1}{2})\}$

b. $\{(7, 9), (3, {}^{-}1), (4, \frac{1}{2}), (4, 2)\}$

c. $\{(3, 2\frac{1}{2}), (6, {}^{-}5), (6, 4), (6, 7), (2, {}^{-}4), (5, 3)\}$

d.



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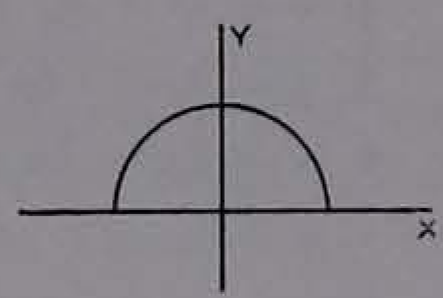

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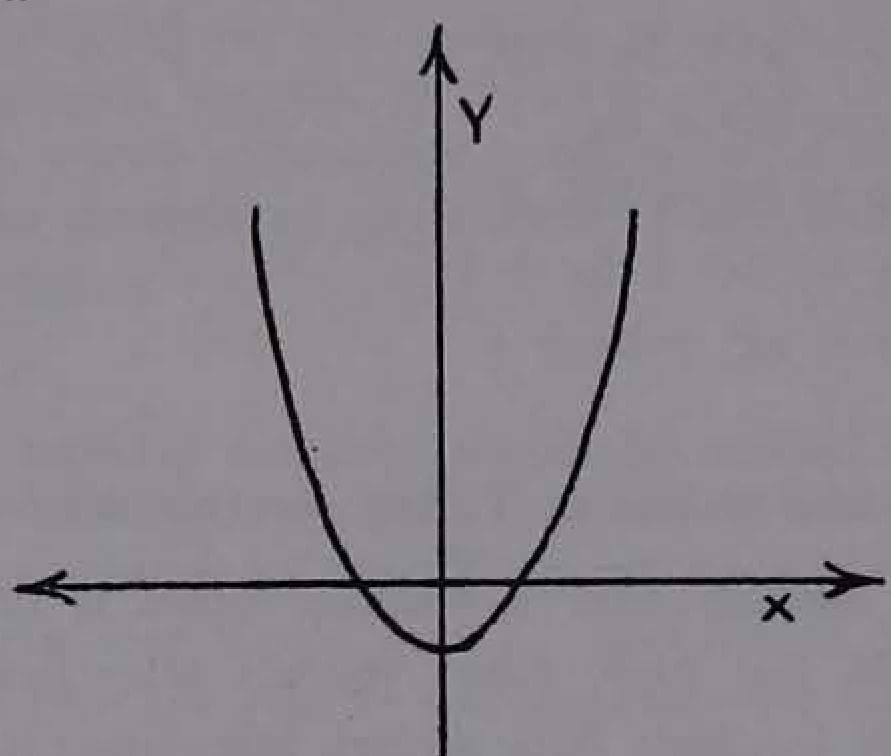

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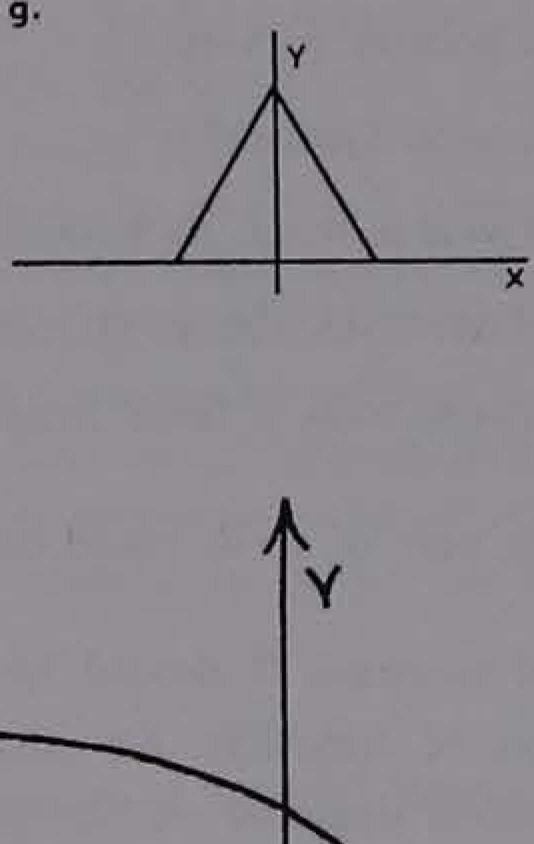

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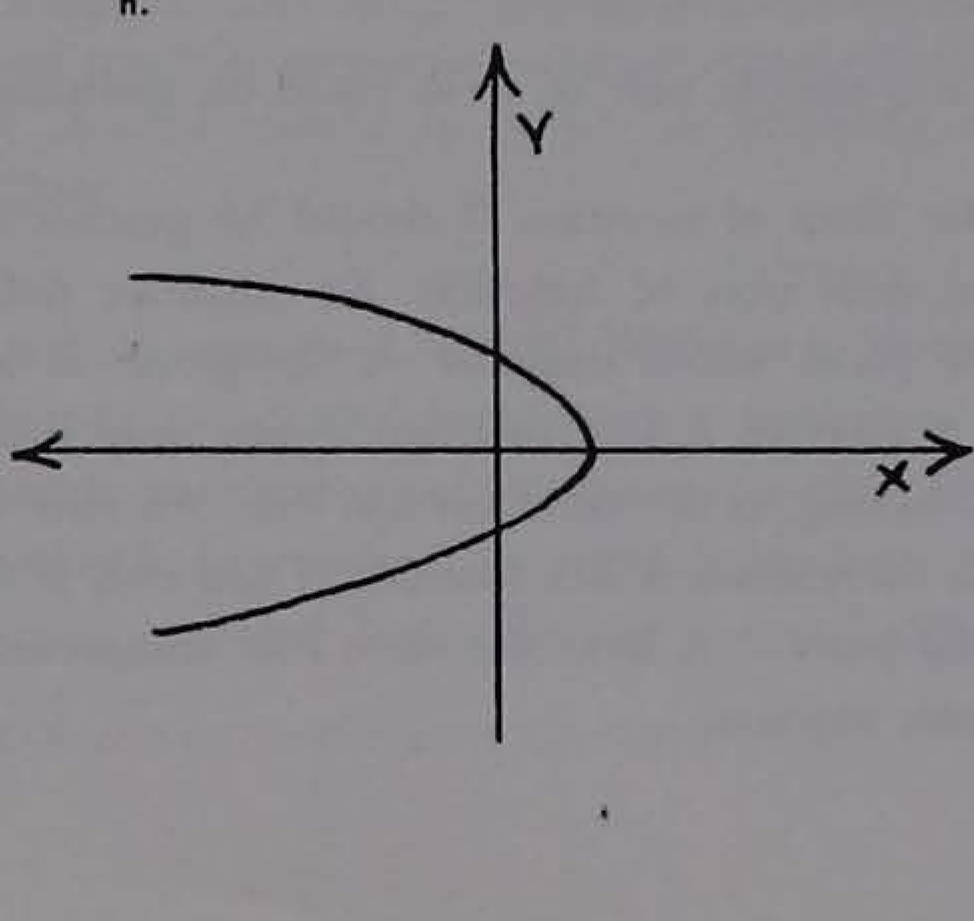

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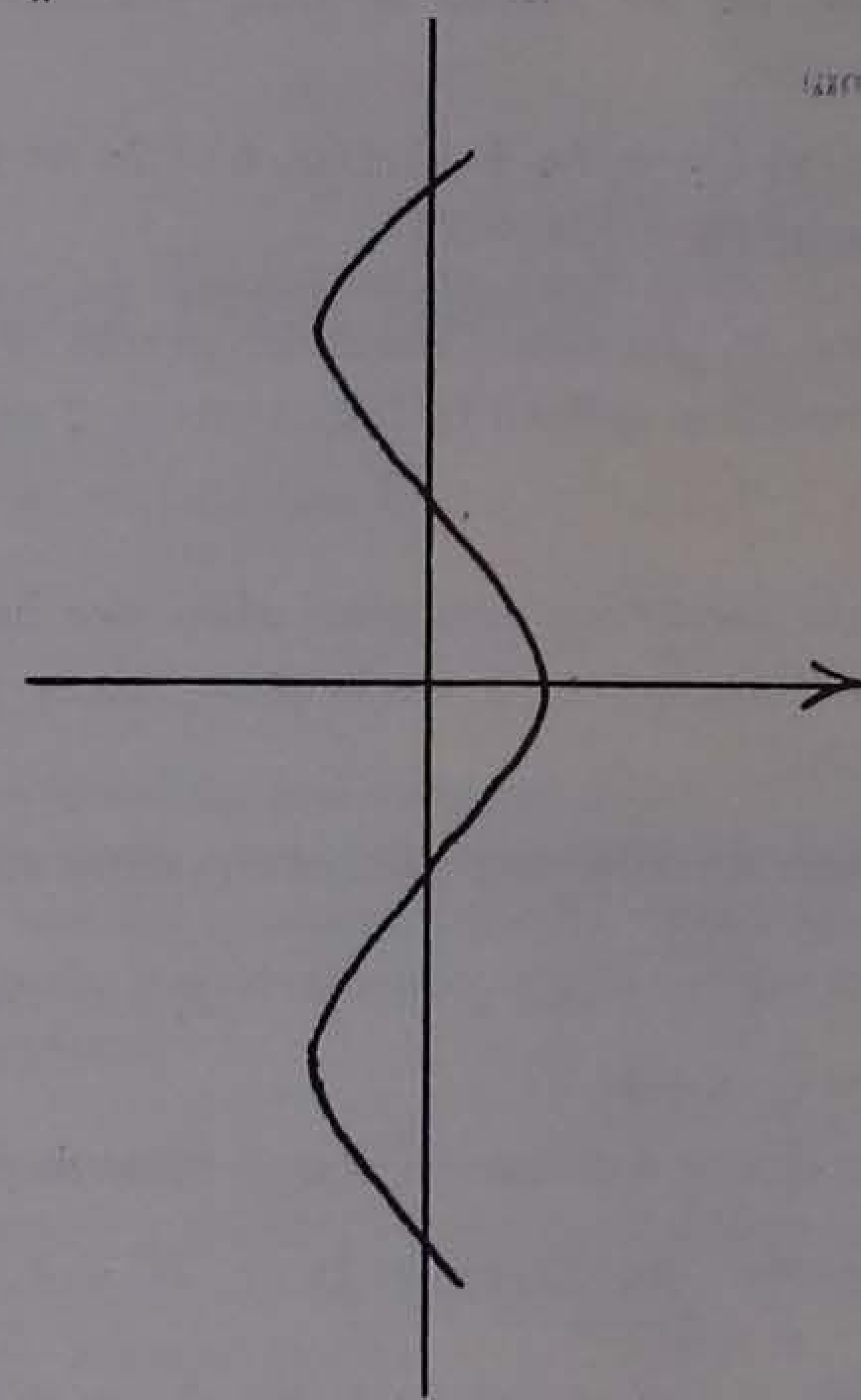

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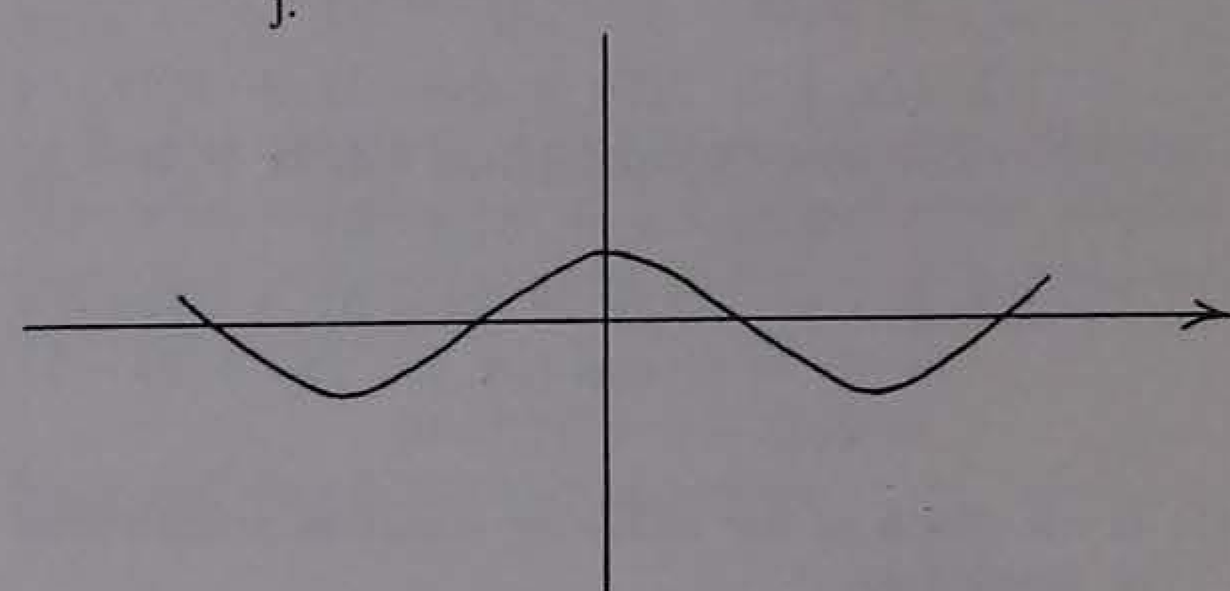


All of the above are relations, but only a, b, d, e, f, g, and j are functions. The relations are: c, h, i. The functions are: a, b, d, e, f, g, j.

The pairs of relations f and h, i and j are also referred to as inverses. If the definition of an inverse of a relation has not already been studied, it can be discussed here in connection with functions. That is, if f is a set of ordered pairs, then the inverse of f (f^{-1}) is the set of ordered pairs (y, x) such that (x, y) belongs to f. For example, if $R = \{(1, 2), (4, 3), ({}^{-}1, {}^{-}3), (8, 15)\}$ then R^{-1} (the inverse of R) $= \{(2, 1), (3, 4), ({}^{-}3, {}^{-}1), (15, 8)\}$. In this case both R and R^{-1} are functions. In the examples above only c, d, h, i have inverses which are functions. Here are some other examples.

$g = \{(x, y) \mid y = 2x + 3\}$ $g^{-1} = \{(y, x) \mid y + 2x + 3\} = \{(x, y) \mid x = 2y + 3\}$

$c = \{(x, y) \mid x + y = 5\}, c^{-1} = \{(y, x) \mid x + y$
$y = 5\}$

$k = \{(x, y) \mid y < x\}, k^{-1} = \{(x, y) \mid x < y\}$

$h = \{(x, y) \mid x^2 + y^2 = 25, x < 0\}, h^{-1} = \{(y, x) \mid$
$x^2 + y^2 = 25, x < 0\}$

Of these g, g^{-1}, c, c^{-1}, and h^{-1} are functions, the others are not. Other exercises of this type may now be provided, the object being to graph them and determine whether or not they are functions. Pupils should also try to analyze some descriptions without graphing and write a description of the inverse of a function and decide if it is a function. One definition is needed: A function f is a linear function if there are numbers a $\neq 0$ and b such that $f = \{(x, y) \mid y = ax + b\}$. A function f is a constant function if there is a number a such that $f = \{(x, y) \mid y = a\}$

The foregoing is preparation for the work suggested in question 7. The objective here is to let pupils review what determines the slope of a function by supplying them many functions with the same slope to graph on the same set of axes. Several problems like the following should be studied:

$\{(x, y) \mid y = 3x + 2\}$
$\{(x, y) \mid y = 3x + 1\}$
$\{(x, y) \mid y = 3x\}$
$\{(x, y) \mid y = 3x - 4\}$
$\{(x, y) \mid 2y - 6x + 2 = 0\}$
$\{(x, y) \mid y = 2x - 2\}$

The last one might be included as a check on pupil participation in graphing. Then other similar problems should be graphed.

Questions 12, 13 and 14. The purpose of these questions is to provide opportunity for analysis of the graph of the quadratic function.

The problems listed here are chosen to provide the opportunity for pupils to discover accurate and convenient methods of drawing the graph of a quadratic function.

The basic function is $y = x^2$. The first question studies the sliding of the graph up and down the y axis. More problems than these should be provided, and the pupil should find ordered pairs which satisfy the functional relationship.

The second question deals with the fatness or thinness of the graph of the function, as well as the flipover of the graph. Here again more examples should be used to provide adequate understanding.

The third question demonstrates how the graph is slid along the x axis.

It is expected that all of these ideas will be discovered by the pupils. They should also observe the graph's axis of symmetry. They should notice whether the graph has a maximum or minimum value, when this value occurs, and what the value is.

Question 15. This question introduces pupils to the form $(x - a)^2 - c = y$ for graphing the quadratic function.

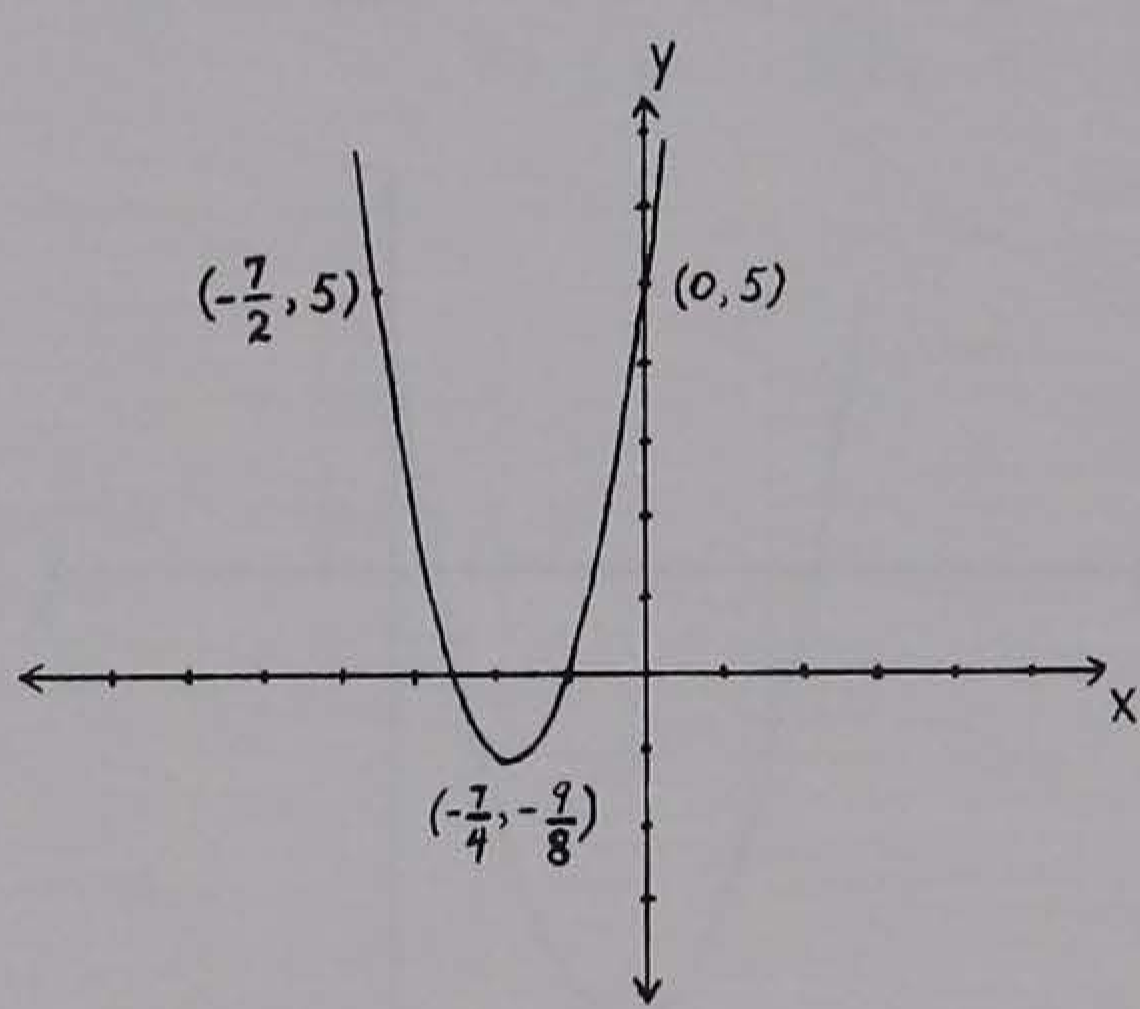
After working with the above material, pupils should now have insight into another method of drawing the graph of a quadratic function. This study should utilize the approach which was introduced in TOPIC XIV. Here we can analyze the axis of symmetry and maximum or minimum values for graphing the function.

Here is one example:

Draw the graph of the function $y = 2x^2 + 7x + 5$

$$y = 2\left(x^2 + \frac{7x}{2}\right) + 5$$
$$y = 2\left(x^2 + \frac{7x}{2} + \frac{49}{16}\right) + 5 - 2\left(\frac{49}{16}\right)$$
$$y = 2\left(x + \frac{7}{4}\right)^2 - \frac{9}{8}$$

From previous work, the pupil should have in mind the graph of $x^2 = y$. He can locate the point $(\frac{-7}{4}, \frac{-9}{8})$ and then draw the graph of twice x^2. He can also use symmetry to locate the point $(0, 5)$ and $(\frac{-7}{4} + \frac{-7}{4}, 5)$ or $(\frac{-14}{4}, 5)$



Here again, time permitting, much can be done to bring further understanding of the quadratic function.

For some it will be difficult, but surely all can recognize the power of their previous work when applied to problems of this type.

Question 16. The following discussion deals with the location of the zeros of the quadratic function.

Looking at the graphical solution of the quadratic equation, the zeros are the points where the graph cuts the x axis.

At this point a review of solution by factoring and by completing the square (TOPIC XIV) is appropriate. The quadratic formula should be developed again as a review of algebraic techniques.

Now the pupil should look at the quadratic function and quadratic equation and make many observations, for example:

Draw the graph of $y = x^2 + 6x + 3$.

State the maximum or minimum value of the function.

State the equation of the axis of symmetry.

State the value of x which makes the $y = 0$.

Solution:

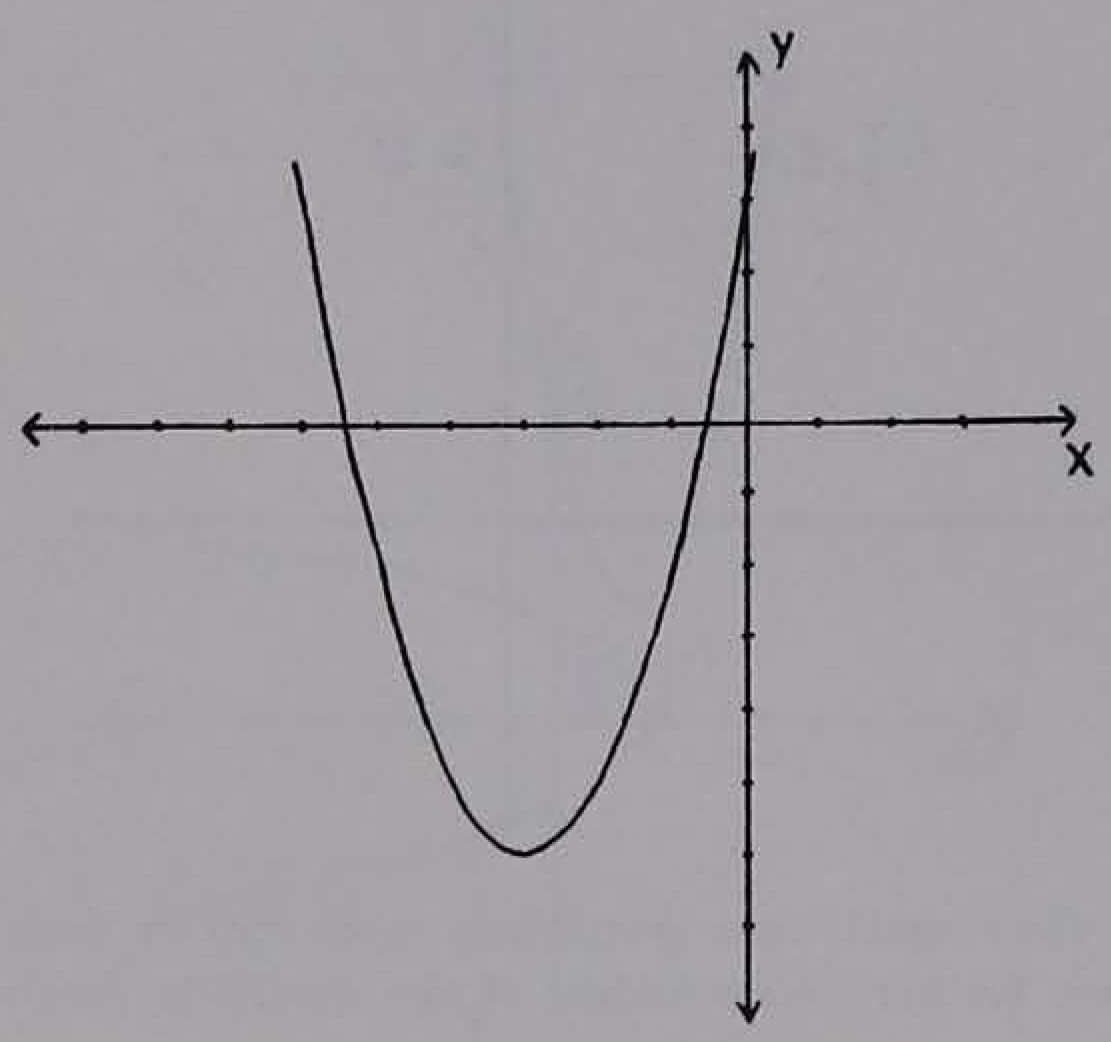
$y = x^2 + 6x + 3$
$y = (x^2 + 6x + 9) + (3 - 9)$
$y = (x + 3)^2 - 6$

Minimum values $^-6$

Axis of symmetry $x = {}^-3$

Zero of the function

$0 = (x + 3)^2 - 6$
$0 = (x + 3 + \sqrt{6})\ (x + 3 - \sqrt{6})$
$x = {}^-3 - \sqrt{6}$ or $x = {}^-3 + \sqrt{6}$



When examining the quadratic formula, time can now be spent in analyzing the discriminant.

Question 17. This question illustrates a problem application of the quadratic function.

Working through the given problem should illustrate many ideas. First of all, the given points should be graphed on an appropriate scale.

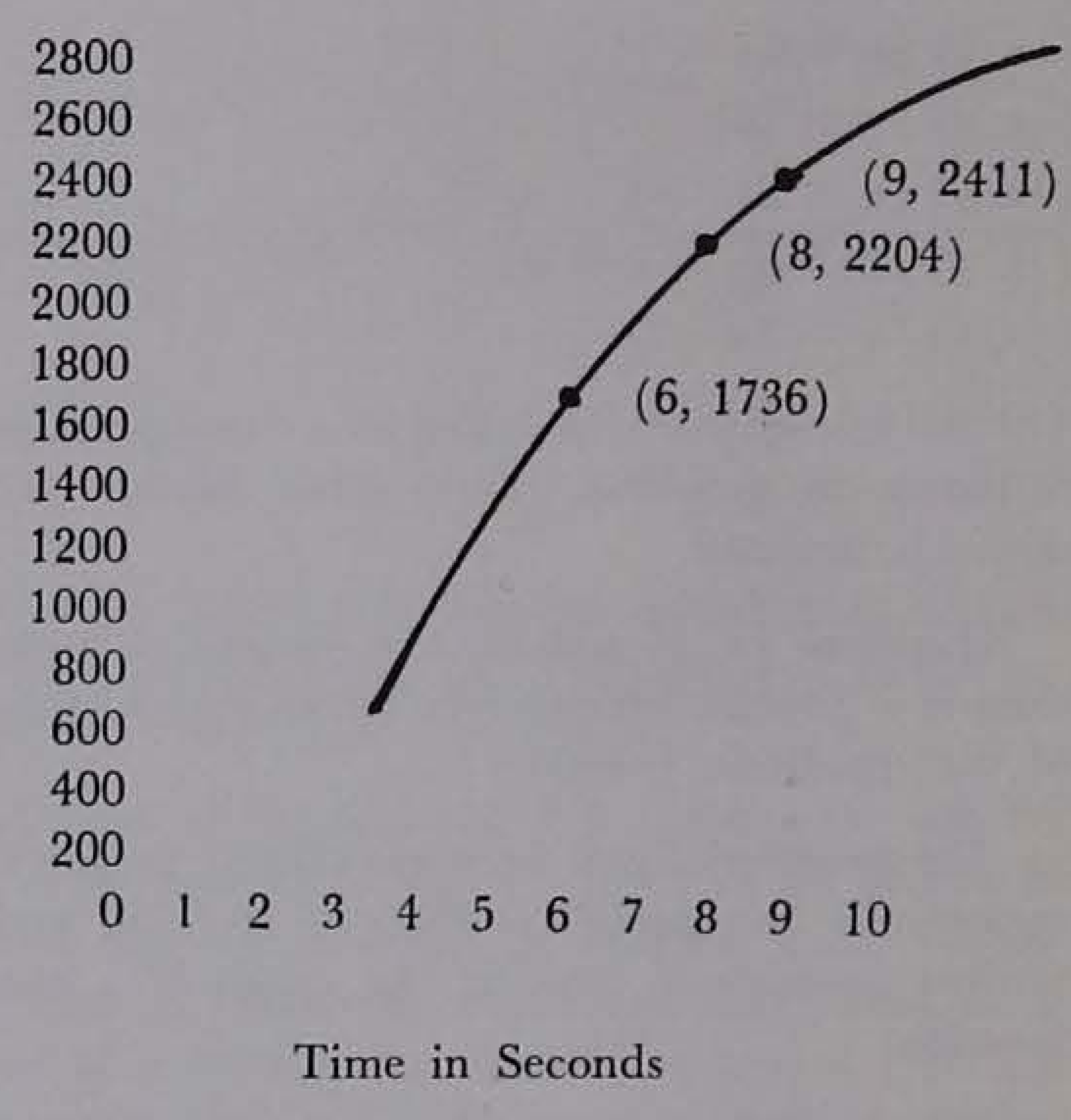
The quadratic function can be determined as follows:

$$y = ax^2 + bx + c$$
$$1736 = a \cdot 6^2 + b \cdot 6 + c$$
$$2204 = a \cdot 8^2 + b \cdot 8 + c$$
$$2411 = a \cdot 9^2 + b \cdot 9 + c$$

Solving the three equations should show:

$a = {}^-9$
$b = 360$
$c = {}^-100$

The function then is $\{(x, y) \mid y = {}^-9x^2 + 360x - 100\}$



Then, using the function we can find the remaining solutions.

$y = {}^-9x^2 + 360x - 100$
$y = {}^-9(x^2 - 40x) - 100$
$y = {}^-9(x^2 - 40x + 400) - 100 + 9 \cdot (400)$
$y = {}^-9(x - 20)^2 + 3500$

Therefore, the rocket will climb to a height of 3500 feet. It will reach this after 20 seconds. It will strike the ground when y = 0, so $0 = {}^{-}9(x-20)^2 + 3500$

$$9(x-20)^2 - 3500 = 0$$

$$(x-20)^2 - \frac{3500}{9} = 0$$

$$\left[x - 20 + \frac{\sqrt{3500}}{3}\right]\left[x - 20 - \frac{\sqrt{3500}}{3}\right] = 0$$

$$x = 20 + \frac{\sqrt{3500}}{3}$$

The time then is $(20 + \frac{10\sqrt{35}}{3})$ or nearly 40 seconds when it strikes the earth.

The rocket was fired from 100 feet below ground level since at 0 time y = -100.

Summary

The latter part of this topic certainly should demonstrate many applications of the pupils' earlier work in mathematics. Undoubtedly, some pupils will not be able to do all the work suggested here, but it is hoped it will be challenging to many and that most classes will have opportunity to study material of this type One function of modern mathematics is to stimulate pupils to do better mathematical thinking. This topic on relations and functions surely provides unusual opportunities for this.

TOPIC XVI

Mathematics As A Logical Structure*

Suggested time allotment: 3-4 weeks

This topic provides opportunity to review the logical development of the mathematics studied since the seventh grade. It should arouse an appreciation for the way that the structure of mathematics grows out of the basic assumptions underlying its foundation.

The topic should begin with a listing of undefined terms—number, addition, multiplication, etc.—and the postulates for the natural numbers. A few of the definitions and theorems should be discussed and proven.

The integers may then be "invented" after showing the lack of a solution in the set of natural numbers for equations like $4 + x = 2$. The additive inverse of natural numbers can be postulated (i.e., for each number x there is a number ^-x such that $x + {}^-x = 0$). The rules for computing with the positive and negative integers may be proved as theorems after it has been agreed (postulated) that the integers should have the same properties as the natural numbers.

The rational numbers may be introduced through the study of an equation like $3x = 5$ and the postulates for each number, a, there exists a number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} - 1$. Theorems on computation, definitions, equivalence of fractions and repeating decimals, and ordering the rational numbers can be included.

The existence of still another gap in the number system may be found by examining the equation $x^2 = 2$. $\sqrt{2}$ should be defined as the number whose square is 2. It has been proved that there is no rational number $\frac{a}{b}$ such that $(\frac{a}{b})^2 = 2$.

*Topic References: Numbers 1 ,2, 6, 7

The real numbers may be defined as numbers represented by infinite decimals and postulates, and the theorems governing their behavior may be investigated.

The complex numbers are motivated by attempting to solve an equation like $x^2 = {}^-1$. The simple $a + bi$ form of complex numbers can be studied briefly.

The developments suggested above have been described by Bertrand Russell as creating numbers by "theft" instead of by "honest toil." With a class of sufficient ability, the creation of the integers, rational numbers, etc., might be attempted by "honest toil." For example, the integers may be defined as ordered pairs (a, b) of natural numbers where $(a, b) = (c, d) \leftrightarrow a + d = b + c$, $(a, b) + (c, d) = (a + c, b + d)$, and $(a,b) \cdot (c,d) = (ac + bd, ad + bc)$ and it can then be proved that the integers have the desired properties. Similarly, the rationals can be defined as ordered pairs of integers where $(a, b) = (c, d) \leftrightarrow ad = bc$, $(a, b) + (c, d) = (ad + bc, bd)$ and $(a, b) \cdot (c, d) = (ac, bd)$.

Thorough study of this topic should provide for investigation of such questions as:

1. What are some properties that are common to all of these number systems?
2. Are there number systems which do not possess these properties?
3. What properties are characteristic of only some of these systems?
4. What are some purposes for organizing mathematics in this way?
5. How are new mathematical structures developed?
6. What are some of the new fields which mathematicians have attempted to structure?

Bibliography

1. Adler, Irving. *The New Mathematics.* New York: The John Day Company, 1958. Pp. 187.

2. Aiken, Daymond J. and Charles A. Beseman. *Modern Mathematics, Topics and Problems.* New York: McGraw-Hill Book Company, 1959. Pp. 140. (Booklet)

3. Brown, John A. and others. *Mathematics, First Course.* Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Pp. 314.

4. Brown, John A. and others. *Mathematics, Second Course.* Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Pp. 365.

5. Brumfiel, Charles and others. *Algebra I.* Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1961. Pp. 371.

6. College Entrance Examination Board. *Program for College Preparatory Mathematics.* New York: College Entrance Examination Board, 1959. Pp. 63 (Paperbound)

7. College Entrance Examination Board. *Report of the Commission on Mathematics, Appendices.* New York: College Entrance Examination Board, 1959. Pp. 231. (Paperbound)

8. Evenson, A. B. *Modern Mathematics.* Chicago: Scott, Foresman and Company, 1962. Pp. 216. (Paperbound)

9. Glenn, William H. and Donovan A. Johnson. *Adventures in Graphing.* St. Louis: Webster Publishing Company, 1961. Pp. 64. (Paperbound)

10. Glenn, William H. and Donovan A. Johnson. *Computing Devices.* St. Louis: Webster Publishing Company, 1961. Pp. 55. (Paperbound)

11. Glenn, William H. and Donovan A. Johnson. *Fun With Mathematics.* St. Louis: Webster Publishing Company, 1961. Pp. 43. (Paperbound)

12. Glenn, William H. and Donovan A. Johnson. *Invitation to Mathematics.* St. Louis: Webster Publishing Company, 1960. Pp. 64. (Paperbound)

13. Glenn, William H. and Donovan A. Johnson. *Number Patterns.* St. Louis: Webster Publishing Company, 1960. Pp. 47. (Paperbound)

14. Glenn, William H. and Donovan A. Johnson. *The Pythagorean Theorem.* St. Louis: Webster Publishing Company, 1960. Pp. 48. (Paperbound)

15. Glenn, William H. and Donovan A. Johnson. *Sets, Sentences, and Operations.* St. Louis: Webster Publishing Company, 1960. Pp. 63. (Paperbound)

16. Glenn, William H. and Donovan A. Johnson. *Short Cuts in Computing.* St. Louis: Webster Publishing Company, 1961. Pp. 46. (Paperbound)

17. Glenn, William H. and Donovan A. Johnson. *Topology.* St. Louis: Webster Publishing Company, 1960. Pp. 40. (Paperbound)

18. Glenn, William H. and Donovan A. Johnson. *Understanding Numeration Systems.* St. Louis: Webster Publishing Company, 1960. Pp. 56. (Paperbound)

19. Glenn, William H. and Donovan A. Johnson. *The World of Measurement.* St. Louis: Webster Publishing Company, 1961. Pp. 64 (Paperbound)

20. Glenn, William H. and Donovan A. Johnson. *The World of Statistics.* St. Louis: Webster Publishing Company, 1961. Pp. 64. (Paperbound)

21. Greenleaf, Newcomb and Robert J. Wisner. "The Unique Factorization," *The Mathematics Teacher,* LII (December, 1959), 600-603.

22. Hartung, Maurice L. *A Teaching Guide for Slide Rule Instruction.* Chicago: Pickett and Eckel, Inc., 1960. Pp. 43.

23. Hayden, Dunstan and E. J. Finan. *Algebra One.* Boston: Allyn and Bacon, Inc., 1961. Pp. 440.

24. Hohlfeld, Joseph F. "An Analysis of the Quadratic," *The Mathematics Teacher,* LIV (March, 1961), 138-141.

25. Lichtenberg, Donovan and Marilyn Zweng, "Linear Programming Problems for First-Year Algebra," *The Mathematics Teacher,* LIII (March, 1960), 171-176.

26. Mayor, John R. and Marie S. Wilcox. *Algebra, First Course*. Second edition. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1961. Pp. 440.

27. McMahon, Della. "An Experimental Comparison of Two Methods of Teaching Per Cent to Seventh Grade Pupils." Unpublished Master's thesis. Columbia, Missouri: University of Missouri, 1959. Pp. 149.

28. Moshan, Ben. "Primitive Pythagorean Triples," *The Mathematics Teacher,* LII (November, 1959), 541-545.

29. Nichols, Eugene D. and Wagner G. Collins. *Modern Elementary Algebra*. New York: Holt, Rinehart and Winston, Inc., 1961. Pp. 402.

30. Rosskopf, Myron F. and others. *Modern Mathematics for Junior High School, Book 1*. Morristown, New Jersey: Silver Burdett Company, 1961. Pp. 424.

31. Rosskopf, Myron F. and others. *Modern Mathematics for Junior High School, Book 2*. Morristown, New Jersey: Silver Burdett Company, 1961. Pp. 424.

32. School Mathematics Study Group. *Mathematics for Junior High School*. 4 vols. New Haven, Connecticut: Yale University Press, 1961.

33. School Mathematics Study Group. *Mathematics for High School: First Course in Algebra*. 2 units. New Haven, Connecticut: Yale University Press, 1961.

34. University of Illinois Committee on School Mathematics. *High School Mathematics*. 5 units. Urbana: University of Illinois Press, 1959.

35. Van Engen, Henry; Maurice L. Hartung; and Lois Knowles. *Seeing Through Arithmetic*. Special Book B. Chicago: Scott, Foresman and Company, 1961. Pp. 320.

36. Van Engen, Henry and others. *Seeing Through Mathematics, Book* 1. Chicago: Scott, Foresman and Company, 1962. Pp. 508.

37. Van Engen, Henry and others. *Seeing Through Mathematics, Book* 2. Chicago: Scott, Foresman and Company, 1962. Pp. 512.