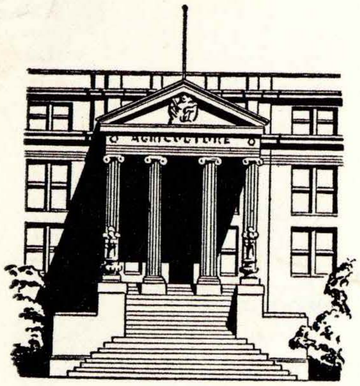

Allocation of Milk Among Products to Maximize Gross Income of the Nation's Dairy Farmers under 1964 Demand Functions

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SUMMARY

Linear equations of farm level demand were obtained for milk used in six different products. These demand equations were used in several quadratic-programming analyses to determine levels of farm marketings of milk and cream or allocation among products that would have maximized farmers' cash receipts from marketings of milk and cream in 1964. Each analysis computed farm and retail prices for milk used in various products, quantity used in each product and total cash receipts. In one quadratic program, the solution was unconstrained; i.e., no upper limit was imposed on prices, and no lower limit was imposed on quantities. The solutions of several quadratic programs were required to satisfy certain constraints. These constraints were upper limits on prices or lower limits on quantities available.

Most quadratic-programming solutions called for increases in retail prices. To estimate the effect of these on consumer welfare, average compensating variations in per-capita income were computed. Given a change in one or more retail prices, compensating variation for an individual consumer is the amount by which his income must change to leave him exactly as well off after the price change as before.

Two analyses found that farmers could have more than doubled their 1964 cash receipts from milk and cream by cutting production by more than a third. This would have caused farm prices of milk used in fluid milk and ice cream to quadruple and other prices to rise substantially. Per-capita compensating variation for this solution was about $45.

Other analyses showed that, even while marketing the quantity of milk actually marketed in 1964, total cash receipts of producers could have been increased by 82 percent by allocating less milk to fluid milk products and ice cream and more milk to other products. Per-capita compensating variation for this solution was about $35. That such a large increase in income is possible and that optimal allocations differ so much from actual allocations can be partly explained by the principles of price discrimination. Price discrimination theory says that, if the elasticities of demand in markets differ, the higher prices will be charged in the less-elastic markets (in this instance, the fluid-milk market is less elastic than all other product markets), assuming separation of these markets. And, according to price-discrimination theory, to get the greatest total revenue from any given total volume of sales from two or more separated markets, marginal revenue in each of the separated markets must be equal. This equating of marginal revenues did not hold for actual 1964 market conditions. Further, strict separation of markets does not exist because many handlers are multiproduct. It would be possible, however, to obtain strict separation of the markets for the various dairy products. This separation could be accomplished in the same way that the market for grade A milk for fluid milk and cream products is now separated from the market for grade A milk for processed dairy products by federal milk-marketing orders. Federal milk-marketing orders accomplish this by requiring dealers to account for their milk utilization and by auditing provisions.

Several analyses containing lower limits on quantities used indicated that 1964 cash receipts could have been increased by 33 to 43 percent. The per-capita compensating variation for these problems ranged from $4 to $9.

Most federal milk-marketing orders contain two milk classes: Class I milk is milk used in fluid milk products; class II milk is all other milk, including milk used in ice cream. In every problem analyzed, the farm price for milk used in fluid milk products is close to the farm price for milk used in ice cream, and these two farm prices are higher than the other farm prices for milk. These results raise the possibility that income to federal-order producers could be increased by reclassifying ice cream from class II to class I. An alternative would be to put ice cream in a class II by itself and put all other manufactured dairy products into class III.

ALLOCATION OF MILK AMONG PRODUCTS TO MAXIMIZE GROSS INCOME OF THE NATION'S DAIRY FARMERS UNDER 1964 DEMAND FUNCTIONS[1]

by George W. Ladd and Gail E. Updegraff

One purpose of this study was to determine prices and marketings of milk and milk products that would have yielded maximum cash receipts to United States dairy farmers for 1964. Prices allocate resources among alternative uses and distribute income among the owners of the various factors of production. In this study, prices will be considered as equity instruments rather than as allocation instruments.

In the dairy region of the central-northeastern United States, one of the best milk-producing regions in the country, average net income of dairy farmers dropped from $4,567 in 1958 to $4,178 in 1964. These same farmers had an average investment of $45,500 and an average herd size of 33 in 1964 (13). The average American family had an income of $6,556 in 1964. Grade A milk producers in eastern Wisconsin, the top income receivers among United States dairy farmers, received an average of $6,541 in return for an average investment of $71,950 in 1964. This amount of money invested at 4 percent interest and compounded semiannually would return $2,906.78. If an average eastern Wisconsin grade A milk producer chose such an alternative, he need only earn $3,634.22 annually at a full-time job to equal the income from dairy farming.

The Dairy Marketing Advisory Committee (4) found that, of the 13 farm enterprises showing lower returns per $100 invested (after family labor was deducted from gross returns at hired labor rates), only three required as much total capital per farm as grade A milk production. These were hog - beef fattening in the Corn Belt, cattle ranches in the Southwest and sheep ranches in the Southwest.

Dissatisfaction with low milk prices prompted the National Farmers Organization to vote in favor of a holding action on milk in December 1966 in an attempt to obtain an increase of 2 cents per quart in the price of fluid milk.

The Dairy Marketing Advisory Committee (4) concluded that the price necessary to guarantee a safe and adequate milk supply in the short run has ceased to be a practical criterion for determining an acceptable milk price because the present price does not return equitable incomes to milk producers. Kelley and Knight (7) also question the equity of the pricing standards of the Agricultural Marketing Agreement Act of 1937; much of the grade A milk produced in this country is priced in federal orders set up pursuant to this Act.

Maximizing cash receipts to dairy farmers involves higher prices to consumers.

The second purpose of this study was to estimate the effect of the higher prices on consumer welfare. And the third purpose was to meet the need recently expressed by Iowa Farmers Union president Sydney Gross. He wants "the Iowa Cooperative Extension Service to do much more to educate farmers on the price-weakening consequences of overproduction of corn, soybeans and other commodities" (5). One way to study the "price-weakening consequences of overproduction" is to compare the actual situation with the situation that would maximize gross income.

PROCEDURE

Six different uses of milk considered in this study and the values of the subscript i used to identify them are:

Fluid milk and cream, $i = 1$
Evaporated and condensed milk, $i = 2$
Cheese, $i = 3$
Ice cream, $i = 4$
Butter, $i = 5$
Other uses, $i = 6$

Farm level demands were estimated for each of these six products. Since these different uses of milk can be considered as separate markets, the cash receipts of dairy farmers can be maximized by controlling the amount of milk offered to each market. This method of analysis, similar to the analyses carried out by Ladd and Kuang (9) and Ladd and Hallberg (8), is analogous to the analysis of a price-discriminating monopolist who has more than one outlet for his product.

Two basic problems were studied. One involved maximizing cash receipts of milk producers when both the total quantity of milk and the quantity of milk allocated to each product can be varied. The second problem involved maximizing cash receipts of milk producers when the total quantity of milk was fixed and allocation among uses was varied. Several variations of these two basic problems were also analyzed.

[1] Project 1466 of the Iowa Agriculture and Home Economics Experiment Station.

These variations involved placing lower limits on production or upper limits on price increases. All problems were solved by using quadratic programming. Louwes et al. (10) performed a similar study for The Netherlands.

Farm Level Demand

For each product, the annual farm level demand for domestic use can be written

$$q_i = a_i + \sum_{j=1}^{6} b_{ij}p_j$$

where q_i represents number of hundredweights of milk used in the i-th product and p_j represents farm price per hundredweight for milk used in the j-th product. The values of slope coefficients b_{ij} were obtained from Brandow (3, pp. 112-115). Published data on 1964 values of q_i and p_j were used to compute 1964 intercept a_i as a residual $a_i = q_i - \sum_j b_{ij}p_j$.

For i = 2, 3 and 5, export-demand equations are

$$x_i = f_i + e_{ii}p_i$$

where x_i represents number of hundredweights of milk used in the i-th product for export. Values of e_{ii} were obtained from Brandow (3, p. 115); these values and published data on 1964 values of x_i and p_i were used to compute intercept f_i as $f_i = x_i - e_{ii}p_i$.

Total farm level demand is the sum of domestic plus export demand; for the i-th product

$$q_i + x_i = (a_i + f_i) + (b_{ii} + e_{ii})p_i + \sum_{j \neq i} b_{ij}p_j = c_i + \sum_j d_{*ij}p_j$$

Let

$$q = (q_1, q_2 + x_2, q_3 + x_3, q_4, q_5 + x_5, q_6)'$$

a 6-element column vector of total demands.

$$p = (p_1, p_2 \ldots, p_6)'$$

a 6-element column vector of farm prices.

$$c = (c_1, c_2, \ldots, c_6)'$$

a 6-element column vector of intercepts.

$$d_{ij} = (d_{*ij} + d_{*ji})/2$$

$D = (d_{ij})$, where i = 1, 2, . . . , 6; j = 1, 2, . . . , 6. D is a 6-by-6 matrix of slope coefficients.[2] Values of c_i and d_{ij} are presented in table 1. The six functions for 1964 total farm level demand can be written in matrix notation as

$$q = c + D\,p$$

[2] Making D symmetric does not affect the solution to the quadratic program and does make it easier to solve the quadratic program.

Quadratic Programming

Gross farm income for 1964 from the sale of milk is $\sum p_iq_i = p'q$. $R = p'q = c'p + p'Dp = \sum_i c_ip_i + \sum_i \sum_j d_{ij}p_ip_j$. Since D is negative definite, R has a proper maximum. The maximum value of R can be computed by using quadratic programming. We used the Zrubek (18) machine program of the Van de Panne and Whinston algorithm (16, 17) to solve the quadratic programs. Boot (2) provides an extensive discussion of theory and methods of solution of quadratic programs.

The total differential of R if prices vary independently is

$$dR = \sum_j \sum_i (\partial p_iq_i/\partial p_j)dp_j$$

At a maximum of R, dR = 0. Elasticity of demand for the i-th product with respect to the j-th price is

$$E_{ij} = (\partial q_i/\partial p_j)\,(p_j/q_i)$$

Hence

$$\partial p_iq_i/\partial p_j = q_i\,(\delta_{ij} + E_{ij}p_i/p_j)$$

where

$\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, and

$$(1)\quad dR = 0 = \sum_j \sum_i q_i\,(\delta_{ij} + E_{ij}p_i/p_j)dp_j$$

If we solve for those values of q_i that maximize R and if we do not impose any explicit constraints, we find the values of q_i (and of p_i) at which own-price and cross-price elasticities of demand satisfy equation 1. (If we have but one product, $dR = q_i(1 + E_{ii})\,dp_i = 0$ if and only if $E_{ii} = -1$.) The maximum possible value of R in problems without explicit constraints is limited by the negativity of each d_{ii} and by the negative definiteness of D. In the problems containing explicit constraints, the maximum value of R is limited also by the constraints.

Recently, artificial milk and filled milk have received a great deal of publicity. Whether these two products have become or will become sufficiently important to affect the farm-level demand for other dairy products is not now known. To the extent that these two products have significantly altered farm level demand in recent years, the 1964 results of this study do not provide reliable guides to the future. Whether our decision not to pay special attention to these products has made the 1964 results biased or inaccurate depends mainly upon the effects the development of these products between 1955-57 (period

Table 1. Values of c_i and of d_{ij}[a].

i	c_i	d_{i1}	d_{i2}	d_{i3}	d_{i4}	d_{i5}	d_{i6}
1	661.25	-15.99	1.062	0.11506	0.05145	0.11417	
2	60.76	1.062	-4.896	0.017093	0.0082265	0.028025	
3	223.55	0.11506	0.017093	-19.27	0.02955	0.09073	
4	110.62	0.05145	0.0082265	0.02955	-2.808	0.044095	
5	469.96	0.11417	0.028025	0.09073	0.044095	-70.19	
6	51.87						-5.073

[a] These are parameters in the equation: $R = \sum_i c_ip_i + \sum_i \sum_j d_{ij}p_ip_j$, where p_i = farm price per hundredweight of milk used in product i and R = gross farm income from sale of milk.

covered by Brandow's equations) and 1964 had upon the d_{ij} slope coefficients. Our method of computing the c_i (as residuals) adjusted the demand functions for any changes in intercepts that occurred between 1955-57 and 1964.

Constraints

Several types of constraints were used.

TOTAL QUANTITY

One constraint required that the total of the quantities allocated to various uses in the solution equal the acutal total amount of milk available in 1964; i.e., $\sum_i q_i = Q_{64}$.

MINIMUM CONSUMPTION LEVEL

This constraint was designed to put lower limits on the domestically consumed quantities of the six dairy products. This was deemed desirable as a welfare constraint because of the threat that large cutbacks in quantities would pose to nutrition and the standard of living.

The minimum consumption levels for this constraint are based on a major food consumption survey, taken in 1955, of households in the United States (14). The minimum household consumption levels were selected as those levels consumed by households in the $2,000 to $3,000 income bracket. In this particular survey, this group of households appeared to have adequate nutritional levels of dairy products, according to the standards of Tobey (12). Per-capita consumption figures were found by dividing household consumption figures by the average family size given in the survey for this group.

Total minimum-consumption levels could have been found for all dairy products by multiplying per-capita figures by total population. But this method would fail to take account of changes in tastes for dairy products. For all uses except "other uses," changes in tastes were accounted for by a production-ratio method. Treating 1955 as the base year, the 1955 total minimum-consumption figure was multiplied by the ratio of 1964 production to 1955 production. The minimum consumption for "other uses" represents a downward adjustment of actual consumption. This adjustment is partly based on the total minimum consumption for nonfat dry milk and partly on an arbitrary figure assigned for the remaining uses of milk in this category. The minimum consumption quantities are given in table 2. The minimum-consumption contraint for each product is expressed by setting each farm-level domestic-use demand equation greater than or equal to its respective minimum consumption.

Table 2. Actual and minimum consumption levels in million hundredweight, 1964.

	Actual	Minimum
Fluid	592.00	558.28
Evaporated	47.83	41.64
Cheese	177.00	57.50
Ice cream	102.44	89.74
Butter	307.64	267.76
Other	42.08	25.52

Table 3. Percentage ratios of actual 1964 consumption minus minimum 1964 consumption, to actual 1964 consumption.

Fluid	5.7
Evaporated	12.9
Cheese	67.5
Ice cream	13.0
Butter	12.4

Table 4. Price constraint weights, w_i[a].

w_1	6.632
w_2	0.378
w_3	1.159
w_4	0.759
w_5	0.874
w_6	0.198

[a] w_i = 10 times the ratio of 1964 consumer expenditures on i-th product to 1964 consumer expenditures on all dairy products.

CHEESE PRODUCER EQUITY

Table 3 shows that the minimum-consumption constraints allow a much larger proportional reduction in production of cheese than in production of other products. To guard against the possible inequity caused by drastic reduction in production of one commodity, a constraint quantity for cheese was determined by taking the average of the percentages given in table 3 for evaporated and condensed milk, cheese and butter, and multiplying 1 minus this average percentage, times actual cheese consumption. The result is a minimum-cheese quantity of 153.99 million hundredweight. Other constraints are the same as in the minimum-consumption constraint.

PRICE CONSTRAINT

The price constraint used in this study is the same as one used in a study of the optimal use of milk in The Netherlands (10). The constraint is

$$I = \sum_i w_i(p_i - p_{ia})/p_{ia} = \sum_i w_i\,(p_i/p_{ia}) - 10$$

where

w_i = 10 times the ratio of 1964 consumer expenditures on i-th product to 1964 consumer expenditures on all dairy products.[3]

p_{ia} = actual 1964 price of i-th product.

The values of w_i are shown in table 4. I is an index of changes in dairy prices. I = 1 means a weighted-average price change of 10 percent.

[3] Before computing w_i for butter, the ratio of expenditures on butter was halved; this adjustment was made to allow for the substitution of margarine for butter.

Problems Solved

Different combinations of the constraints were added to the objective function to form these different maximization problems, each of which will be identified by initials in the subsequent discussion:

Problem 1, NoC: No constraints were added before maximizing the value of the objective function. That is, the problem is to maximize cash receipts of producers when total quantity of milk and its allocation among the six products is allowed to vary.

Problem 2, TQC: The total quantity constraint is added; the problem is to maximize cash receipts when only the allocation of milk among the products is allowed to vary.

Problem 3, MCLC: The minimum-consumption level constraints are added before maximizing cash receipts of producers.

Problem 4, MCLC + TQC: The minimum-consumption level and total quantity constraints are added before maximization.

Problem 5, ChPE: The cheese producer equity constraints are added before maximization.

Problem 6, ChPE + TQC: The cheese producer equity and total quantity constraints are added.

Problem 7, Pr, I = 0: The price constraint with I = O is added.

Problem 8, Pr, I = 0 + TQC: The price constraint with I = 0 and the total quantity constraint are added.

Problem 9, Pr, I = 1: The price constraint with I = 1 is added.

Problem 10, Pr, I = 1 + TQC: The price constraint with I = 1 and the total quantity constraint are added.

Problem 11, Pr, I = 2: The price constraint with I = 2 is added.

Problem 12, Pr, I = 2 + TQC: The price constraint with I = 2 and the total quantity constraint are added.

Problem 13, Pr, I = 3: The price constraint with I = 3 is added.

Problem 14, Pr, I = 3 + TQC: The price contraint with I = 3 and the total quantity constraint are added.

Problem 15, Pr, I =0.6: The price constraint with I = 0.6, which represents the approximate average change in the consumer price index between 1951 and 1964, is added.

Problem 16, Pr, I = 28: The price constraint with I = 28, which is the approximate average I needed to give results as obtained in problem 1, is added.

Compensating Variation In Consumer Income

An equation relating farm to retail prices was developed from Brandow (3). This equation was used to compute retail prices corresponding to the farm prices obtained in the solutions to the quadratic programs. These solution retail prices were higher than actual retail prices; we considered some of the welfare implications of these price increases by computing compensating variations in consumer income.

Consider a consumer who has been in equilibrium and who moves to a new equilibrium in response to an increase in prices while his money income remains constant. The compensating variation in income is the amount by which this consumer's income must increase to leave him on the same level of utility as he was before the price increase. Let dY represent the amount of the compensating variation, and let p_i and q_i represent price paid and quantity purchased of i-th commodity. Then

$$dY = \sum_i (\partial Y/\partial p_i)dp_i + \tfrac{1}{2} \sum_i \sum_j (\partial^2 Y/\partial p_i \partial p_j)(dp_i dp_j)$$

Hicks (6, pp. 329-333) has shown that when utility is kept constant

$$dY = \sum_i q_i dp_i + \tfrac{1}{2} \sum_i \sum_j S_{ij}\, dp_i dp_j$$

where S_{ij} is the substitution term in the Slutsky equation.

By definition

$$S_{ij} = \partial q_i/\partial p_j + q_j(\partial q_i/\partial Y)$$

Suppose we have computed values of the B_{ij} and of B_{iM} in the demand function for an individual person

$$Q_i = B_{io} + \sum_i B_{ij}P_j + B_{iM}\, M$$

where Q_i is the logarithm of quantity of i-th product purchased, P_j is the logarithm of price paid for the j-th product and M is the logarithm of his money income Y. Then

$$\partial q_i/\partial p_j = B_{ij}\, q_i/p_j$$
$$\partial q_i/\partial y = B_{iM}\, q_i/Y$$

and S_{ij} can be computed from

$$S_{ij} = B_{ij}(q_i/p_j) + B_{iM}(q_i q_j/Y).$$

This equation provides a "point substitution term" analogous to a "point elasticity." We estimated a point substitution term as follows

$$ES_{ij} = B_{ij}(q_{ia}/p_{ja}) + B_{iM}(q_{ia}q_{ja}/2{,}272)$$

where p_{ja} and q_{ja} represent actual 1964 retail price and per-capita consumption of the j-th product and 2,272 is per-capita disposable income in 1964 dollars.

Since B_{ij} and B_{iM} are constants, S_{ij} varies as the ratios q_i/p_j and q_iq_j/Y vary. We also estimated an "arc substitution term," analagous to an "arc elasticity." These were computed as

$$ET_{ij} = (B_{ij}/2)(q_{ia}/p_{ja} + q_{i1}/p_{j1}) + (B_{iM}/2)([q_{ia}q_{ja} + q_{i1}q_{j1}]/2{,}272)$$

where q_{i1} and p_{j1} represent per-capita consumption and retail price in the solution to problem 1.

Brandow's (3) retail price and income elasticities B_{ij} and B_{iM} were used to compute ES_{ij} and ET_{ij}. These values of ES_{ij} and ET_{ij} were then used to compute dY for 1964 for each problem studied. Letting p_{it} and q_{it} represent the values of retail price and per-capita consumption in the solution to problem t, we computed the measures of per-capita compensating variation for each problem

$$SCV_t = \sum_i (p_{it} - p_{ia}) q_{ia} + \tfrac{1}{2} \sum_j \sum_i ES_{ij} (p_{it} - p_{ia}) (p_{jt} - p_{ja})$$

$$TCV_t = \sum_i (p_{it} - p_{ia}) (q_{ia} + q_{i1})/2 + \tfrac{1}{2} \sum_j \sum_i ET_{ij} (p_{it} - p_{ia}) (p_{jt} - p_{ja})$$

QUADRATIC PROGRAMMING RESULTS

Actual and solution values of farm prices, farm quantities and farm income are presented in tables 5, 7 and 8. Actual and solution values of retail prices are shown in table 6.

Total farm level demand equations, total quantity constraints, minimum consumption level constraints and cheese producer equity constraints were computed for 1951, 1955 and 1960, as well as for 1964. Problems 1 through 16 were also solved for each of these years. Because the solutions showed the same basic patterns for each year, it is appropriate to present only the 1964 solutions. Results for all 4 years are presented in Updegraff (15).

Problems 1 and 2

The results of problem 1 show that, by decreasing the total quantity of milk available by 38 percent and allocating the milk among the six products in a specified way, milk producers as a whole could have raised their total cash receipts by 103 percent in 1964. This increase in total cash receipts would have involved an increase in domestic cash receipts for every product (all product quantities are in milk equivalent). Cash receipts for milk used as fluid, evaporated and ice cream would have increased 162, 248 and 257 percent. Cash receipts from net exports would have declined for all exported products. The farm price for milk used in fluid milk would have increased 370 percent, and the farm price for milk used in ice cream would have increased 566 percent. In this solution, as in the solutions to problems 2 through 6, and problem 16, farm prices for milk used in fluid milk products and for milk used in ice cream are nearly equal. In all solutions, these two prices are much higher than prices for other products. The smallest percentage increase in farm price was that of milk used in butter (47 percent).

The solution called for reducing the amount of milk used in fluid milk from 592 million to 331 million

Table 5. Farm prices in dollars per hundredweight for 1964: acutal and solution values.

Type	Fluid	Evaporated	Cheese	Ice cream	Butter	Other	Weighted average
Actual	4.59	3.04	2.79	3.04	2.32	1.93	3.53
Problem 1	21.54	10.95	5.99	20.24	3.41	5.11	13.31
Problem 2	16.18	5.55	0.68	14.97	0.0	0.0	6.40
Problem 3	6.99	6.64	6.05	7.65	2.91	5.11	6.05
Problem 4	6.95	6.57	3.76	7.59	1.13	2.83	4.74
Problem 5	6.57	5.44	3.65	7.61	2.90	5.11	5.37
Problem 6	6.88	5.49	3.65	7.59	1.23	2.93	4.70
Problem 7	4.71	2.72	1.99	3.90	2.40	1.47	3.50
Problem 8	4.66	2.75	2.07	3.85	2.51	1.56	3.53
Problem 9	5.24	2.97	2.12	4.41	2.43	1.59	3.83
Problem 10	5.26	2.96	2.07	4.53	2.35	1.54	3.81
Problem 11	5.77	3.23	2.24	4.93	2.46	1.70	4.15
Problem 12	5.88	3.16	2.06	5.12	2.19	1.51	4.07
Problem 13	6.30	3.49	2.37	5.44	2.49	1.82	4.47
Problem 14	6.51	3.34	2.04	5.64	2.04	1.47	4.32
Problem 15	5.03	2.87	2.07	4.21	2.42	1.54	3.70
Problem 16	19.51	9.96	5.50	18.27	3.29	4.67	12.21

Table 6. Retail prices in dollars per hundredweight for 1964: actual and solution values.

Type	Fluid	Evaporated	Cheese	Ice cream	Butter
Actual	10.09	7.11	4.79	9.92	3.26
Problem 1	28.90	15.45	8.04	25.97	3.97
Problem 2	22.95	9.67	2.41	20.91	0.70
Problem 3	12.75	10.83	8.10	13.88	3.49
Problem 4	12.70	10.76	5.68	13.83	1.78
Problem 5	12.28	9.55	5.56	13.85	3.48
Problem 6	12.63	9.60	5.56	13.83	1.88
Problem 7	10.22	6.64	3.80	10.28	3.00
Problem 8	10.16	6.67	3.88	10.24	3.11
Problem 9	10.81	6.91	3.94	10.77	3.03
Problem 10	10.83	6.90	3.88	10.89	2.96
Problem 11	11.39	7.19	4.06	11.27	3.06
Problem 12	11.52	7.11	3.87	11.46	2.80
Problem 13	11.98	7.46	4.20	11.76	3.09
Problem 14	12.22	7.30	3.85	11.95	2.66
Problem 15	10.57	6.80	3.88	10.58	3.02
Problem 16	26.65	14.39	7.52	24.08	3.86

hundredweight while increasing the retail fluid milk price to $28.90 per hundredweight. Decreases in the quantities of evaporated milk, cheese, ice cream, butter and other uses would have been 2, 25, 47, 23 and 38 percent, respectively.

The solution to problem 2 (TQC) shows that producers could have increased total cash receipts by 82 percent in 1964 from the total quantity of milk actually marketed. This increase could have been made possible by increasing the amounts of milk allocated to the production of evaporated milk, cheese, butter and other uses by 13, 17, 53 and 23 percent, respectively, while decreasing the amounts of milk allocated to the production of fluid milk and ice cream by 31 and 32 percent, respectively.[4] This reallocation of quantity would cause large increases in the farm prices of milk used in fluid milk, evaporated milk and ice cream and large decreases in the prices of milk used in the other products. Butter and other uses are abundant products (farm prices equal zero). Butter was also abundant in the solutions to problem 2 for 1955 and 1960.

[4] Ladd and Hallberg (8) obtained similar results in a study of Chicago and Detroit markets. In an analysis in which the total quantity of milk was fixed, they found that returns to milk producers supplying these markets would be increased by charging a higher price for class I milk and reducing the price for class II milk.

Problems 3, 4, 5 and 6

Problems 1 and 2 called for changes in quantities and prices that would probably be unacceptable socially and politically. Problem 3 (MCLC) on the other hand, sets an acceptable minimum level for quantities, which in turn leads to prices much less undesirable from the consumer standpoint than the prices of problems 1 and 2. In evidence of this: for fluid milk, the solution farm price is $6.99 per hundredweight in problem 3 compared with $21.54 per

Table 7. Farm unit quantities in million hundredweight for 1964: actual and solution values.[a]

Type	Fluid	Evap.[b]	Cheese	I.C.[c]	Butter	Other	X-E[d]	X-C[e]	X-B[f]	Total
Actual	592.00	47.83	177.00	102.44	307.64	42.08	3.27	-6.70	.42	1,265.98
Problem 1	331.04	46.96	132.79	54.87	235.39	25.95	-15.71	-22.70	-0.89	787.70
Problem 2	409.43	54.23	207.87	69.15	469.21	51.87	-2.75	3.85	3.20	1,266.06
Problem 3	558.31	41.64	130.78	89.73	267.81	25.95	-5.37	-23.00	-0.29	1,085.56
Problem 4	558.24	41.63	163.33	89.74	390.35	37.51	-5.20	-11.55	1.84	1,265.89
Problem 5	563.36	44.11	165.00	89.73	268.17	25.95	-2.49	-11.00	-0.28	1,142.55
Problem 6	558.24	44.25	164.90	89.73	383.42	37.01	-2.61	-11.00	1.72	1,265.66
Problem 7	589.68	48.75	188.44	100.00	302.09	44.41	4.04	-2.70	0.32	1,275.03
Problem 8	590.53	48.63	187.30	100.15	294.50	43.96	3.97	-3.10	0.19	1,266.08
Problem 9	581.52	48.72	186.63	98.59	300.11	43.80	3.44	-3.35	0.28	1,259.74
Problem 10	581.18	48.76	187.34	98.25	305.63	44.06	3.47	-3.10	0.38	1,265.97
Problem 11	573.38	48.66	184.96	97.16	298.14	43.25	2.82	-3.95	0.25	1,244.67
Problem 12	571.49	48.94	187.52	96.61	316.76	44.21	2.99	-3.05	0.57	1,266.04
Problem 13	565.23	48.60	183.15	95.75	296.16	42.64	2.19	-4.60	0.21	1,229.33
Problem 14	561.62	49.18	187.85	95.16	327.20	44.41	2.55	-2.95	0.75	1,265.77
Problem 15	584.75	48.73	187.33	99.15	300.77	44.06	3.68	-3.10	0.30	1,265.67
Problem 16	362.24	47.17	139.61	60.31	243.31	28.18	-13.33	-20.25	-0.75	846.49

[a] Since inventory variation is ignored, entries in the first six columns are quantities (measured in farm equivalent units) consumed by domestic consumers.
[b] Evaporated.
[c] Ice cream.
[d] Net export of evaporated milk.
[e] Net export of cheese.
[f] Net export of butter.

Table 8. Cash receipts in millions of dollars for 1964: actual and solution values.

Type	Fluid	Evap.[a]	Cheese	I.C.[b]	Butter	Other	X-E[c]	X-C[d]	X-B[e]	Total
Actual	2717.28	145.40	493.83	311.42	713.72	81.21	9.94	-18.69	0.97	4,455.08
Problem 1	7130.59	514.21	795.41	1110.57	802.68	132.60	-172.02	-135.97	-3.0	10,175.02
Problem 2	6624.58	300.98	141.35	1035.18	0.0	0.0	-15.26	2.62	0.0	8,089.42
Problem 3	3902.59	276.49	791.22	686.43	779.33	132.60	-35.66	-139.15	-0.84	6,393.00
Problem 4	3879.77	273.51	614.12	681.13	441.09	106.15	-34.16	-43.43	2.08	5,920.25
Problem 5	3701.27	239.96	602.25	682.85	777.69	132.60	-13.55	-40.15	-0.81	6,082.11
Problem 6	3840.69	242.93	601.88	681.05	471.61	108.44	-14.33	-40.15	2.12	5,894.22
Problem 7	2777.39	132.60	375.00	390.00	727.02	65.28	10.99	-5.37	0.77	4,471.66
Problem 8	2751.87	133.73	387.71	385.58	739.19	68.50	10.92	-6.42	0.48	4,471.55
Problem 9	3047.16	144.70	395.66	434.78	729.27	69.64	10.22	-7.10	0.68	4,824.99
Problem 10	3057.01	144.33	387.79	445.07	718.23	67.85	10.27	-6.42	0.89	4,825.02
Problem 11	3308.40	157.17	414.31	479.00	733.42	73.52	9.11	-8.85	0.61	5,166.69
Problem 12	3360.36	154.65	386.29	494.64	693.70	66.76	9.45	-6.28	1.25	5,160.80
Problem 13	3560.95	169.61	434.07	520.88	737.44	77.60	7.64	-10.90	0.52	5,497.80
Problem 14	3656.14	164.26	383.21	536.70	667.49	65.28	8.52	-6.02	1.53	5,477.11
Problem 15	2941.29	139.86	387.77	417.32	727.86	67.85	10.56	-6.42	0.73	4,684.92
Problem 16	7067.30	469.81	767.85	1101.86	800.49	131.60	-132.77	-111.37	-2.4	10,092.29

[a] Evaporated.
[b] Ice cream.
[c] Net export of evaporated milk.
[d] Net export of cheese.
[e] Net export of butter.

hundredweight in problem 1; and for ice cream, farm price is $7.65 per hundredweight compared with $20.24 per hundredweight in problem 1.

The solution quantities of fluid milk, evaporated milk, ice cream, butter and other uses are equal to their minimum consumption levels, whereas the solution quantity of cheese is greater than its minimum level. The constraint on cheese was trivial for this problem and, therefore, did not affect the value of the objective function. The total quantity of milk used in the solution was 14 percent below total actual use in 1964.

The additional milk made available in problem 4 by adding the total quantity constraint to problem 3 caused a decrease of $473 million in total cash receipts for 1964. The additional milk was allocated to the production of butter, cheese and other uses, 68 percent of it being used for butter production.

Setting a lower limit on cheese production (in problem 5) led to a decrease of $311 million in total cash receipts from problem 3 to problem 5. Involved were decreases in cash receipts for fluid milk, evaporated milk and cheese of 5, 13 and 24 percent, respectively. The change in cash receipts for ice cream, butter and other uses was negligible. Solution farm prices of fluid milk, evaporated milk and cheese in problem 5 were appreciably different from the prices of problem 3, the cheese price having the largest difference (a 40-percent decrease). The total quantity of milk in problem 5 was 5 percent greater than the total quantity of milk in problem 3. Quantities used in ice cream, butter and other uses equaled constraint quantities; quantities used in fluid milk, evaporated milk and cheese slightly exceeded constraint quantities. The results of problem 6 (ChPE + TQC) were very similar to those of problem 4.

Problems 7 through 16

The solution to problem 7 (Pr, I = 0) did not result in any appreciable increase in total cash receipts of dairy farmers. The increase in those receipts was only $17 million. The addition of the total quantity constraint to problem 7 to obtain problem 8 did not produce any significant changes in the solutions.

Setting the price constraint index at 1 in problem 9 (recall that this represents a 10-percent increase in the relative farm price level) resulted in an increase of 8 percent in total cash receipts over actual 1964 cash receipts. Cash receipts for cheese exports increased, but cash receipts for exports of evaporated milk and butter stayed about the same. Cash receipts for fluid milk and ice cream increased 12 and 40 percent, respectively; cash receipts for evaporated milk and butter changed only slightly; and cash receipts for cheese and other uses decreased 20 and 14 percent, respectively. Also, significant changes in farm prices occurred for fluid milk (14 percent increase), ice cream (45 percent increase), butter (18 percent decrease) and cheese (25 percent decrease). Total quantity of milk used was about the same as was actually available in 1964. As in problem 7 (Pr, I = 0) and problem 8 (Pr, I = 0 + TQC), results for problem 10 (Pr, I = 1 + TQC) were very similar to those of problem 9 (Pr, I = 1). A trend has begun to develop, and it continues in the solutions to problems 11 (Pr, I = 2) and 13 (Pr, I = 3).

In problem 13, there is a 23-percent increase in total cash receipts over actual 1964 cash receipts. As in problem 11, cash receipts for milk used in fluid milk and ice cream increased substantially in problem 13, but cash receipts for milk used in cheese decreased. Problem 13 also shows the cumulative effect of having continually raised the price index from I = 0 to I = 3. All farm prices rise smoothly as the index is increased without the total quantity constraint, but the solution farm prices for cheese and other uses in problem 13 are still 12 and 6 percent less than the actual farm prices for 1964. In problem 13, farm prices for milk used in fluid, evaporated, ice cream and butter have increased 37, 15, 79 and 7 percent, respectively, over actual prices. Total quantity used in the solution was only 3 percent below actual 1964 utilization, with a 5-percent decrease in the quantity of fluid milk, a 6-percent decrease in the quantity of ice cream, a 4-percent decrease in butter and a negligible change for evaporated milk and other uses. Net export quantities were about the same as actual quantities.

Problems 8, 10, 12 and 14 show that adding the total quantity constraint to the price constraint problem forces an increase in the prices of fluid, evaporated and ice cream and a decrease in the quantities of these products. Conversely, adding the total quantity constraint forces a decrease in the prices of cheese, butter and other uses and an increase in the quantities of these products. In addition, as the price index is increased in these problems, prices of fluid, evaporated and ice cream are increased and those of cheese, butter and other uses decreased.

The solution to problem 15 (Pr, I = 0.6) shows that, if a weighted average of the relative farm prices increased at the same rate as did the weighted average of all consumer prices, then cash receipts of milk producers would have increased an average of 5 percent between 1951 and 1955, between 1955 and 1960, and again between 1960 and 1964. Problem 16 was designed to approximate, by using the price constraint, the 4-year average of the solutions to problem 1. Problem 1 implies price constraint index of 31.9.

The National Farmers' Organization voted in December 1966, in favor of a holding action to obtain a 2-cents-per-quart ($0.93 per cwt.) increase in the farm price of fluid milk. At the same time, the National Milk Producer's Federation was lobbying Congress to establish a 1-cent-per-quart ($0.47 per cwt.)

raise in the farm price of fluid milk. An increase of 93 cents per cwt. in 1964 would have resulted in an increase in total cash receipts of \$551 million (a 12-percent increase) and a 20-percent increase in cash receipts for fluid milk.

The 1964 solutions to problem 15 would give an increase in total cash receipts approximately equal to the target level of the National Milk Producer's Federation. The increase in total cash receipts, which would result from the fulfillment of the NFO's plan, could be achieved by adopting as instrument variables quantities about midway between the solution quantities of problems 9 and 11; i.e., solutions to a problem with price constraint index about equal to 1.5.

Sensitivity Analysis

In the quadratic programming problem:

maximize $R = c'p + p'Dp$ subject to $Ap = b$ and to $p \geqq 0$ the elements of c, D, A and b are usually not known exactly. It is useful to determine, therefore, how the solution changes with a change in the values of these elements. Such an investigation is termed a sensitivity analysis or a check on the robustness of the solutions.

Detailed expositions of the theory of sensitivity analysis, such as those of Boot (1, 2) and Theil (11), are available for the interested reader. However, given that D is negative semidefinite, the more general results of this theory are

$\partial R/\partial c_i \geqq 0$, $\partial R/\partial d_{ij} \leqslant 0$, $\partial R/\partial a_{ij} \leqslant 0$ and $\partial R/\partial b_i \geqq 0$.

The last two results apply only when the problem contains nonnegativity restrictions and all constraints are inequalities. Note the offsetting effects that certain changes in the elements will have. For instance, changing the elements of c and D in the same direction will have offsetting effects on the solution.

The sensitivity analysis performed in this study involved a study of problem 2 in hopes of finding solution values free of abundancy; i.e., of zero prices. Next, the minimum consumption levels of problems 3 and 4 were systematically varied. Lastly, by using the Monte Carlo approach, a sensitivity analysis was carried out on problems 1 and 9. Problem 1 was chosen for analysis because it is one of the two basic problems studied, and problem 9 was chosen because it is fairly representative of the many problems solved with the price constraint added.

The analysis of problem 2 consisted of experimenting with different combinations of farm-demand slope coefficients for butter and other uses. The slope coefficient for butter was allowed to vary from -100.0 to -40.0 in steps of 20.0, and that of other uses was allowed to vary from -6.5 to -3.5 in steps of 1.0. All 16 possible combinations of these ranges and steps were introduced into problem 2. Most alternative farm-demand slope coefficient combinations attempted (because of the way in which the model is set up, a change in the slope coefficient of the farm-demand equation changes the constant term of the equation) left butter and other uses abundant for each year and resulted in cheese becoming abundant. In addition, attempts at fixing the prices of abundant products at some value greater than zero were futile because fixing the price of an abundant product at an arbitrary positive level caused previously unabundant products to become abundant. Such price fixing can be done only at cost to the value of the objective function. Abundancy in 1955, 1960 and 1964 and no abundancy in 1951 was partly because the total quantity of milk available for the year 1951 was about 90 million cwt. less than the average available for the other three years.

Since the final form of the minimum-consumption level constraint is a set of weak inequalities, the previously cited theorem

$\partial R/\partial b_i \geqq 0$

is applicable. Lowering the minimum-consumption levels for the products corresponds to increasing the right-hand sides of the inequality constraints, and vice versa. Therefore, lowering the minimum consumption levels increases the value of the objective function, and vice versa.

Four different variations of the minimum-consumption levels for each product were used in the analysis: a 5-percent increase, a 5-percent decrease, a 15-percent decrease, and a 25-percent decrease. For a 5-percent increase in levels, there was a 16-percent average decrease in cash receipts for problem 3 and a 15-percent decrease for problem 4. Prices of all products in problem 3 were greater than their actual prices, and prices in problem 4 were greater except for butter, whose price was less than the actual price. For a 5-percent decrease in minimum-consumption, there was an increase in cash receipts for problems 3 and 4. As the minimum-consumption levels were further decreased, the solutions approached those arrived at in problems 1 and 2. With a 25-percent decrease in minimum levels, cash receipts rose 33 percent and 27 percent, respectively, for problems 3 and 4, and butter became abundant in problem 4. In summary, cash receipts for problems 3 and 4 are fairly sensitive to variations in the minimum-consumption.

The Monte Carlo analysis of problems 1 and 9 consisted of solving these problems with different sets of slope coefficients. The own price slope coefficients were selected from a range of coefficients derived from all available elasticity studies, except for the coefficients for other uses of milk and exports of milk products. The ranges for these were set up by allowing a 30-percent deviation from the slopes used previously in this study. Table 9 gives the ranges arrived at for each product. Slopes used in the original problems are also given. Since the values of cross price slope coefficients are also uncertain, variations in these were added. These variations were made up of a

20-percent decrease in all cross-elasticity slope coefficients, a 10-percent decrease in all these slope coefficients, no change in them and 10- and 20-percent increases for all cross-price slope coefficients.

Of all possible different sets of slope coefficients, 20 were selected at random, and problems 1 and 9 were solved with each of the 20 sets.

The ranges (R), means ($\bar{x}$), standard deviations (s) and t-values for Monte Carlo results on prices and total cash receipts are shown in table 10.[5] The t-values are defined separately for each problem as

$$t_i = (\bar{x}_i - p_i)/(s_i/20^{1/2})$$

where the p_i are the solution prices to problems 1 and 9 listed in table 5, x_i is the mean of the prices of the i-th product in the 20 solutions and s_i is the standard deviation of these 20 prices. The t-values show that none of the mean prices differs greatly from the original solution prices when the differences are considered in relation to the standard deviations of these mean prices. In essence, the solutions are robust with respect to the uncertainty of the values of the elasticities.

For all problems solved in this study, a computation of row errors was made. That is, the computed values for the row restraints were compared with the original restraint values. This check showed that accuracy was good to the second or third decimal place for all problems. However, had accuracy for a particular problem been at an undesirable level, a forced inversion could have been executed. This forced inversion reduces inaccuracies associated with large problems and many iterations. On the contrary, the problems in this study were small, and all required less than 10 iterations.

[5] More detail on the sensitivity analysis is contained in Updegraff (15).

COMPENSATING VARIATION RESULTS

In the section on "Compensating Variation in Consumer Income" two sets of substitution terms were discussed. One set, whose elements are indicated by ES_{ij}, was computed by using actual 1964 retail prices (first line of table 6) and quantities (first line of table 7). These values are presented in table 11. Another set, whose elements are indicated by ET_{ij}, was computed by using an average of actual 1964 prices and solution prices for problem 1 and an average of 1964 actual quantities and solution quantities for problem 1. These values are shown in table 12.

Economic theory of consumer behavior, see Hicks (6), for example, contains several hypotheses concerning substitution terms. The estimated terms in tables 11 and 12 are consistent with some of these and inconsistent with others: (a) One hypothesis is $S_{ii} < 0$ and is confirmed since every ES_{ii} and ET_{ii} is negative. (b) A second hypothesis is the symmetry relation, $S_{ij} = S_{ji}$. But, this is not confirmed since for every i and j $ES_{ij} \neq ES_{ji}$ and $ET_{ij} \neq ET_{ji}$. One possible explanation for the discrepancy (assuming the hypothesis true) is that statistical errors account for the discrepancy and that ES_{ij} and ES_{ji} (or

Table 11. Substitution terms evaluated at actual 1964 prices and quantities: ES_{ij}.

i \ j	1	2	3	4	5
1	-0.088573	0.005419	0.001360	0.000749	0.001112
2	0.004716	-0.010038	0.000133	0.000081	0.000185
3	0.001082	0.000108	-0.143138	0.000431	0.000956
4	0.000483	0.000049	0.000364	-0.030278	0.000433
5	0.001346	0.000137	0.001001	0.000570	-0.427283

Table 12. Substitution terms evaluated at means of 1964 actual values and problem 1 solution values: ET_{ij}.

i \ j	1	2	3	4	5
1	-0.052891	0.003420	0.000922	0.000460	0.000806
2	0.003516	-0.008268	0.000122	0.000062	0.000200
3	0.000755	0.000094	-0.107640	0.000292	0.000804
4	0.000302	0.000036	0.000245	-0.018234	0.000310
5	0.000915	0.000114	0.000752	0.000376	-0.348574

Table 9. Alternative slope ranges.

Product	Low	High	Original
Fluid	-22.0	-10.0	-15.99
Evaporated	-6.7	-1.7	-2.496
Cheese	-18.0	-10.0	-14.27
Ice Cream	-4.0	-2.0	-2.808
Butter	-100.0	-40.0	-68.99
Other	-6.5	-3.5	-5.073
X-Evaporated	-2.1	-1.7	-2.4
X-Cheese	-6.5	-3.5	-5.0
X-Butter	-1.6	-.8	-1.2

Table 10. Sensitivity analysis price and total cash receipts results, in dollars per hundredweight, for Monte Carlo approach.

Statistic	Fluid	Evaporated	Cheese	Ice cream	Butter	Other	Total cash receipts
Problem 1							
1964							
R	16.04-34.40	6.50-19.10	5.25-7.35	14.57-28.40	2.89-4.97	4.47-6.98	8,465.20-14,686.96
$\bar{x}$	22.10	9.71	6.07	20.72	3.64	5.27	10,382.94
s	6.11	3.24	0.68	4.58	0.60	0.75	1,890.79
t	0.41	-1.71	0.53	0.47	1.70	0.93	0.49
Problem 9							
1964							
R	4.77-5.68	2.27-3.53	1.84-2.55	3.01-6.23	2.12-3.35	1.38-2.01	4,798.60-4.870.21
$\bar{x}$	5.19	2.71	2.14	4.53	2.57	1.63	4,833.50
s	0.21	0.31	0.16	0.84	0.34	0.20	20.24
t	-1.04	-3.67	0.59	0.63	1.77	0.95	2.04

ET_{ij} and ET_{ji}) are not significantly different. Errors of aggregation provide a second possible explanation. The symmetry relation is derived for an individual consumer. ES_{ij} and ET_{ij} are estimated average substitution terms derived from aggregate demand functions. Errors of aggregation may cause asymmetry even though the symmetry relation holds for each individual. (c) A third hypothesis is that

$$\sum_{j=1}^{m} \sum_{i=1}^{m} S_{ij} h_i h_j < 0$$

for all nonzero h_i and h_j for all values of m up to n, where n is the total number of goods and services. We did not perform a general test of this relation. But for every problem t, $\sum_{j=1}^{5} \sum_{i=1}^{5} ES_{ij}(p_{it} - p_{ia})(p_{jt} - p_{ja})$ was negative and $\sum_{j=1}^{5} \sum_{i=1}^{5} ET_{ij}(p_{it} - p_{ia})(p_{jt} - p_{ja})$ was also negative.

Measures of per-capita compensating variation are presented for each problem in table 13. There is a positive correlation between the size of per-capita compensating variation for a problem and the increase in cash receipts for that problem. The solutions to problems 1 (NoC) and 16 (Pr, I = 28) call for 128 percent increases in cash receipts, and per-capita compensating variation for these problems is in the range of \$43 to \$55. The solutions to problems 3 (MCLC), 4 (MCLC + TQC), 5 (ChPE) and 6 (ChPE + TQC) call for 33- to 43-percent increases in cash receipts; per-capita compensating variation is in the range of \$4 to \$10. The correlation is not perfect; the solution to problem 3 increases cash receipts by more than the solution to problem 4 increases cash receipts, but the per-capita compensating variation is more for problem 4 than for problem 3. Figures in this table are interpreted as follows, with TCV_3 for an example: The solution to problem 3 called for higher retail prices for each dairy product and a 43-percent increase in cash receipts. If retail prices rose by the amounts indicated for problem 3 in table 6 and the average consumer adjusted to the new level of higher dairy products prices, after his adjustment it would require a \$5.77 increase in his income to return to the same level of utility as he was before the price increase. This \$5.77 compares with the \$12.61 by which his income would have to increase to permit him to purchase the same collection of goods after the price increase as he purchased before the price increase. The per-capita compensating variation is an indicator of the reduction in consumer welfare occasioned by price increases, *ceteris paribus*. Specifically, it is computed on the assumption that prices not considered in the study are constant or that $S_{ij} = 0$ for each product i included and each product j excluded.

Table 13. Measures of compensating variation in per-capita consumer income.

Problem t	SCV_t (\$)	TCV_t (\$)
1	54.91	47.55
2	40.64	31.10
3	4.25	5.77
4	9.58	7.69
5	9.43	7.83
6	9.10	7.21
7	-1.98	-1.70
8	-1.08	-1.02
9	0.99	0.66
10	1.20	0.76
11	2.86	2.17
12	3.30	2.40
13	6.12	4.68
14	5.89	4.37
15	0.45	0.15
16	50.28	42.99

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