Experiments With

Autoregressive Error Estimation

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SUMMARY

Autocorrelated errors are recognized as potentially troublesome in regression analysis. Because of the computational problems encountered, however, few economists have estimated equations under the assumption of autocorrelated errors. Recently, relatively economical procedures have been developed for estimating equations containing autocorrelated errors. In this study, one of these procedures—autoregressive least squares (A.L.S.) —is applied to equations describing the behavior of various economic agents, by using different unit observation periods—year, quarter and month. Some of the results have been published elsewhere; some are published here. In addition to presenting some results of autoregressive error estimation, this report summarizes experience with the use of A.L.S. Some equations presented here were estimated by a simultaneous equations method under the assumption of autocorrelated errors.

The results of four different tests for autocorrelation in errors were compared: Durbin-Watson d statistic, Theil-Nagar d, Hart-von Neumann ratio and A.L.S. Essentially, the Theil-Nagar d test classes as significant those values of d that are significant or inconclusive in the Durbin-Watson test. The Theil-Nagar d yielded evidence of autocorrelated errors most frequently; A.L.S., second most frequently; Hart-von Neumann ratio, third most frequently; and Durbin-Watson test, least frequently. The proportions of the equations in which each test provided significant evidence of autocorrelated errors are: Theil-Nagar d, 66 percent; autoregressive least squares, 51 percent; Hart-von Neumann ratio, 37 percent; Durbin-Watson test, 26 percent.

Each test provided evidence of significant autocorrelation more frequently in equations not containing the lagged dependent variable, y_{t-1}, than in equations containing the lagged dependent variable. In equations not containing y_{t-1}, the Theil-Nagar d appears to be a reasonably efficient test, with the disadvantage, however, of fairly frequent Type-I error. In equations containing y_{t-1}, none of the three tests using residuals (estimated errors) to test for autocorrelation seems satisfactory. Theil-Nagar d appears to make frequent Type-I errors and also frequent Type-II errors. The other two make frequent Type-II errors.

There appears to be no good way to use residuals to compute the autoregressive properties of errors. Autoregressive coefficients computed from residuals appear inefficient and biased toward zero.

When using L.S. or some simultaneous equations procedure and finding a significant (or inconclusive) value of d, econometricians commonly conclude that caution is necessary in interpreting the results from that equation. The results of this study indicate that this is insufficient. We do not know what bias or inefficiency exists in the coefficients or in the tests of significance. Re-estimation by a procedure that allows for temporal dependence in the disturbances will, in many cases, make substantial differences in the coefficients and in their levels of significance.

In equations in which A.L.S. produced significant evidence of autocorrelated errors, one-fourth of the A.L.S. coefficients differed from the corresponding least squares (L.S.) coefficients by two or more L.S. standard errors; one-fourth differed by one to two L.S. standard errors; half differed by less than one L.S. standard error. In these same equations, one-fourth of the estimated coefficients were significant by one method of estimation and nonsignificant by the other.

It is known that, under certain assumptions, L.S. coefficients are unbiased even through errors are autocorrelated. The empirical A.L.S. results raise a question as to whether the necessary assumptions are generally satisfied. The results suggest the possibility that autocorrelated errors are not distributed independently of the independent variables. Because of the intercorrelations among time series, autocorrelated errors arising from the omission of relevant variables are likely correlated with included independent variables. Autocorrelated errors arising from incorrect specification of the functional form may be similarly correlated with included independent variables.

Omission of relevant variables is recognized as a possible source of autocorrelated errors. Under certain conditions, the addition of a variable can introduce autocorrelation into the errors. The addition of y_{t-1} introduces autocorrelation into the errors fairly regularly; the addition of other variables has this effect infrequently. The coefficient of y_{t-1} is highly sensitive to the presence of autocorrelated errors.

Several equations which had been estimated by assuming first-order autoregressive errors were re-estimated by assuming second-order autoregressive errors. In half, there was significant evidence of second-order autoregression. Differences between results obtained by assuming second-order autoregression and those obtained by assuming first-order autoregression were much smaller than the differences between results obtained by assuming first-order autoregression and those obtained by L.S.

In a nonlinear regression problem such as that created by the presence of autoregressive errors, there exists the possibility of multiple minima in the residual sum of squares (multiple maxima in the likelihood function). Twenty-one different equations were investigated for the existence of multiple minima: multiple minima were found in four, all containing y_{t-1}. Multiple minima are rare in equations that do *not* contain y_{t-1}, but not so rare in equations that do contain y_{t-1}. Here is evidence of another kind of interaction between autocorrelated errors and y_{t-1}.

Experiments With Autoregressive Error Estimation[1]

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When estimating behavioral equations or production functions from time series data, economists usually use an estimation procedure that assumes the errors to be temporally independent. Work of Orcutt (36) and Cochrane and Orcutt (6) suggested that this assumption frequently is not satisfied. Recent empirical work by Hildreth and Lu (23) provides evidence that autocorrelated errors may be common. Even though autocorrelated errors are common, econometricians need not be concerned about them unless their presence seriously affects the statistical results. If the lagged value of the dependent variable is not among the independent variables, the presence of autocorrelated errors does not bias least-squares estimates of the coefficients (19; 59, p. 211), although it does make least-squares coefficients inefficient (19, 57, 58), and it does lead to biased (57) but consistent (59, pp. 211-212) estimates of the error variance and standard errors.[2] If the lagged value of the dependent variable is among the independent variables, autocorrelated errors bias the least-squares estimates of the coefficients (14, 17).

Little work has been done to analyze the effects of autocorrelated errors on simultaneous equations estimates. The results describing the undesirable effects of autocorrelated errors on least-squares estimates are asymptotically applicable to two-stage least squares. Examination of the work of Sargan (38) indicates that autocorrelated errors will bias limited-information single-equation estimates through the effect of autocorrelation on the two residual sums of squares whose ratio is minimized.

Granted that autocorrelated errors do exist and do have undesirable effects, two questions remain: (a) How common are autocorrelated errors? (b) Is the magnitude of the undesirable effects generally negligible or sizable? Each of these questions, in turn, gives rise to several subsidiary questions. Are autocorrelated errors common with certain kinds of equations or certain types of data and uncommon with other equations or data? Are the undesirable effects greater with some kinds of data than with other data? These questions are important because of the computational problems and expense involved in applying estimation procedures that allow for temporal dependence in the errors. If autocorrelated errors are relatively rare, or if their impact is numerically small, it will usually not be worthwhile to assume temporal dependence in the errors and to estimate the equations accordingly.

The research reported here was carried out to provide some evidence on the frequency of autocorrelated errors in various kinds of economic behavioral equations estimated with different unit observation periods—year, quarter and month—and to obtain measures of the magnitude of the effects of autocorrelated errors. Results were obtained by a relatively economical estimation procedure which assumes autocorrelation in the errors.[3]

To accomplish the listed objectives, the first step required was the development of an economical estimation procedure. Such a procedure—autoregressive least squares—was developed by Fuller and Martin (see next section). This procedure was applied to a number of equations and various types of data. (a) Annual aggregate consumer demands for several groups of food items in the United Kingdom were estimated and published (15). All other equations were estimated with United States data. (b) Annual aggregate consumer demands for several foods were analyzed. Some results are published in this report. (c) Annual demands for automobiles and housing were analyzed. The results are published here. (d) Quarterly and monthly consumer demand was studied. Quarterly and monthly estimates of food demands of the Michigan State University consumer panel were analyzed (29, 30). Aggregate quarterly beef and pork demand and quarterly consumer's expenditures on durable goods, nondurable goods and services were studied. The beef and pork results were published previously (14). (e) Equations describing annual aggregate factor demands and product supplies by farmers were estimated. Some results are reported here. (f) Business behavior was studied with the use of quarterly aggregate data. Beef and pork inventories and prices were studied and reported (14). Business plant and equipment expenditures and nonfarm inventory investment components of gross national product and department store inventories were analyzed.

The general procedure was to estimate an equation by least squares and compute the d statistic. Then auto-

[2] Summaries of the effect of autocorrelated errors can be found in (12) and (24).

[3] A temporally dependent error is an error in which each observation is correlated with previous values of itself, with errors in other equations or with both. An autocorrelated error is correlated with previous values of itself.

regressive least-squares estimates were obtained. Various comparisons were made between the least-squares and the autoregressive least-squares results. This was the general, not the universal, procedure; some equations were estimated only one way.

Since economic data are generated by a dynamic economic system, it is possible that static equilibrium theory is not adequate for explaining observed behavior. To investigate this possibility, both static and dynamic versions of a number of equations were estimated. Here, a static equation is one in which all variables refer to the same time period; one or more lagged variables appear in a dynamic equation. The most commonly used dynamic equation was the type of equation proposed by Koyck and Nerlove for the estimation of long-run elasticities (26, 34). In some contexts, this equation can also be interpreted as representing behavior affected by expected price or expected income (34). In a number of cases, a static equation was estimated by least squares and autoregressive least squares; then, its dynamic correspondent also was estimated both ways. Results from these four regressions were then compared.

ESTIMATION PROCEDURES

Four different estimation procedures were used in this study. Each corresponds to a different set of assumptions about the error terms in the structural equations estimated. Detailed discussion of the procedures and their properties are not presented here because they are covered elsewhere (12, 24, 15, 16, 30).

Least Squares (L.S.)

The L.S. model may be written, using matrix notation, as

$(1.1) \quad Y_t = X_t A + \epsilon_t$

$(1.2) \quad E(\epsilon_t) = 0$

$(1.3) \quad E(\epsilon_t \epsilon_{t-i}) = 0, \quad \text{all } i \neq o$

$(1.4) \quad E(\epsilon_t^2) = \sigma^2 \qquad \text{for all } t$

$(1.5) \quad$ Elements of X_t distributed independently of ϵ_t.

Y is a column vector of N observations on the dependent variable; X is an $N \times (M + 1)$ vector of observations on m independent variables and a column of ones; A is an $(m + 1) \times 1$ column vector of coefficients; ϵ is an $N \times 1$ vector of errors.

Autoregressive Least Squares (A.L.S.)

As we are concerned with autocorrelated errors, assumption 1.3 is the part of the model of interest here. The simplest way to generalize 1.3 is to assume

$(1.6) \quad \epsilon_t = \beta_1 \epsilon_{t-1} + u_t, \quad -1 \leq \beta_1 \leq 1$

$(1.7) \quad E(u_t) = 0$

$(1.8) \quad E(u_t u_{t-j}) = 0 \quad \text{for } j \neq 0$

$(1.9) \quad E(u_t^2) = \sigma^2 \quad \text{for all } t$

$(1.10) \quad u_t$ distributed independently of X_t

$(1.11) \quad u_t$ distributed independently of $Y_{t-j}, j \geqq 1$.

A generalization of this is to replace equation 1.6 by 1.12 and 1.13.

$(1.12) \quad \epsilon_t = \beta_1 \epsilon_{t-1} + \beta_2 \epsilon_{t-2} + u_t$

$(1.13) \quad$ Roots of $x^2 = \beta_1 x + \beta_2$ less than unity in absolute value.

Write the t-th row of equation 1.1 as

$(1.14) \quad Y_t = X_t A + \epsilon_t$

Substituting equation 1.6 and equation 1.14 lagged one period into equation 1.14 we obtain

$(1.15) \quad Y_t = X_t A - X_{t-1} A \beta_1 + \beta_1 Y_{t-1} + u_t.$

Substituting equation 1.12 and equation 1.14 lagged one period and two periods into 1.14, we obtain

$(1.16) \quad Y_t = X_t A - X_{t-1} A \beta_1 - X_{t-2} A \beta_2 + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + u_t.$

Equations 1.15 and 1.16 illustrate why the assumption of autocorrelated errors creates serious computational problems: the equations are nonlinear in the parameters. First-order autoregressive least squares (A.L.S.-1) is an iterative procedure for obtaining simultaneous estimates of A and β_1 in equation 1.15. Second-order autoregressive least squares (A.L.S.-2) is an iterative procedure for obtaining simultaneous estimates of A and of β_1 and β_2 in equation 1.16. These procedures are discussed in detail in (15), (16) and (30). A brief exposition is contained in the Appendix of this report. Both procedures are special cases of modified Gauss-Newton nonlinear least squares (21).

A.L.S. is an iterative procedure which starts with an initial set of estimates of the parameters and proceeds to improve on these estimates. Usually the L.S. estimates were used as the initial set of estimates of A, and the initial estimate of β_1 was computed from d. Several equations were started with two different sets of initial estimates to see whether both would converge to the same final solution. The ones of these equations that are presented later are footnoted.

Hildreth and Lu (23) developed a method for obtaining maximum likelihood estimates of A and β_1. Ladd and Martin estimated some equations with this procedure and with A.L.S.-1. Estimates obtained from the two methods were virtually identical (29).

Other estimation procedures have been proposed by Theil and Nagar (41), Durbin (10) and Klein (25, pp. 85-89).

Two-Stage Least Squares (T.S.L.S.)

Various methods of estimation have been developed to apply to equations in which assumption 1.5 is not met because the equation under consideration is part of a system of simultaneous equations. One is the two-stage least-squares procedure. Some of the equations in this study were estimated by T.S.L.S.

Autoregressive Two-Stage Least Squares (A.T.S.)

This procedure is a synthesis of A.L.S. and T.S.L.S. appropriate for situations where the errors are believed to satisfy equations 1.6 to 1.11 or equations 1.7 to 1.10 and 1.12 and 1.13 and where the endogenous variables are generated by a system of simultaneous equations. A brief exposition is presented in the Appendix.

Sargan (38) has developed a procedure for estimating simultaneous equations having auto- and serial-correlated errors.

FOOD DEMAND

All tables of results in this report follow the same basic format. All equations are numbered. If the equation is copied from another study, the number is followed by the final initial of the original investigator. All equations containing the same observed independent variables have the same number. A.L.S. and A.T.S. estimates are denoted by a number followed by A.1 or A.2 to indicate first- or second-order autoregressive error assumption, respectively. Equations estimated by me by L.S. or T.S.L.S. are identified only by number. Some equations not shown in tables will be discussed. They will be numbered as though they were in the tables. For each equation, coefficients are presented. A single superscript asterisk, *, by a coefficient indicates significance of the coefficient at the 10-percent level; ** indicates significance at the 5-percent level (referred to in the text as significant); *** indicates significance at the 1-percent level (referred to in the text as highly significant); superscript s indicates that the coefficient exceeds its standard error in absolute value. For each equation estimated by A.L.S. or A.T.S., the number of iterations required for the solution is shown in the last column. The IBM program used to obtain these estimates does not compute the value of the intercept. Some intercepts were computed on a desk calculator. A blank indicates that the intercept was not computed.

The two-tailed Durbin-Watson d test was used in tests of autocorrelation in the errors of L.S. and T.S.L.S. equations. The results are presented in the columns labelled d and in footnotes

The sample period used is given in footnotes, using the time subscript on the dependent variable as reference. If, for example, 1921-41 (N = 21) is indicated as the sample period, this means that the first observation on the dependent variable was for 1921 and that the last was for 1941. Observations on some variables for 1919 or 1920 may have been used in the estimation process. Where I have refitted equations estimated by other investigators, my sample period is shorter than theirs because of the data requirements of the A.L.S. procedure.

For all foods discussed in this section, except oranges, the sample period was 1921-41, 1947-58 (N = 32). With the exception of oranges, the estimates were obtained by T.S.L.S. and A.T.S. Demand equations for oranges were estimated by L.S. and A.L.S.

The analyses presented here constitute an updating with revised data of T.S.L.S. analyses carried out for the sample period 1921-41, 1947-49 by Tedford (40).

Table 1 presents results of analyses of annual per-capita demand for beef, pork and lamb and mutton. The variables are:

C_{Bt} = per-capita beef consumption, pounds, carcass-weight equivalent (44, 45); 1934-36 data adjusted to exclude relief distribution (60, p. 91).

C_{Pt} = per-capita pork consumption, pounds, carcass-weight equivalent (44, 45); 1933-34 and 1939-41 data adjusted to exclude relief distribution (60, p. 91).

C_{Lt} = per-capita lamb and mutton consumption, pounds, carcass-weight equivalent (44, 45); 1935 figure adjusted to exclude relief distribution (60, p. 91).

P_{Bt} = deflated average retail price per pound, retail-weight equivalent, all grades of beef (50, p. 24). (The deflator used throughout this section was the Bureau of Labor Statistics consumer price index, 1947-49: 100.)

P_{Pt} = deflated average retail price per pound of pork (43, p. 272; 47).

Table 1. Selected statistical results from annual per-capita beef, pork and lamb and mutton demand analyses.

Equation number	Dependent variable	Coefficients											Number of iterations
		P_{Bt}	P_{Pt}	P_{Lt}	I_t	t	Lagged consumption	1	β_1	β_2	d	R^2	
1 ....C_{Bt}		—0.88***	0.26**	—0.10	0.058***	....	....	35.76	....	....	0.66[a]	0.905	....
1A.2[b] ....C_{Bt}		—1.02***	0.25[s]	—0.06	0.062***	....	....	42.74	0.50***	—0.31**	....	0.927	4
2 ....C_{Bt}		—1.09***	0.16[s]	0.46*	0.023**	0.59***	0.23**	33.00	....	....	1.35[c]	0.932	....
3A.2[b] ....C_{Bt}		—0.99***	0.22[s]	....	0.058***	....	0.06	39.96	0.47***	—0.30**	....	0.927	6
4 ....C_{Pt}		—0.04	—0.77***	0.90***	—0.006	....	....	63.34	....	....	0.76[a]	0.531	....
4A.1 ....C_{Pt}		0.10	—0.86***	0.28	0.008	....	....	67.53	0.86***	....	....	0.764	8
5 ....C_{Pt}		0.40*	—0.61***	0.09	0.020*	—0.59***	0.36***	37.58	....	....	1.48[c]	0.724	....
5A.1 ....C_{Pt}		0.21[s]	—0.85***	0.30	0.015[s]	—0.14[s]	0.14[s]	63.28	0.72***	....	....	0.797	8
7 ....C_{Lt}		0.067***	0.047***	—0.16***	—0.0019***	....	....	10.58	....	....	1.14[d]	0.902	....
7A.1 ....C_{Lt}		0.061***	0.036***	—0.15***	—0.0018***	....	....	9.25	0.31*	....	....	0.913	5
8 ....C_{Lt}		0.056***	0.039***	—0.14***	—0.0016***	....	0.17[s]	8.79	....	....	1.30[c]	0.908	....
8A.1 ....C_{Lt}		0.057***	0.031*	—0.17***	—0.0018**	....	—0.17	0.13	0.48***	....	....	0.915	10

[a] Significant at 2-, 5- and 10-percent levels.
[b] Estimated twice with two different sets of initial estimates. Both sets converged to the same final solution.
[c] Inconclusive at 2-, 5- and 10-percent levels.
[d] Inconclusive at 2- and 5-percent levels, significant at 10-percent level.

P_{Lt} = deflated average retail price per pound, choice grade lamb (43, p. 273; 47).

I_t = deflated disposable personal income per capita (45).

t = year minus 1920.

The d statistic in equation 1 and the estimates of β_1 and β_2 both indicate the presence of autocorrelated errors in the static beef demand equation. The d statistic in equation 4 and the estimate of β_1 in equation 4A.1 likewise agree. The d statistics in equations 2 and 5 and estimated β_1 in equations 2A.1 and 5A.1 are not in similar agreement. Equations 2 and 5 are the first of many examples to be presented of inconsistency between d and A.L.S. Inconsistency is more frequent in equations containing the lagged dependent variable.

Coefficients of lagged consumption and time were always significant or highly significant in T.S.L.S. estimates of beef and pork demand equations. They were always nonsignificant in A.T.S. equations. A.T.S equations always contained significant evidence of autocorrelated errors. F tests indicated that adding lagged consumption and time to A.L.S. beef and pork demand equations did not significantly increase the value of R^2. Equations 4A.1 and 5A.1 are an example. Equation 6A.2 (not shown) was obtained by deleting time and lagged consumption from equation 5A.1. The resulting estimate of β_2 was significant, although estimated β_2 was nonsignificant in 5A.2. Evidently there is a lag in annual beef demand and in annual pork demand. The lag in beef demand is explainable by the use of time and lagged consumption or by the use of a static equation and second-order autocorrelation in the errors. The lag in pork demand is also explainable by lagged consumption and time or by a static equation with second-order autocorrelation in the errors.

Estimated β_1 in equation 5A.1 is almost equal to the sum, estimated $(\beta_1 + \beta_2)$, in equation 5A.2. Estimated $(\beta_1 + \beta_2) = 0.73$. This near equality almost invariably holds in A.L.S.-1 and A.L.S.-2 equations.

The shift variable $D_t = 0$ for $1921 \leqq t \leqq 1941$ and $D_t = 1$ for $t \geqq 1947$ was added to a pork demand equation containing time to test for a sharp change in consumer demand for pork during World War II. The coefficient of D_t had a t ratio of only 0.04.

The evidence that beef and pork are competitive with lamb and mutton in lamb and mutton demand is much stronger than is the evidence that lamb and mutton are competitive with beef and pork in demand for the latter two foods. Although Tedford (40) obtained a highly significant coefficient of C_{Lt-1} for a sample

period ending with 1949, the coefficient of C_{Lt-1} was nonsignificant in every equation fitted to the longer sample period used in this study. The coefficients of time were also nonsignificant.

Estimated β_1 was significant at only the 10-percent level in equation 7A.1, and the coefficients were only slightly different from those in equation 7. Judging from equation 8A.1, the introduction of lagged consumption increased the autocorrelation in the errors. Some cases will be presented later in which estimated β_1 was nonsignificant in the static equation (equation not containing the lagged dependent variable—y_{t-1}) and was significant in the dynamic equation (equation containing y_{t-1}). One reason for the presence of autocorrelated errors is commonly agreed to be the omission of a relevant variable. It appears that the addition of a variable may also introduce autocorrelation.

Results of analyses of per-capita chicken demand are shown in table 2. The variables are:

C_{ft} = per-capita chicken consumption, pounds, ready-to-cook basis (44, 45).

P_{ft} = deflated average retail price per pound of chicken (3, 51).

P_{cFt} = deflated average retail price of canned fish: 1921-34, canned red salmon price divided by 1935-36 average price (53); 1935-58, pink canned salmon divided by 1935-36 average (53).

P_{Mt} = deflated average retail price per pound, retail-weight equivalent, for pork, all beef, veal, lamb and mutton, (50); 1919-20 values estimated from 1921-30 regression of P_{Mt} on average retail price per pound of all red meat in 1935-39 prices (p_{mt}) and 1919-20 values of p_{mt} (60, p. 93).

The addition of C_{ft-1} effectively eliminated the autocorrelation in the errors of equation 1. C_{ft-1} was nonsignificant in Tedford's analysis of a shorter period (40). A.T.S. estimation of equation 2 resulted in a nonsignificant estimate of β_1 and made the coefficients little different from their values in equation 2. A time trend variable was included in a few equations and was nonsignificant.

A large number of iterations was required for a stable solution in equation 1A.1. This is five-and-one-half times as many iterations as the mean number of iterations required for a stable solution in equations estimated to date with A.L.S. The large number of iterations required evidently results from the magnitude of β_1. This is one of only two equations estimated to date by A.L.S.-1 in which estimated β_1 exceeds unity in absolute value. A.L.S.-1 equations with large estimates of β_1 do not

Table 2. Selected statistical results from annual per-capita chicken demand analyses.

| Equation number | Coefficients | | | | | | β_1 | d | R^2 | Number of iterations |
	P_{ft}	P_{cFt}	P_{Mt}	I_t	C_{ft-1}	1				
1	−0.23***	0.026***	−0.063	0.0059***	....	19.03	....	0.92[a]	0.933	....
1A.1	−0.15*	−0.009[s]	0.050	0.0049[s]	....	16.72	1.06***	....	0.951	39
2	−0.14**	0.012*	−0.013	0.0022	0.60***	9.46	....	2.11[b]	0.959	....

[a] Significant at 2-, 5- and 10-percent levels.

[b] Nonsignificant at 2-, 5- and 10-percent levels.

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Table 3. Selected statistical results from annual per-capita evaporated and condensed milk demand and per-capita fluid milk demand analyses.

Equation number	Dependent variable	Coefficients							d	R^2	Number of iterations
		P_{vt}	P_{mt}	P_{kt}	I_t	Lagged consumption	1	β_1			
1	C_{vt}	−0.35[s]	1.98***	−0.05[s]	0.0036[s]	....	−20	....	0.71[a]	0.794	....
1A.1	C_{vt}	−0.17	0.17	0.03[s]	−0.0030[s]	....	15	0.91***	....	0.953	14
2	C_{vt}	0.08	0.46[s]	....	−0.0017***	0.90***	−7	....	2.16[b]	0.961	....
3	C_{mt}	3.34***	−1.43	....	0.033***	....	292	....	1.08[c]	0.680	....
3A.1	C_{mt}	1.49[s]	−3.72**	....	0.031***	....	352	0.52***	....	0.859	3
4	C_{mt}	1.29[s]	−0.91	....	0.016***	0.48***	161	....	2.06[b]	0.868	....
4A.1[d]	C_{mt}	1.30[s]	−0.97	....	0.016***	0.47***	169	0.02	....	0.868	4

[a] Significant at 2-, 5- and 10-percent levels.

[b] Nonsignificant at 2-, 5- and 10-percent levels.

[c] Inconclusive at 2-percent level, significant at 5- and 10-percent levels.

[d] Estimated twice with two different sets of initial estimates of the coefficients. Both sets converged to the same final solutions.

always require a much larger than average number of iterations. A.L.S.-1 equations that require an unusually large number of iterations frequently do have large absolute values of the estimate of β_1.

Results of analyses of per-capita demand for evaporated and condensed milk and fluid milk are presented in table 3. Variables are:

C_{vt} = per-capita consumption of evaporated and condensed milk, pounds (44, 45); 1935-40 data adjusted to exclude relief distribution (52).

P_{vt} = deflated average retail price per 14½ ounce can of evaporated milk in leading cities (42, 46).

P_{mt} = deflated average retail price per quart of fresh home delivered milk in leading cities (42, 46).

P_{Kt} = deflated average retail price per pound of coffee in leading cities: 1919-21 data from (37); 1922-56 data from (8); 1957-58 prices—average of reported prices of coffee in bags and in vacuum packs (53).

C_{mt} = pounds of milk consumed as fluid milk and cream per capita (44, 45); 1918-23 data estimated from post-1923 relation between this series and fresh whole milk consumption and cream consumption and 1918-23 values of these latter two variables (44).

In analyses using the sample period 1920-41, 1947-49, Tedford (40) found a significant negative coefficient of P_{Kt} in evaporated and condensed milk demand equations. In the analyses for the longer period ending with 1958, the coefficient was never significant.

C_{vt-1} was highly significant in every equation; estimated β_1 was highly significant in 1A.1 and nonsignificant in 2A.1. The use of C_{vt-1} eliminated the autocorrelation in the errors. The use of C_{mt-1} also eliminated the autocorrelation in the errors in the fluid milk demand equation. But, apparently, there is some interaction between C_{mt-1} and β_1. The addition of β_1 to an equation containing C_{mt-1} results in a nonsignificant estimate of β_1 and a nonsignificant increase in the R^2. The addition of C_{mt-1} to an equation containing β_1 does not significantly increase the value of R^2 but drops the estimate of β_1 to nonsignificance.

We have two examples in this table of the way in which the presence of autocorrelated errors can have a sizable impact on the estimated coefficients: equations 1 and 1A.1 and equations 3 and 3A.1. A.L.S. estimation reduced the coefficient of P_{mt} to nonsignificance in the static evaporated milk demand equations, while it raised the coefficient of P_{mt} to significance and reduced the coefficient of P_{vt} to nonsignificance in the static fluid milk demand equation.

We also have examples of interaction between lagged consumption and other coefficients. In equations 1 and 2, the addition of C_{vt-1} reduced the coefficient of P_{mt} to nonsignificance and raised the coefficient of I_t to significance. In equations 3 and 4 the addition of C_{mt-1} reduced the coefficient of P_{vt} to nonsignificance.

In equations fitted by T.S.L.S., time had a significant coefficient when C_{vt-1} was not in the equation and a nonsignificant coefficient when C_{vt-1} was in the equation. C_{vt-1} had a significant coefficient when both were included. The coefficient of time was nonsignificant in the fluid milk demand equations.

Under the federal government's low-cost milk program (1937-43), school lunch milk program (since 1940) and special milk program (since 1954), recipients obtain milk at special low prices. Per-capita consumption from these three sources (48) was included as an exogenous variable in some regressions to see if these programs have had a measurable effect on total consumption. Apparently they have not because the coefficient was negative and smaller than its standard error.

Results of analyses of per-capita cheese and egg demand are presented in table 4. The variables are:

C_{ct} = per-capita consumption of all cheese, pounds (44, 45).

C_{et} = per-capita egg consumption, number of eggs (44, 45).

P_{et} = deflated average retail price per dozen eggs as computed by the Agricultural Marketing Service (51).

P_{ct} = deflated average retail price per pound of cheese in leading cities (42, 46).

P_{MFPt} = deflated Bureau of Labor Statistics index of retail prices of meat, poultry and fish, 1947-49: 100 (53).

The d statistic and A.T.S. estimation agreed in in-

Table 4. Selected statistical results from annual per-capita cheese and egg demand analyses.

Equation number	Dependent variable	Coefficients					Lagged consumption					Number of iterations
		P_{et}	P_{ct}	P_{MFPt}	I_t	t		1	β_1	d	R^2	
1 ....C_{et}	−0.032**	−0.024[s]	0.040***	0.0029***	....	....	2.83	....	0.99[a]	0.956	....
1A.1 ....C_{et}	−0.038**	−0.015	0.038***	0.0027***	....	....	5.17	0.57***	....	0.970	5
2 ....C_{et}	0.018[s]	−0.003	−0.005	0.0003	0.056**	0.55**	0.86	....	2.32[b]	0.980	....
2A.1 ....C_{et}	0.023**	−0.004	−0.008[s]	−0.0005	0.060**	0.86***	−2.91	−0.42**	....	0.983	4
3 ....C_{et}	−0.46	3.01***	0.56[s]	0.047*	....	....	122	....	0.73[a]	0.856	....
3A.1 ....C_{et}	−1.34***	1.26*	0.56[s]	0.045[s]	....	....	333	0.96***	....	0.954	7
4 ....C_{et}	−1.22**	1.16*	1.15***	0.054***	−2.13***	0.56***	55	....	1.45[c]	0.969	....

[a] Significant at 2-, 5- and 10-percent levels.
[b] Nonsignificant at 2-percent level, inconclusive at 5- and 10-percent levels.
[c] Inconclusive at 2-, 5- and 10-percent levels.

dicating autocorrelated errors present in both static demand equations. In the dynamic equations, both d tests were inconclusive at the 5-percent level, whereas estimated β_1 was significant in equation 2A.1 but was nonsignificant in equation 4A.1. The addition of lagged consumption and time to the egg demand equation eliminated the autocorrelation in the errors. The addition of lagged consumption and time to the cheese demand equation did not eliminate the autocorrelation in the errors; it changed the estimate of β_1 from positive to negative and had an unfavorable effect on the other coefficients.

The addition of β_1 to equations 1 and 2 had a relatively small effect on the coefficients; its addition to equation 3 had a larger effect on the coefficients.

In several of the postwar years, there have been substantial amounts of cheese distributed from United States Department of Agriculture stocks through relief and welfare agencies. The variable G_{ct}, which equals per-capita consumption from United States Department of Agriculture stocks or bought wholly or partially with government funds (48), was included as an exogenous variable in some equations. Its coefficient was positive but nonsignificant. Evidently consumption from government stocks has not been large enough to significantly affect total cheese consumption.

Some results of analyses of lard and shortening demand are presented in table 5. The variables are.

C_{Lt} = per-capita lard consumption, pounds (44, 45).

C_{st} = per-capita shortening consumption, pounds (44, 45).

P_{st} = deflated average retail price per pound of shortening: 1921-52 Agricultural Marketing Service price in all communities (1); 1953-58, prices in leading cities (1, 49).

P_{Lt} = deflated average retail price per pound of lard in leading cities (1, 49).

C_{pt} = per-capita consumption of white and whole-wheat flour, corn flour and cornmeal, potatoes and sweetpotatoes (44, 45).

In equations 1 and 1A.1 and equations 3 and 3A.1, the results of the d statistic and A.T.S. estimation were in agreement at the 5-percent level of significance. C_{Lt-1} was significant in equations 2 and 2A.1. Estimated β_1 was smaller than its standard error in equation 2A.1. C_{st-1} was significant in equations 4 and 4A.1, and estimated β_1 was smaller than its standard error in equation 4A.1.

Potato and bakery products consumption has been undergoing a downward trend. In a previous study, C_{pt} was used successfully in explaining part of the downward trend in butter and margarine consumption (27). C_{pt} was included in these equations since potatoes and bakery products may be complementary with lard and shortening. The coefficients in the lard demand equations are consistent with this hypothesis; the coefficients in the shortening demand equations are not.

In these analyses, there were examples of: (a) omission of a relevant variable not causing autocorrelated errors, (b) omission of a relevant variable causing autocorrelated errors and (c) introduction of a nonsignificant variable introducing autocorrelation. The omission of C_{Lt-1} from equations 1 and 1A.1 did not introduce autocorrelation into the errors. The omission of C_{pt} from equation 5A.1 did introduce autocorrela-

Table 5. Selected statistical results from analyses of per-capita lard and shortening demand.

Equation number	Dependent variable	Coefficients				Lagged consumption					Number of iterations
		P_{st}	P_{Lt}	C_{pt}	I_t		1	β_1	d	R^2	
1 ....C_{Lt}	0.39***	−0.39***	0.052***	0.0062***	....	−15.0	....	1.75[a]	0.772	....
2 ....C_{Lt}	0.33***	−0.34***	0.044***	0.0054***	0.25**	−13.9	....	2.10[a]	0.802	....
3 ....C_{st}	−0.16*	0.26***	−0.050***	−0.0059**	....	31.2	....	1.00[b]	0.462	....
3A.1 ....C_{st}	−0.13*	0.20***	−0.005	0.0005	....	10.1	0.78***	....	0.696	6
4 ....C_{st}	−0.15**	0.21***	−0.029**	−0.0039*	0.60***	17.6	....	1.98[a]	0.700	....
5A.1 ....C_{st}	−0.14*	0,19***		0.0009	0.12	9.3	0.76***	....	0.700	4

[a] Nonsignificant at 2-, 5- and 10-percent levels.
[b] Significant at 2-, 5- and 10-percent levels.

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tion into the errors. It also reduced C_{st-1} to nonsignificance. An index of the prices of butter and margarine was added to equation 4A.1. Its coefficient was only one-fourth the size of its standard error. Estimated β_1, which was nonsignificant in equation 4A.1, became highly significant. The addition of the butter and margarine price index to equation 2A.1 had little effect on the other coefficients; its coefficient was nonsignificant.

Table 6 presents results of analyses of demand for oranges. All analyses used the data published by Nerlove and Waugh (35), who estimated the first equation in the table. The variables are:

V_t = log of per-capita farm value of sales of oranges deflated by consumer price index.
  = dependent variable
Y_t = log of per capita personal disposable income deflated by the consumer price index.
Q_t = log of per-capita marketings of oranges, boxes.
A_t = log a_t.

$$\overline{A}_t = \log \frac{1}{10} \sum_{i=1}^{10} a_{t-i}$$

a_t = per-capita advertising expenditures for oranges

by Sunkist Growers and the Florida Citrus Commission, deflated by consumer price index.

Although estimated β_1 is significant in equation 1A.1 and the coefficient of V_{t-1} is significant in equation 2, neither is significant in equation 2A.1. The sum of the two, however, is significant in 2A.1. There is a lag, but the data do not permit us to identify it as a lag in consumer adjustment or as autocorrelation in the errors. Equations 2A.1 and 3A.1 provide an example of a case in which the omission of a relevant variable introduces autocorrelation into the errors. An F test indicated that the elimination of A_t and $\overline{A}_t$ from equations 2 and 2A.1 did significantly reduce the value of R^2. Equations 3 and 3A.1 are an example of what Griliches (17) and Fuller and Ladd (14) discussed: The L.S. coefficient of the lagged dependent variable picked up the autocorrelation in the errors.

CONSUMER DURABLES DEMAND

Chow (5) and Muth (32) have published studies of demand for automobiles and nonfarm housing. I used their data, which they published, and re-estimated some of their equations. My L.S. results differ from theirs because I had to use a shorter sample period in order to apply A.L.S.

Four automobile demand functions obtained by

Table 6. Selected statistical results from annual orange demand analyses.

Equation number	Coefficients						β_1	d	R^2	Number of iterations
	Y_t	Q_t	A_t	$\overline{A}_t$	V_{t-1}	1				
1.N & W[a]	0.92***	−0.39**	0.23**	0.10**	....	−2.94	....	....	0.85	....
1	0.92***	−0.46**	0.21[s]	0.19***	....	−1.88	....	1.39[b]	0.720	....
1A.1	0.90***	−0.39*	0.18[s]	0.20**	....	−0.94	0.32**	....	0.751	4
2	0.73***	−0.43**	0.17[s]	0.16**	0.33**	−1.59	....	1.94[c]	0.757	....
2A.1	0.77***	−0.43**	0.17[s]	0.17*	0.26	−1.37	0.10	....	0.758	5
3	0.50**	0.04	....	....	0.48***	−1.00	....	1.92[c]	0.682	....
3A.1	1.35***	−0.04	....	....	−0.20[s]	−0.15	0.71***	....	0.712	4

[a] From: Marc Nerlove and Frederick V. Waugh. Advertising without supply control. Jour. Farm Econ. 43:813-837. 1961. Based on sample period of 1909-10 to 1940-41, 1946-47 to 1958-59. Other equations were estimated using 1911-12 to 1940-41, 1948-49 to 1958-59 data.

[b] Inconclusive at 2-, 5- and 10-percent levels.

[c] Nonsignificant at 2-, 5- and 10-percent levels.

Table 7. Selected statistical results from automobile demand analyses.

Equation number	Dependent variable	Coefficients					β_1	β_2	d	R^2	Number of iterations
		P_t	I_{dt}	I_{et}	X_{t-1}	1					
1.C[a]	X_t	−0.040***	0.021***	....	....	1.17	....	....	....	0.850	....
1	X_t	−0.030***	0.021***	....	....	0.22	....	....	0.73[b]	0.889	....
1A.1	X_t	−0.008[s]	0.018***	....	....	5.23	0.61***	....	....	0.969	8
2.C[a]	X_t	−0.049***	....	0.025***	....	−0.72	....	....	....	0.895	....
2	X_t	−0.055***	....	0.026***	....	0.43	....	....	1.96[c]	0.931	....
2A.2	X_t	−0.010[s]	....	0.020***	....	5.64	0.98***	−0.30**	....	0.973	4
3.C[a]	X_t^1	−0.020***	0.012***	....	−0.23***	0.08	....	....	....	0.858	....
3	X_t^1	−0.022***	0.012***	....	−0.22***	−0.02	....	....	1.40[d]	0.872	....
3A.1	X_t^1	−0.015***	0.014***	....	−0.39***	1.48	0.63***	....	....	0.884	12
4.C[a]	X_t^1	−0.026***	....	0.014***	−0.30***	0.40	....	....	....	0.628	....
4	X_t^1	−0.038***	....	0.017***	−0.35***	−1.00	....	....	1.44[d]	0.719	....
4A.1	X_t^1	−0.011[s]	....	0.019***	−0.62***	1.33	0.70***	....	....	0.754	7

[a] From: Gregory C. Chow. Statistical demand functions for automobiles and their use for forecasting. In: Arnold C. Harberger (ed.). The demand for durable goods. Univ. of Chicago Press, Chicago. 1960. Chapter IV. Sample period was 1921-53 excluding 1942-46. For other equations, the sample period was 1923-57, excluding 1942-48.

[b] Significant at 2-, 5- and 10-percent levels.

[c] Nonsignificant at 2-, 5- and 10-percent levels.

[d] Inconclusive at 2-, 5- and 10-percent levels.

Chow are presented in table 7. The variables are (5, pp. 156-157, 164):

X_t = per-capita stock of automobiles = weighted per-capita sum of registrations of passenger automobiles of various ages at end of year t, in hundredths of a unit.

X_t^1 = per-capita number of new automobiles purchased in year t, in hundredths.

P_t = price index of automobiles deflated by Gross National Product deflator and set at 100 in 1937.

I_{dt} = per-capita disposable personal income deflated by GNP deflator, 1937 = 1.00.

I_{et} = real expected per-capita income in 1937 dollars.

Equations 2.C and 4.C differ from 1.C and 3.C in the income variable used. Equations 1.C and 3.C use current income; 2.C and 4.C use expected income, Friedman's empirical approximation of permanent income (13). Expected income is defined as a weighted average of current and past real disposable incomes per capita.

The results in equations 1, 2, 3 and 4 agree with Chow's finding that expected income performs better than disposable income in explaining variations in stocks but disposable income performs better in explaining variations in purchases.

The value of R^2 for equation 1 is less than the value of R^2 for equation 2. The values of R^2 for equations 1A.1 and 2A.1 are nearly identical, as are the values of R^2 for equations 1A.2 and 2A.2. Estimated β_2 is nonsignificant in 1A.2. In every case, A.L.S. estimation yielded evidence of autocorrelation in the errors. It reduced the absolute size and level of significance of the coefficient of P_t in the stock demand equations and in equation 4.

Three housing equations estimated by Muth (32) are presented along with some comparisons in table 8. The variables are (32, p. 84):

h_{ft} = end-of-year per-capita nonfarm housing stock.

p_t = Boeckh index of residential construction cost (brick).

y_{pt} = Friedman's per-capita expected income series.

r_t = Durand's basic yield of 10-year corporate bonds.

h'_{gt} = per-capita gross rate of nonfarm residential construction.

y_{ct} = per-capita current income.

Monetary magnitudes were deflated by BLS consumer price index, 1935-39 = 100.

Muth did not estimate stock demand equations using y_{ct}, and neither did I.

About all we can conclude from the A.L.S. estimates is that housing demand adjusts to changing conditions with a lag, and that the housing demand functions possess highly autocorrelated errors. This is much less than Muth could conclude: that p_t, y_{pt} or y_{ct}, r_t and h_{ft-1} affect housing demand. The A.L.S. estimates suggest that the significant estimates obtained by Muth were spuriously significant.

In an iterative procedure such as A.L.S., the final results will be affected by the choice of initial estimates of the parameters if the likelihood function has multiple maxima (i.e., if the residual sum of squares possesses multiple minima). It seemed possible that such had happened here. Equations 1A.2 and 2A.2 were each estimated twice, using greatly different start vectors each time. For each equation, both sets of initial estimates yielded final results that were equal in coefficients, standard errors and R^2 to 3 significant digits or more. Hence, the A.L.S. results here do not appear to be the result of an unfortunate selection of initial estimates.

Muth (32, p. 54) estimated an equation like 2.M containing a time trend. The coefficient of time was only significant at the 30-percent level. It is possible that including a time trend in equations 2A.2 and 3A.2 would have reduced (but not have eliminated) the autocorrelation in the errors, yielded a significant coefficient of time, and improved the estimates of the other coefficients.

A.L.S. estimation more than doubled the size of the coefficient of lagged stock.

FARM FACTOR DEMAND

Cromarty (7) has analyzed demand for tractors and farm machinery, and Hildreth and Jarrett (22) studied demand for protein feed. Their results and re-

Table 8. Selected statistical results from nonfarm housing demand analyses.

Equation number	Dependent variable	Coefficients						β_1	β_2	d	R^2	Number of iterations
		p_t	y_{pt}	y_{ct}	r_t	h_{ft-1}	1					
1.M[a]	h_{ft}	—4.66***	0.82***	....	—24.7**	....	....	....	....	....	0.448	....
1	h_{ft}	—4.57**	0.80***	....	—21.5s	....	841	....	....	0.39[b]	0.424	....
1A.2[c]	h_{ft}	0.002	—0.06	....	—4.7*	....	....	1.69***	—0.84***	....	0.983	5
2.M[a]	h'_{gt}	—2.49***	0.44***	....	—8.3*	—0.28***	....	....	....	....	0.621	....
2	h'_{gt}	—2.32***	0.47***	....	—5.4	—0.31***	268	....	....	0.85[d]	0.712	....
2A.2[c]	h'_{gt}	0.002s	0.05	....	—2.3	—0.66**	....	1.70***	—0.85***	....	0.967	6
3.M[a]	h'_{gt}	—1.49***	....	0.25***	—0.11	—0.12*	....	....	....	....	0.607	....
3	h'_{gt}	—1.45**	....	0.27***	2.79	—0.16**	151	....	....	0.89[d]	0.760	....
3A.2	h'_{gt}	0.003*	....	0.06**	—0.87	—0.78***	124	1.72***	—0.87***	....	0.977	10

[a] From: Richard F. Muth. The demand for nonfarm housing. In: Arnold C. Harberger (ed.). The demand for durable goods. Univ. of Chicago Press, Chicago. 1960. Chapter II. Sample period was 1915-41 excluding 1917-18. Other equations used 1921-41 sample period.

[b] Significant at 2-, 5- and 10-percent levels.

[c] Estimated twice with two different sets of initial estimates. Both converged to the same final solution.

[d] Inconclusive at 2-percent level, significant at 5- and 10-percent levels.

Table 9. Selected statistical results from farm machinery demand analyses.

Equation number	Coefficients Z_{2t}	Z_{3t}	Z_{4t}	Z_{5t}	Z_{6t}	Z_{7t}	Z_{9t}	Y_{6t}	Y_{1t-1}	1,000	β_1	d	R^2	Number of iterations
1.Cª	236	−1,206***	29	16***	39*	1,233	−433***	− 703ˢ	....	2,398	....	....	0.95ᵇ	....
1	66	−1,043***	−7ˢ	16***	−1	2,727ˢ	−288ˢ	− 957ˢ	....	1,911	....	1.32ᶜ	0.918	....
1A.1ᵈ	−96	− 965ˢ	−7ˢ	9*	−5	1,711	− 5	− 895	....	....	0.70***	....	0.938	14
2	38	−1,098***	....	16***	13	247	−308***	−1,444**	....	4,043	....	1.48ᶜ	0.913	....
2A.1ᵈ	−90	−1,011ˢ	....	6ˢ	19	− 68	− 30	1,122ˢ	....	1,575	0.75***	....	0.933	11
3	40	− 649**	....	7*	7	−1,308ˢ	−128ˢ	− 814ˢ	0.63***	3,130	....	2.26ᶜ	0.952	....
3A.1ᵈ	77	− 623**	....	7**	6	−1,340ˢ	−141ˢ	− 727ˢ	0.67***	3,390	−0.23	....	0.954	5

ª From: William A. Cromarty. The demand for farm machinery and tractors. Mich. Agr. Exp. Sta. Tech. Bul. 275. 1959. Estimated using sample period of 1923-54. Other equations in table were estimated with a sample period of 1926-56.

ᵇ Adjusted for degrees of freedom.

ᶜ Inconclusive at 2-, 5- and 10-percent levels.

ᵈ Estimated with two different sets of initial estimates. Both converged to the same final solution.

sults obtained from L.S. and A.L.S. estimation for a shorter sample period are presented in tables 9, 10 and 11.

Results of analyses of demand for farm machinery are presented in table 9. The variables are (7, pp. 38-39, 70):

Y_{1t} = value of manufacturers' sales of farm machinery and equipment for use on farms deflated by wholesale price index for farm machinery including tractors, 1947-49 = 1,000.

= dependent variable.

Y_{6t} = wholesale price index for farm machinery including tractors, 1947-49 = 1,000.

Z_{2t} = index of prices received by farmers for crops and livestock, 1910-14 = 1,000.

Z_{3t} = index of prices paid by farmers for items used in production, excluding wages, farm machinery and motor vehicles, 1910-14 = 1,000.

Z_{4t} = value of farm machinery on farms at the beginning of the year in millions of dollars.

Z_{5t} = asset position of farmers at beginning of year in millions of dollars.

Z_{6t} = realized net farm income for the previous year, in millions of dollars.

Z_{7t} = average acreage of cropland per farm, in tenths of acres.

Z_{9t} = an index of farm labor costs, 1910-14 = 1,000.

Y_{6t}, Z_{2t}, Z_{3t}, Z_{5t}, Z_{6t} and Z_{9t} were deflated by the wholesale price index for all commodities, 1947-49 = 100.

There are substantial differences between the coefficients in equations 1 and 1.C. These are due to differences in the sample period. The d statistics for equations 1 and 2 are inconclusive, whereas estimated β_1 in equations 1A.1 and 2A.1 are highly significant. A.L.S. estimation made a number of changes in the size and significance of the coefficients in equations 1 and 2. Equation 1 is of the same type of equation used by Chow (5) and Muth (32) in analyzing purchases of automobiles and new houses: an incomplete adjustment model. Equation 2 might be termed a complete adjustment model. Equation 3 represents another type of incomplete adjustment model: the Koyck-Nerlove type. Adding the lagged dependent variable significantly increased the value of R^2; it also reduced the level of significance of most of the coefficients. It is not uncommon to have the addition of the lagged dependent variable do this. Equation 2, a static equation, contains autocorrelated errors. Equation 1, a dynamic equation, also contains autocorrelated errors. Equation 3, also a dynamic equation, does not contain autocorrelated errors.

In the tractor demand analyses, table 10, the situation is somewhat different. Equation 3, which contains the lagged dependent variable, has autocorrelated errors; however, equation 2 does not.

Table 10. Selected statistical results from tractor demand analyses.

Equation number	Coefficients $\left(\dfrac{Y_2}{X_1}\right)_t$	X_{2t}	X_{3t}	X_{6t}	X_{9t}	Y_{1t-1}	1	β_1	d	R^2	Number of iterations
1.Cª	−1.69*	0.092ˢ	1.43***	....	−0.99***	....	2,211	....	....	0.78ᵇ	....
1	−2.29***	0.105*	1.27***	....	−0.97***	....	2,906	....	1.87ᶜ	0.885	....
2	−1.50**	0.040ˢ	1.35***	220***	−0.78***	....	2,337	....	2.64ᵈ	0.943	....
2A.1ᵉ	−1.49***	0.036ˢ	1.41***	215***	−0.81***	....	2,374	−0.39ˢ	....	0.951	7
3	−0.86	0.08ˢ	....	218**	....	0.25ˢ	1,626	....	1.61ᶠ	0.849	....
3A.1	−1.07ˢ	0.16*	....	302***	....	−0.26	....	0.49**	....	0.867	2

ª From: William A. Cromarty. The demand for farm machinery and tractors. Mich. Agr. Exp. Sta. Tech. Bul. 275. 1959. Estimated using sample period of 1926-56, excluding 1943. Other equations in table estimated using sample period of 1929-42, 1946-56.

ᵇ Adjusted for degrees of freedom.

ᶜ Nonsignificant at 2-, 5- and 10-percent levels.

ᵈ Inconclusive at 2-, 5- and 10-percent levels.

ᵉ Estimated with two different sets of initial estimates. Both converged to the same final solution.

ᶠ Nonsignificant at 2-percent level, inconclusive at 5- and 10-percent levels.

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Table 10 presents results of analyses of demand for tractors. The variables are (7, pp. 43, 47, 72):

Y_{1t} = manufacturer's shipments of wheel-type tractors (excluding garden) for domestic farm use, in hundreds.

= dependent variable.

$(Y_2/X_1)_t$ = ratio of retail price of farm tractors (1937-41 = 1,000) to the prices received by farmers for crops and livestock (1910-14 = 1.00).

X_{2t} = net cash receipts received by farmers during the previous year, thousands of dollars.

X_{3t} = 8-year weighted average of number of tractors on farms, in thousands.

X_{6t} = a quantified measure of farm price-support programs.

X_{9t} = average tractor sales for the previous 5 and 6 years, in thousands.

The differences between equations 1 and 1.C are due to the change in the sample period. The coefficients in equation 1A.1 were almost identical to those in equation 1, which might be expected because of the small size and nonsignificance of estimated β_1.

In some other equations which he ran, Cromarty obtained significant positive coefficients for X_{6t}. He concluded ". . . farm purchases have tended to be higher when a combination of high, fixed price supports, no soil bank and a Democratic president are in operation" (7, p. 43). The results of equations 3 and 3A.1 are in agreement with his findings. These coefficients of X_{6t} impute a greater effect to government programs than did Cromarty's results, being much larger than his coefficient of X_{6t}.

Equations 2A.1 in table 10 and 3A.1 in table 9 illustrate a common result of A.L.S. estimation. When estimated β_1 is nonsignificant, A.L.S. estimates differ little from L.S. estimates.

Hildreth and Jarrett (22) estimated farmer's demand for protein feed by using L.S. and limited-information single-equation methods of estimation. Their L.S. results are presented in table 11 along with other results.

The variables are (22, pp. 60-63):

Y_{3t} = log of price of protein feeds in dollars per 1,000 pounds total digestible nutrients.

= dependent variable.

Y_{2t} = log of price of feed grains in dollars per 1,000 pounds total digestible nutrients.

Y_{5t} = log of index of the price of livestock and livestock products.

Y_{7t} = log of total quantity of protein feeds fed in million pounds total digestible nutrients.

Z_{1t} = log of Jan. 1 inventory of livestock in million dollars of estimated potential production.

Z_{3t} = log of quantity of roughage fed in million pounds total digestible nutrients.

Hildreth and Jarrett interpreted the results of their livestock supply equation (presented in next section) to represent farmers' reactions to anticipated prices (22, pp. 104-106). Equations 2 to 5 in table 11 were estimated in a search for anticipatory elements in protein feed demand. These equations are consistent with the hypotheses that anticipatory elements do play a role. An increase in livestock prices ($\triangle Y_{5t} > 0$) generates anticipations of further increases, or at least of no immediate decreases. Farmers, therefore, are willing to pay more for protein feed. The coefficient of $\triangle Y_{2t}$ may be similarly interpreted. Neither Y_{2t} nor Y_{2t-1} are significant, though both Y_{5t} and Y_{5t-1} are significant.

A.L.S. found evidence of autocorrelation in the errors of equation 1 but not in the errors of equation 2. The main effect of A.L.S. estimation of equation 1 was to increase the size of the coefficient of Z_{1t}. A.L.S. estimation of equation 3 also changed the coefficient of Z_{1t} and also affected some of the other coefficients.

Neither equation 2 nor equation 2A.1 contain evidence of autocorrelated errors. Neither does equation 3, and the coefficient of Y_{3t-1} in equation 3 is significant at only the 10-percent level. Yet both estimated β_1 and the coefficient of Y_{3t-1} are highly significant in equation 3A.1. This situation has also been observed elsewhere (29). An equation not containing y_{t-1} shows no evidence of autocorrelated errors, and L.S. estimation of the corresponding equation containing y_{t-1} yields a nonsignificant coefficient of y_{t-1} and a nonsignificant or inconclusive value of d. But the A.L.S. estimates of both β_1 and the coefficient of y_{t-1} are significant. Such situations are examples of the interrelation between

Table 11. Selected statistical results from protein feed demand analyses.

Equation number	Coefficients														Number of iterations
	Y_{2t}	Y_{2t-1}	$\triangle Y_{2t}$	Y_{5t}	Y_{5t-1}	$\triangle Y_{5t}$	Y_{7t}	Z_{1t}	Z_{3t}	Y_{3t-1}	1	β_1	d	R^2	
1.H & J[a]	0.12s	....	....	0.74***	....	....	—0.53***	0.34s	0.08	....	1.21	....	....	0.965	....
1	0.09s	....	....	0.79***	....	....	—0.56***	0.40s	0.06	....	1.95	....	1.66b	0.963	....
1A.1	0.12s	....	....	0.77***	....	....	—0.54***	0.64*	—0.20	....	2.06	0.37*	....	0.965	7
2	....	—0.02	0.24**	....	0.91***	0.59***	—0.48***	—0.12	0.23	....	2.87	....	1.87c	0.976	....
3	....	—0.11	0.23**	....	0.73***	0.60***	—0.30**	—0.16	0.06	0.31*	2.75	....	2.65d	0.979	....
3A.1	....	—0.10s	0.28***	....	0.54***	0.46***	—0.20**	—0.43**	0.31s	0.50***	1.79	—0.65***	....	0.986	4
4A.1	....	....	0.24***	....	0.90***	0.61***	—0.45***	....	....	....	3.23	0.14	....	0.975	4
5A.1	....	....	0.26***	....	0.56***	0.51***	—0.28***	....	....	0.38**	2.64	—0.28s	....	0.979	3

[a] From: Clifford Hildreth and F. G. Jarrett. A statistical study of livestock production and marketing. John Wiley and Sons, Inc., New York. 1955. Sample period was 1920-49; for other equations the sample period was 1922-49.

[b] Nonsignificant at 2-percent level; inconclusive at 5- and 10-percent levels.

[c] Nonsignificant at 2-, 5- and 10-percent levels.

[d] Inconclusive at 2-, 5- and 10-percent levels.

y_{t-1} and autocorrelated errors that was discussed by Griliches (17) and by Fuller and Ladd (14). These results show how sensitive the coefficient of y_{t-1} can be to the assumption made about the properties of the errors.

Deleting Y_{2t-1}, Z_{1t} and Z_{3t} from equations 2 and 2A.1 had a negligible effect on the other coefficients, on the value of R^2 and on the estimate of β_1 (see equation 4A.1). Deleting these variables from equations 3 and 3A.1 eliminated the autocorrelation in the errors (see equation 5A.1). Here is another case in which the inclusion of nonsignificant variables introduced autocorrelation into the errors.

SUPPLY OF FARM PRODUCTS

Hildreth and Jarrett (22) estimated the supply of livestock and livestock products by L.S. and limited-information single-equation methods. Their L.S. results are summarized in table 12. The variables are (22, pp. 60-63):

Y_{4t} = log of sales of livestock and livestock products in million dollars at average prices.
  = dependent variable.
Y_{1t} = log of production of livestock and livestock products in million dollars at average prices.
Y_{2t} = log of price of feed grains in dollars per 1,000 pounds total digestible nutrients.
Y_{3t} = log of price of protein feeds in dollars per 1,000 pounds total digestible nutrients.
Y_{5t} = log of index of prices of livestock and livestock products.
Z_{1t} = log of Jan. 1 inventory of livestock in million dollars of estimated potential production.
Z_{8t} = log of cash farm wage in cents per hour.

The values of d in equations 1 and 2 are inconclusive, but the estimates of β_1 in equations 1A.1 and 2A.1 are both significant. The coefficients in equation 1A.1 are not appreciably different from those in equation 1. There are substantial differences between the coefficients of Y_{3t} and Z_{1t-1} in equations 2 and 2A.1, however. There are a number of differences between the coefficients in equations 3 and 3A.1, notably in the coefficients of Y_{3t-1}, Z_{1t-2} and Z_{8t}.

Hildreth and Jarrett (22, pp. 105-106) interpret the coefficients of Y_{2t}, Y_{5t} and Z_{8t} as representing farmers' reactions to anticipated prices. Anticipated prices were assumed to be functions of current prices. As future prices of feed grains and labor are expected to rise, current marketings increase. As future prices of livestock products are expected to rise, current marketings decline. The negative coefficient of Y_{3t} is interpreted in a different way.

Equations 3 and 3A.1 are derived from a Nerlove (34) type of price expectation model. Several variants of this type of equation were estimated, of which equation 3 is one. None was an improvement over equation 1, either in terms of the size of the R^2 or the magnitude and significance of the coefficients. The coefficients of Y_{4t-1} were nonsignificant.

In every A.L.S. equation, the estimate of β_1 was negative. It was highly significant in all but equation 1A.1.

Table 13 presents results on spring farrowings in the United States. The variables are (9, p. 578):

Y_t = number of spring farrowings, United States, in 1,000 litters.
  = dependent variable.
X_{1t} = United States average hog-corn price ratio, October, November and December of year $t-1$.
$X_{2t} = S_{t-1} - S_{t-2} + 15$.
S_t = oats, barley and grain sorghum as a percent of corn production.
X_{4t} = ratio between average price of 500-800 pound good-choice stockers and feeders at Omaha and the average United States hog price in October, November and December of year $t-1$.

Equations 1 and 2 are like the equations that Dean and Heady estimated (9). The differences are that they used $\triangle Y_t$ as the dependent variable, obtained larger values of R^2 (0.93 and 0.76 for equations 1 and 2, respectively) and obtained a nonsignificant coefficient of Y_{t-1} in equation 2. In equation 2, they tested a hypothesis that farrowings respond to expected price ratios.

Although d in equation 1 is inconclusive, estimated β_1 is significant in equation 1A.1. The absence of autocorrelated errors in equation 2 is probably due to the

Table 12. Selected statistical results from analyses of supply of livestock and livestock products.

Equation number	Coefficients									β_1	d	R^2	Number of iterations
	Y_{1t}	Y_{2t}	Y_{3t}	Y_{5t}	Z_{1t}	Z_{1t-1}	Z_{8t}	Y_{4t-1}	1				
1.H & J[a]	0.80***	0.14***	−0.13**	−0.14**	0.08s	....	0.12**	....	0.99	....	....	0.992	....
1	0.71***	0.13***	−0.15***	−0.10s	0.18s	....	0.12***	....	0.35	....	2.89b	0.992	....
1A.1	0.76***	0.14***	−0.15***	−0.11*	0.12s	....	0.11***	....	....	−0.51**	....	0.994	3
2	0.84***	0.14***	−0.10*	−0.16***	....	0.06	0.11**	....	0.23	....	3.06c	0.991	....
2A.1	0.82***	0.12***	−0.10***	−0.11**	....	0.12**	0.08***	....	....	−0.62***	....	0.994	3
	Y_{2t-1}	Y_{3t-1}	Y_{5t-1}	Z_{1t-2}									
3	0.74***	0.02	−0.15*	0.08	−0.04	0.42***	0.06s	−0.12	0.16	....	3.20d	0.986	....
3A.1	0.63***	0.04*	−0.14**	0.01	−0.16*	0.34***	0.09***	0.08s	....	−0.77***	....	0.993	8

[a] From: Clifford Hildreth and F. G. Jarrett. A statistical study of livestock production and marketing. John Wiley and Sons, Inc., New York. 1955. Fitted to sample period of 1920-49; other equations fitted to sample period of 1922-49.

[b] Inconclusive at 2-, 5- and 10-percent levels.

[c] Significant at 10-percent level, inconclusive at 2- and 5-percent levels.

[d] Significant at 5- and 10-percent levels, inconclusive at 2-percent level.

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Table 13. Selected statistical results from analyses of spring farrowings.[a]

Equation number	Coefficients					β_1	d	R^2	Number of iterations
	X_{1t}	X_{2t}	X_{4t}	Y_{t-1}	1				
1	235***	3,709s	973**	....	3,855	....	1.22b	0.540	....
1A.1	269***	4,368*	1,389***	....	4,824	0.37**	....	0.708	6
2	261***	....	573s	0.51**	−212	....	1.98c	0.647	....
2A.1	266***	....	594s	0.52**	−1,184	−0.09	....	0.652	9

[a] Equations estimated with sample period of 1939-41, 1945-56.

[b] Inconclusive at 2-, 5- and 10-percent levels.

[c] Nonsignificant at 2-, 5- and 10-percent levels.

Table 14. Selected statistical results from analyses of farm supply of all crops, 1925-57.

Equation number	Coefficients						β_1	d	R^2	Number of iterations
	P_t	W_t	t	A_{t-1}	S_{t-1}	1				
1	0.071s	0.56***	0.005***	....	....	0.61	....	1.59a	0.870	....
2	0.084s	0.52***	0.004***	....	0.23**	0.24	....	2.30a	0.888	....
2A.1	0.069*	0.45***	0.003***	....	0.41***	....	−0.45**	....	0.900	5
3	0.13*	0.44***	....	0.65***	....	−0.46	....	2.10b	0.775	....
4	0.19**	0.52***	....	−1.09s	1.70*	−0.63	....	2.30a	0.801	....
4A.1	0.14**	0.36***	....	−0.84s	1.60**	....	−0.46**	....	0.821	7

[a] Nonsignificant at 2- and 5-percent levels; inconclusive at 10-percent level.

[b] Nonsignificant at 2-, 5- and 10-percent levels.

addition of Y_{t-1}, although, as previous experience has shown, it could be due to the exclusion of X_{2t}.

Results of analyses of farm supply of all crops are presented in table 14. The variables are:

S_t = log of index of output of all crops (56, p. 31).
   = dependent variable.

$$P_t = \log \left[\frac{100 \, P_{ct}}{P_{pt}} \right].$$

P_{ct} = index number of prices received for all crops, March 15, year t (55, p. 15).

P_{pt} = index number of prices paid, interest, taxes and wage rates, March 15, year t (54, p. 58).

W_t = log of index of influence of weather on total index of crop production (39).

t = year minus 1923.

These variables are intended to duplicate the variables Griliches used (18), although the sample period used here is quite different from the ones he used.

A_{t-1} = log of index number of crop production per acre, year $t-1$ (56, p. 52).

Griliches obtained a significant coefficient of S_{t-1} for 1911-58 but not for 1911-34 or 1935-58. The coefficient of S_{t-1} is significant here, but the addition of S_{t-1} introduced negative autocorrelation into the disturbances. The results of equations 4 and 4A.1 are hardly credible, however. The coefficients of S_{t-1} exceed unity. This means that the ultimate response to an increase in prices or weather index is an explosive, unlimited expansion of output and the response to a decline in these variables is an explosive decline in output. The d statistic and A.L.S. estimation yielded nonsignificant evidence of autocorrelation in the errors in equations 1 and 3. Estimates of β_1 in equations 2A.1 and 4A.1 are both significant, although the d statistics in equa-

tions 2 and 4 are nonsignificant. This is illustrative of the low power of the d statistic when used on equations containing the lagged dependent variable.

A_{t-1} was tried as a replacement for trend. It yielded a smaller value of R^2, made the coefficients of P_{t-1} significant and had little effect on the autoregressive properties of the errors. The addition of S_{t-1} made the coefficient of A_{t-1} nonsignificant.

Cromarty's political price program variable (X_{6t} in the tractor demand equations) was included in some analyses. Its coefficient was about equal to or smaller than the standard error in both static and dynamic equations.

COMPARISONS OF RESULTS

This section summarizes results from several different studies that used A.L.S. In addition to the results discussed previously in this report, results from the following other studies are summarized: (a) studies of monthly and quarterly demand for seven food items by the cooperators on the Michigan State University consumer panel, (b) analyses of annual food demands in the United Kingdom, (c) a quarterly model of the national income accounts of the United States and (d) analyses of demand for commercial fertilizer and additional analyses of supply of livestock.

The results obtained in the Michigan consumer panel studies have been reported elsewhere (29, 30). In these analyses, static and dynamic equations, like the annual food demand equations discussed earlier in this report, were estimated by L.S. and A.L.S. The dependent variable was per-capita consumption. Independent variables were per-capita income, own price, prices of related products and seasonal 0-1 shift variables.

In the United Kingdom analyses, static demand

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equations were estimated by L.S.; dynamic equations were estimated by L.S. and A.L.S. Per-capita consumption was the dependent variable; per-capita income and own price were independent variables. These results have been published in (15).

In the quarterly national income model, the equations estimated were: (a) durable goods, nondurable goods and services, static and dynamic consumption functions, (b) a depreciation equation, (c) static and dynamic capital investment equations and (d) inventory investment equations. Quarterly, seasonally adjusted, data were used.

These studies are all covered in this statistical summary to report cumulative experience to date with A.L.S. estimation. This summary covers some 150 equations that have been estimated by A.L.S. or A.T.S. No distinction will be made between A.L.S. and A.T.S. results; nor will any be made between L.S. and T.S.L.S. results.

Comparison of Tests for Autocorrelated Errors

Two tests commonly used to check for autocorrelation in errors are the von Neumann-Hart ratio, $\triangle^2/S^2$ (20) and the Durbin-Watson d statistic (11). To apply these tests, coefficients are first estimated under the assumption of zero autocorrelation in the errors, and the residuals (estimated errors) are used to test for autocorrelation. In the A.L.S. procedure, testing the significance of β_1 (in A.L.S.–1) or of β_1 and β_2 (in A.L.S.–2) tests for autocorrelation in the errors. Table 15 compares results from the three different tests applied to 97 different equations.

One disadvantage of the tabulated Durbin-Watson d test is that it may not yield a definite answer. Some values of d are in an inconclusive range. For any equation whose d value is in this range, the tabulated tests permit neither acceptance nor rejection of the null hypothesis. To avoid this indeterminacy, Theil and Nagar (41) have published an alternative set of significance levels that does not contain an inconclusive range. Their significance values are almost exactly equal to the limits of the inconclusive range of the Durbin-Watson test. Hence, the Theil-Nagar test classes as significant all values of d that are significant in the Durbin-Watson tables plus virtually all values of d that are in the inconclusive range in the latter tables. The Theil-Nagar table is set up only for testing the null hypothesis against the alternative hypothesis of positive autocorrelation. By assuming symmetry about 2.00, the expected value of d if the null hypothesis is true, the Theil-Nagar table can be used to make a two-tailed test, and it was so used here. In the remainder of this discussion, those values of d that are inconclusive in table 1 will be treated as significant, as they would in the Theil-Nagar test.

In equations not containing the lagged dependent variable y_{t-1}, autocorrelation showed up much more frequently in the annual analyses than in the quarterly and monthly analyses. Of the equations estimated with annual data, 76 percent had significant autocorrelation in the errors according to the A.L.S. results; of the equations estimated with monthly or quarterly data, 33 percent had autocorrelated errors. Of the equations containing the lagged dependent variable, the difference was not so great, the proportions being 45 and 38 percent, respectively. The other tests also indicated autocorrelation in a larger proportion of the annual equations. The comparative performance of the tests for autocorrelated errors did not vary appreciably between the longer and shorter unit observation periods. The Theil-Nagar d yielded evidence of autocorrelation more often than did the other tests. The Durbin-Watson d yielded inconclusive values more frequently in dynamic than static equations. A.L.S. and the other tests were in agreement more often in static than in dynamic equations.

Both d and $\triangle^2/S^2$ appear to be fairly reliable tests for autocorrelated errors in equations not containing the lagged dependent variable. Suppose we had used the following research strategy on the equations not containing the lagged dependent variable: (a) Compute the regression, assuming temporally independent errors, (b) compute d, (c) if Theil-Nagar d is significant, re-estimate by A.L.S. We would have applied A.L.S. estimation to 30 of the 32 equations in which estimated β_1 is significant and to 11 of the 22 equations in which β_1 is nonsignificant. Thirteen equations would not have been estimated, two of which have significant values of β_1. Suppose we had used $\triangle^2/S^2$ instead of d in our strategy. We would have re-estimated 26 of the 32 equations in which β_1 is significant and 4 equations in which β_1 is nonsignificant. Twenty-four equations, 6 of which yield significant values of β_1 would not have been re-estimated.

When structural estimation is the objective, either

Table 15. Comparative performance of three different tests for autocorrelation in the errors.

Status of Durbin-Watson d at 5 percent	Equations containing lagged dependent variable				Equations not containing lagged dependent variable			
	Estimated β significant at 5 percent		Estimated β nonsignificant at 5 percent		Estimated β significant at 5 percent		Estimated β nonsignificant at 5 percent	
	$\triangle^2/S^2$ Significant at 5 percent	$\triangle^2/S^2$ Nonsignificant at 5 percent	$\triangle^2/S^2$ Significant at 5 percent	$\triangle^2/S^2$ Nonsignificant at 5 percent	$\triangle^2/S^2$ Significant at 5 percent	$\triangle^2/S^2$ Nonsignificant at 5 percent	$\triangle^2/S^2$ Significant at 5 percent	$\triangle^2/S^2$ Nonsignificant at 5 percent
	(number of equations)							
Significant	3	....	....	....	21	....	2	....
Inconclusive	2	10	2	6	5	4	2	7
Nonsignificant	—	4	....	16	....	2	....	11
Total	5	14	2	22	26	6	4	18

of these strategies would be an improvement over the strategy of computing d and $\triangle^2/S^2$ and then quitting. One's choice between $\triangle^2/S^2$ and Theil-Nagar d would be determined by considerations of costs of Type I and Type II errors and costs of computation. A Type II error would be made if we accepted the null hypothesis of zero autocorrelation in the errors when the errors were autocorrelated. Such an error will lead to inefficient estimates of the coefficients and biased estimates of standard errors and residual mean square. Empirical evidence on the magnitudes of these effects will be presented later. It appears that the use of $\triangle^2/S^2$ will lead to Type II errors more often than will the use of the Theil-Nagar d.

In equations containing y_{t-1}, neither of these strategies would be as useful as they would be in equations not containing y_{t-1}. The use of d to determine which equations to re-estimate would have missed one-fifth of the equations with significant values of β_1. The use of $\triangle^2/S^2$ would have missed four-fifths of them.

We also need to consider Type I errors. Is the probability of making a Type I error (with a nominal 5-percent critical level) substantially greater in A.L.S. estimation of β_1 than in the d or $\triangle^2/S^2$ statistics? Other evidence suggests not: $\triangle^2/S^2$ is designed for testing observed sequences; when applied to residuals, it would be appropriate to make some adjustment to allow for sampling error in the estimated coefficients. The d statistic is based on the assumption of fixed independent variables; it is not appropriate for equations containing the lagged dependent variable as an independent variable (11). There is experimental evidence that the von Neumann-Hart ratio and the d statistic are biased toward too-frequent acceptance of the null hypothesis (6, 28, 33).

One might hope that the inconclusive values of Durbin-Watson d falling close to the nonsignificance limits would be in equations with non-significant estimates of β and that those close to the significance limits would be in equations with significant estimates of β. Such is not the case. There seems to be no relation between the position of d in the inconclusive range and significance of β_1.

In their work with autocorrelated errors in demand equations, which was published before the Theil-Nagar d test became available, Hildreth and Lu (23) studied equations that did not contain the lagged values of the dependent variable. The reasonably good performance of $\triangle^2/S^2$ in their work lead them to suggest the possibility of modifying the von Neumann-Hart test to obtain a test for autocorrelated errors. The results in table 1 indicate that a reasonably good and economical test is now available for equations not containing y_{t-1}: the Theil-Nagar d test.

It does not appear possible to obtain good estimates of the autoregressive parameters from L.S. residuals. (Residuals are estimates of the errors.) The residuals are biased somewhat toward randomness. Estimates of β_1 computed from residuals are not closely correlated with A.L.S. estimates of β_1. Regressing the A.L.S. esti-

mate of β_1 on the L.S. value of d for equations not containing y_{t-1} in which estimated β_1 was significant yielded the results:

$$(3.1) \quad \text{Est } \beta_1^* = 1.30 - 0.61d + v; \quad r^2 = 0.73.$$

In equations containing y_{t-1}, the relation

$$(3.2) \quad \text{Est } \beta_1 = 1.83 - 0.82d + v; \quad r^2 = 0.55$$

was found. F tests indicated that both of these differ significantly from Est $\beta_1 = 1.0 - 0.5d$, which is an approximate relation between β_1 and d obtained by ignoring end-effects. Relations 3.1 and 3.2 do not differ significantly from the relation Theil and Nagar derive (41)

$$(3.3) \quad \hat{\beta}_1 = \frac{N^2 + (m+1)^2}{N^2 - (m+1)^2} - \frac{N^2}{2[N^2 - (m+1)^2]} \, d.$$

However, equation 3.3 is not a very useful estimation procedure. The mean square differences $(\Sigma \, (\hat{\beta}_1 - \text{est } \beta_1)^2/ n)^{1/2}$ were 0.46 for equations containing y_{t-1} and 0.28 for equations not containing y_{t-1}. The mean values of $|\hat{\beta}_1 - \text{est } \beta_1|$ were 0.38 and 0.22, respectively, for equations containing and not containing y_{t-1}.

An F test indicated that relations 3.1 and 3.2 are not significantly different from each other. Pooling the data yielded

$$(3.4) \quad \text{Est } \beta_1 = 1.37 - 0.63d + v; \quad r^2 = 0.64.$$

This does differ significantly from $1 - 0.5d$ and from equation 3.3. The simple correlation between the current and just lagged residual, r_1, was also computed as an estimate of the autoregressive coefficient and compared with β_1 but is not very useful since r_1 consistently underestimated β_1.

Effect of A.L.S. Estimation

Define D_i as the absolute difference between the L.S. and A.L.S. (or T.S.L.S. and A.T.S.) estimates of the i-th coefficient, and define E_i as D_i divided by the L.S. (or T.S.L.S.) estimate of the coefficient. In their study of demand relations with autocorrelated errors, Hildreth and Lu (23) classified equations into three groups according to values of E_i where D_i was the difference between the L.S. estimate of a coefficient and the estimate obtained by their autoregressive error estimation procedure. The groups were:

I. Negligible difference. None of the re-estimated coefficients differ from the corresponding L.S. estimates by as much as 20 percent.

II. Noticeable difference. Some, but fewer than half, of the coefficients change by at least 20 percent.

III. Substantial difference. Half or more of the coefficients change by at least 20 percent.

Of 17 equations, they placed 7 in class I, 5 in class II and 5 in class III. Monthly and quarterly food-demand equations estimated by A.L.S. were classified on

the same basis (29). Of the 15 equations in which estimated β was significant at the 10-percent level, none were in class I, 2 were in class II, and 13 were in class III. Of the 18 equations in which estimated β_1 was nonsignificant at the 10-percent level, 6 were in class I, 9 were in class II, and only 3 were in class III.

Define $\triangle_i$ as D_i divided by the L.S. estimate of the standard error of the i-th coefficient. Tables 16 and 17 classify the values of $\triangle_i$ in equations with significant estimates of β according to the result of the d statistic. About half of the A.L.S. estimates differ from the corresponding L.S. estimates by more than one L.S. standard error. About one-fourth of the coefficients whose $\triangle_i$ exceeds unity are in equations with nonsignificant values of d.

Tables 18 and 19 summarize results on the comparative significance status of L.S. and A.L.S. coefficients. The two methods of estimation lead to different conclusions concerning significance of 23 percent of the coefficients in equations containing y_{t-1} and 37 percent of the coefficients in equations not containing y_{t-1}. In equations containing y_{t-1}, 55 percent of the changes were from nonsignificant L.S. estimates to significant A.L.S. estimates. In equations not containing y_{t-1}, one-third of the changes were of this kind. One-fifth of the changes in significance status occurred in equations in which d was nonsignificant.

Values of D_i were also tabulated separately for equations containing y_{t-1} and equations not containing y_{t-1}. The two distributions of D_i were not significantly different. The mean and median values of D_i were somewhat larger in equations not containing y_{t-1}.

The 17 equations classified by Hildreth and Lu (23) did not contain y_{t-1}. The proportion of coefficients for which $\triangle_i$ exceeds unity is the same in tables 16 and 17. The proportion of coefficients whose significance status was changed is larger in table 18 than in table 19. The empirical evidence all supports the conclusion that the effect of autocorrelated errors is equally serious in equations containing y_{t-1} and in equations not containing y_{t-1}. One might expect the result to be more serious in equations containing y_{t-1}. It has been argued that autocorrelated errors cause L.S. coefficients to be inefficient but unbiased in equations not containing y_{t-1} (19, 57) and to be inefficient and biased in equations containing y_{t-1} (14, 17, 19, 57, 58).

The conclusion that L.S. estimates of equations containing autocorrelated errors are unbiased is derived on the assumption of fixed independent variables or of independence between the independent variables and the errors. One source of errors is the omission of relevant autocorrelated variables. If the intercorrelations among the omitted and the included variables are of the same order of magnitude as the intercorrelations among the included variables, as seems likely, the assumption of independence will not be satisfied, and biased L.S. coefficients will be the result. If the autocorrelated errors arise from incorrect specification of the form of the fitted function, it is again quite possible that the errors will be correlated with the independent

variables, with a resulting bias in the coefficients. This argument suggests that tables 17 and 19 reflect L.S. bias and inefficiency arising from autocorrelated errors and that tables 16 and 18 reflect L.S. bias resulting from correlation between errors and independent variables and also reflect inefficiency resulting from autocorrelated errors.

The proposed hypothesis can be tested. Suppose we wish to estimate

$$(3.5) \quad y_t = \Sigma \alpha_i X_{it} + \epsilon_t$$

under the assumptions 1.6 to 1.11. We can use A.L.S. (or some similar procedure) to estimate the coefficients in

$$(3.6) \quad y_t = \beta_1 y_{t-1} + \Sigma \alpha_i (X_{it} - \beta_1 X_{it-1}) + u_t$$

The ϵ_t can then be estimated from

$$(3.7) \quad e_t = y_t - \Sigma a_i X_{it}$$

Table 16. Values of $\triangle_i$ cross-classified by d-test result, 31 equations not containing y_{t-1}, with estimated β significant at 5-percent level.

Status of d at 5 percent	$\triangle_i \leq 1.0$	$1.0 < \triangle_i \leq 2.0$	$2.0 < \triangle_i$	Total
Significant	18	13	21	52
Inconclusive	35	5	3	43
Nonsignificant	8	6	8	22
Total	61	24	32	117

Table 17. Values of $\triangle_i$ cross-classified by d-test result, 23 equations containing y_{t-1}, with estimated β significant at 5-percent level.

Status of d at 5 percent	$\triangle_i \leq 1.0$	$1.0 < \triangle_i \leq 2.0$	$2.0 < \triangle_i$	Total
Significant	8	2	6	16
Inconclusive	37	19	12	68
Nonsignificant	10	8	9	27
Total	55	29	27	111

Table 18. Coefficients in 31 equations not containing y_{t-1} in which estimated β was significant at 5-percent level classified by values of $\triangle_i$ and changes in significance status of coefficients at 5-percent level.

Value of $\triangle_i$	L. S. estimate significant and A.L.S. estimate nonsignificant or vice versa	Both estimates significant or both nonsignificant	Total
$\triangle_i \leq 1.0$	11	50	61
$1.0 < \triangle_i \leq 2.0$	9	15	24
$2.0 < \triangle_i$	24	8	32
Total	44	73	117

Table 19. Coefficients in 23 equations containing y_{t-1} in which estimated β was significant at 5-percent level classified by values of $\triangle_i$ and changes in significance status of coefficients at 5-percent level.

Value of $\triangle_i$	L.S. estimate significant and A.L.S. estimate nonsignificant or vice versa	Both estimates significant or both nonsignificant	Total
$\triangle_i \leq 1.0$	10	45	55
$1.0 < \triangle_i \leq 2.0$	5	24	29
$2.0 < \triangle_i$	9	18	27
Total	24	87	111

where a_i is the estimate of α_i, and the null hypothesis of $E(X_{it}\epsilon_t) = 0$ can be tested.

It would be fortunate and useful to know if equations having inconclusive values of d and simultaneously having some coefficients with large values of $\triangle_i$ also had significant values of the Hart-von Neumann ratio. Such does not appear to be the case. In the equations estimated in this study in which estimated β_1 was significant and d was inconclusive, there was no tendency for large values of $\triangle_i$ to be concentrated in the equations with significant values of the Hart-von Neumann ratio. The average value of $\triangle_i$ in equations with inconclusive values of d and nonsignificant values of the Hart-von Neumann ratio exceeded the average value of $\triangle_i$ in equations with inconclusive values of d and significant values of the Hart-von Neumann ratio. This was true whether the mean or the median was the average used in the comparison.

In 60 percent of the equations in which estimates of β were significant, A.L.S. made some standard errors larger and some smaller. The proportions varied from equation to equation; about half of the standard errors in these equations rose, and the other half fell. In 20 percent of the equations, A.L.S. made all standard errors larger; in another 20 percent it made all standard errors smaller. In equations with significant estimates of β, A.L.S. increased the size of exactly half of the standard errors. These proportions did not vary appreciably between equations containing y_{t-1} and equations not containing y_{t-1}.

Some insight into the changes of significance status at the 5-percent level for the 68 coefficients in tables 18 and 19 can be obtained by considering the four t ratios: $t = b_i/s_i$; $t' = b_{ai}/s_i$; $t_a = b_{ai}/s_{ai}$; $t'' = b_i/s_{ai}$ where b_i and s_i represent L.S. estimates of a coefficient and its standard error and b_{ai} and s_{ai} denote A.L.S. estimates of the same coefficient and its standard error. There are eight different configurations of these ratios for the cases in which t and t_a lead to different conclusions concerning significance. These eight are shown in table 20. Derivation of the last column in the table will be explained by examples. Take the first row. The difference between t and t' indicates that the difference between the L.S. and A.L.S. coefficients was sufficient to change the significance status of the estimate; the difference between t_a and t'' suggests the

same thing. A comparison of t' and t_a indicates that the change in the standard errors did not change the significance status, as does a comparison of t and t''. Now consider the fifth row. Comparison of t_a and t'' indicates the change in the coefficient to have been responsible for the change in significance status; comparison of t and t' indicates that the change in the coefficient was not sufficient to change the significance. Comparison of t' with t_a and t with t'' is similarly contradictory concerning the role of the change in the standard errors.

These four different ratios were computed for each of 68 coefficients whose significance status was different. By the criteria in table 20: in 43 pairs of coefficients, the difference in the coefficients was responsible for the change in significance; in 10 pairs, the change in the standard errors was responsible; 15 pairs could not be assigned to one cause or the other.

According to the arguments of Griliches (17) and of Fuller and Ladd (14), we can expect to find the estimated coefficients of y_{t-1} sensitive to the presence of autocorrelation in the errors. We do find this. Some examples were presented in earlier tables: equations 5 and 5A.1 in table 1, equations 2 and 2A.1 in table 4, equations 3 and 3A.1 in table 6 and equations 3 and 3A.1 in table 11.

In 80 percent of the equations containing y_{t-1} in which estimated β_1 was significant, $\triangle_i$ for the coefficient of y_{t-1} exceeded unity; in no case was it the only coefficient whose $\triangle_i$ exceeded unity in the equation. In tables 17 and 19, of the 56 values of $\triangle_i$ exceeding unity, one-third are for coefficients of y_{t-1}. Additional results are presented in table 21.

This table presents results from 58 sets of equations. In 52 sets of equations, the static equation was estimated by L.S. and A.L.S. (or by T.S.L.S. and by A.T.S.); the dynamic equation was also estimated both ways. In 6 sets of 3 equations, the static equation was estimated by L.S. and A.L.S. (or by T.S.L.S. and A.T.S.); the dynamic equation was estimated only by A.L.S. (or A.T.S.). The dynamic equation was obtained by adding y_{t-1} as an independent variable. In 30 cases, there was evidence of autocorrelation in the errors of the static equation. In 16 of the 52 quadruples, A.L.S. leads to different conclusions concerning the significance of y_{t-1}. In only 9 of these 16, was estimated β significant. In 2 of the 7 cases in which both estimated β and y_{t-1} were nonsignificant, however, the sum of the two coefficients was significant. In these two equations, there was either an autocorrelated error or a lag in behavior, but the data could not identify which was present. In two other equations in which both estimated coefficients of y_{t-1} were significant, the L.S. coefficient was positive, and the A.L.S. coefficient was negative.

The omission of relevant variables is one possible source of autocorrelated errors. In 14 of the 30 static equations with autocorrelated errors, the addition of y_{t-1} eliminated the autocorrelation. In 9 of the 19 static equations not possessing autocorrelated errors, the addi-

Table 20. Values of t ratios.[a]

$t = \dfrac{b_i}{s_i}$	$t' = \dfrac{b_{ai}}{s_i}$	$t_a = \dfrac{b_{ai}}{s_{ai}}$	$t'' = \dfrac{b_i}{s_{ai}}$	Change in significance of coefficient due to change in
s	n	n	s	coefficient
n	s	s	n	coefficient
s	s	n	n	standard error
n	n	s	s	standard error
s	s	n	s	?
s	n	n	n	?
n	n	s	n	?
n	s	s	s	?

[a] n indicates t ratio nonsignificant at 5 percent; s indicates t ratio significant at 5 percent.

Table 21. Relation between autocorrelation in errors and coefficient of y_{t-1} using 5-percent level of significance.

	Est β significant in static equation		Est β nonsignificant in static equation		
	Est β significant in dynamic equation	Est β nonsignificant in dynamic equation	Est β significant in dynamic equation	Est β nonsignificant in dynamic equation	Total
	(Number of equations)				
y_{t-1} significant in L.S. and A.L.S. equations	4	10	1	2	17
y_{t-1} nonsignificant in L.S. and A.L.S. equations	6	....	2	11	19
y_{t-1} significant in L.S., nonsignificant in A.L.S. equation	2	4	1	....	7
y_{t-1} nonsignificant in L.S., significant in A.L.S. equation	1	....	5	3	9
Dynamic equation not estimated by L.S.					
y_{t-1} nonsignificant in A.L.S.	2	....	....	3	5
y_{t-1} significant in A.L.S.	1	....	....	....	1
Total	16	14	9	19	58

tion of y_{t-1} apparently introduced autocorrelation into the errors.

The lagged dependent variable is not the only variable whose addition may introduce autocorrelation into the errors. Two examples were mentioned earlier: equations 3A.1 and 4A.1 in table 11 and the use of a butter and margarine price index in a shortening-demand equation. In an analysis of quarterly beef demand using consumer panel data, static and dynamic equations containing average quarterly temperature were estimated. The coefficient of temperature and estimated β_1 were highly significant. Deleting temperature led to a nonsignificant estimate of β_i. Hildreth and Lu reported a case in which the introduction of a quadratic trend term introduced positive autocorrelation into the errors (23, p. 22n).

Suppose the true model is, in matrix notation,

$$(3.8) \quad Y_t = X_t A + \epsilon_t = (X_{1t} X_{2t} X_{3t})A + \epsilon_t,$$

where ϵ_t is distributed independently of X_{t-i} and ϵ_{t-i}, for all values of i, and suppose the equation estimated is

$$(3.9) \quad Y_t = X_{1t} A_1 + u_t.$$

Then

$$(3.10) \quad u_t = X_{2t} A_2 + X_{3t} A_3 + \epsilon_t,$$

and the autocovariance of u_t is

$$(3.11) \quad E(u_t' u_{t-1}) = E(A_2' X_{2t}' X_{2t-1} A_2 + A_2' X_{2t}' X_{3t-1} A_3 + A_3' X_{3t}' X_{2t-1} A_2 + A_3' X_{3t}' X_{3t-1} A_3)$$

Suppose the equation is re-estimated with additional variables,

$$(3.12) \quad Y_t = X_{1t} A_1 + X_{2t} A_2 + v_t.$$

Then $v_t = X_{3t} A_3 + \epsilon_t$,
and

$$(3.13) \quad E(v_t' v_{t-1}) = E(A_3' X_{3t}' X_{3t-1} A_3).$$

It is possible for equation 3.11 to be zero or approximately zero, while equation 3.13 is not zero.

Further insight may be gained—at the price of generality—by considering the special case in which X_{2t} and X_{3t} each are a single variable. Then equation 3.11 reduces to

$$(3.14) \quad E(u_t' u_{t-1}) = E[A_2^2 \Sigma x_{2t} x_{2t-1} + A_2 A_3 (\Sigma x_{2t} x_{3t-1} + \Sigma x_{3t} x_{2t-1}) + A_3^2 \Sigma x_{3t} x_{3t-1}].$$

Having the observations on the variables and having A_2, does there exist an A_3 such that $E(u_t' u_{t-1}) = 0$ and $E(A_3^2 \Sigma x_{3t} x_{3t-1}) \neq 0$? Set equation 3.14 equal to zero, assuming the fixed X or regression model, and treat as a quadratic in A_3. Let r_{22} be the autocorrelation between x_{2t} and x_{2t-1}; r_{33} be the autocorrelation between x_{3t} and x_{3t-1}; r_{23} be the serial correlation between x_{2t} and x_{3t-1}; and r_{32} be the serial correlation between x_{3t} and x_{2t-t}. Assume a circular universe so that $\Sigma x_{it}^2 = \Sigma x^2_{it-1}$.

$$(3.15) \quad A_3 = \frac{-A_2 (\Sigma x_{2t}^2 \Sigma x_{3t}^2)^{1/2}(r_{23} + r_{32})}{2 r_{33} \Sigma x_{3t}^2}$$
$$\pm \frac{[A_2^2 \Sigma x_{2t}^2 \Sigma x_{3t}^2 (r_{23}^2 + r_{32}^2 + 2 r_{23} r_{32} - 4 r_{22} r_{33})]^{1/2}}{2 r_{33} \Sigma x_{3t}^2}.$$

Equation 3.15 gives a value of A_3 which will make equation 3.14 equal to zero. A (real number) solution will exist if $r_{23}^2 + r_{32}^2 + 2 r_{23} r_{32} - 4 r_{22} r_{33} \geqq 0$.

Most economic variables will not satisfy this inequality. Since we are dealing with a circular universe, we can set $r_{23} = r_{32}$. Then the inequality is

$$(3.16) \quad 4(r_{23}^2 - r_{22} r_{33}) \geqq 0.$$

The autocorrelation within economic variables usually substantially exceeds the serial correlation among series. Generally, there will exist no (real valued) A_3 that will make equation 3.14 zero.

There is, however, one situation in which the serial correlation may be large enough so that 3.16 will be satisfied: when x_{2t} is the lagged dependent variable

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and a temporal aggregation problem exists. Mundlak (31) studied a Koyck-Nerlove type of model assuming the adjustment period to be a month and assuming a year to contain k months. Let monthly equilibrium demand be q_t^*, a function of observed variables such as current monthly prices and income. Let the monthly adjustment be

$$(3.17.a) \quad q_t - q_{t-1} = \gamma(q_t^* - q_{t-1})$$

$$(3.17.b) \quad q_t = \gamma q_t^* + (1-\gamma)q_{t-1}.$$

If the regression were to be run with annual data, Mundlak showed that the appropriate function would be

$$(3.18) \quad Q_t = (1-C_t)Q_t^* + BQ_{t-1} + B(kq_{t-1,k} - Q_{t-1}).$$

Here Q_t^* is the sum of the k monthly values of q_t^*; $q_{t-1,k}$ is consumption in the last month of the previous year; C_t is a function of time.

In estimating a static demand function with annual data, the first term on the right hand-side of equation 3.18 is the equation we are estimating. Let $Q_{t-1} = x_{2t}$ and $kq_{t-1,k} - Q_{t-1} = Q_{t-1,k} - Q_{t-1} = x_{3t}$. Then in terms of equations 3.15 and 3.16, we have

$$r_{22} = r(Q_{t-1}; Q_{t-2})$$

$$(3.19) \quad r_{33} = r(Q_{t-1,k} - Q_{t-1}; Q_{t-2,k} - Q_{t-2})$$

$$r_{23} = r(Q_{t-1}; Q_{t-2,k} - Q_{t-2})$$

$$r_{32} = r(Q_{t-2}; Q_{t-1,k} - Q_{t-1}).$$

In this particular situation, it is possible that the presence of trend and seasonal components would cause r_{23} and r_{32} to be large enough to satisfy inequality 3.16.

This argument is admittedly oversimplified. For one thing, it takes no account of sampling variation; sample estimates of equations 3.11 and 3.14 may be nonsignificant, even though quite large. If we equate 3.14, not to zero, but to some number ρ, then $[A_2^2 \Sigma x_{2t}^2 \Sigma x_{3t}^2 (r_{23}^2 + r_{32}^2 + 2r_{23}r_{32} - 4r_{22}r_{33}) + 4\rho r_{33} \Sigma x_{3t}^2]^{1/2}$ needs to be non-negative. This term will more frequently be non-negative than will the corresponding term in equation 3.15. Although oversimplified, we may have here the basic explanation of why the addition of the lagged dependent variable to an equation sometimes introduces autocorrelation into the errors, although the addition of other variables rarely introduces autocorrelation.

Effect of A.L.S.-2 Estimation

The previous section presented comparisons between L.S. and A.L.S. results. This section presents a few comparisons between A.L.S.-1 and A.L.S.-2 results.

Twenty-three equations were selected for estimation under the assumption of second-order autoregressive errors. The equations were selected because there was reason to expect the existence of second-order autoregression.

In 13 of these 23 equations, estimated β_2 was significant at the 5-percent level; in 8, estimated β_2 was nonsignificant. The other two possessed multiple minima, with the residual sum of squares being nearly identical at the two minima. Each minimum corresponded to a different set of estimates of the parameters. Since it was not possible to select either set of estimates as superior, these two equations are excluded from further comparisons.

The differences between A.L.S.-1 and A.L.S.-2 estimates were much smaller than the differences between A.L.S.-1 and L.S. estimates. Define $A\triangle_i$ as

$$A\triangle_i = \frac{|\text{A.L.S. 1 coefficient} - \text{A.L.S. 2 coefficient}|}{\text{A.L.S. 1 standard error}}$$

One-fourth of the values of $A\triangle_i$ exceeded unity; 6 percent exceeded two. By contrast, half of the values of $\triangle_i$ exceeded unity and one-fourth exceeded two. Nearly all coefficients for which $A\triangle_i > 1.0$ were significant or nonsignificant under both types of estimation. Under A.L.S.-2, 16 percent of the coefficients had a significance status at the 5-percent level that was different from the significance status under A.L.S.-1; for one-third of these, $A\triangle_i > 1.0$. By contrast, 30 percent of the A.L.S.-1 coefficients had a significance status at the 5-percent level different from the L.S. coefficients. For two-thirds of these, $\triangle_i > 1.0$.

It appears that, in general, the results of an econometrician who assumes first-order autoregressive errors will not suffer appreciably even if the errors follow a second-order autoregressive process. This still leaves the possibility that the errors are generated by a moving-average process.

When an equation was estimated by A.L.S.-1 and A.L.S.-2, the sum of the A.L.S.-2 estimates of β_1 and β_2 almost invariably was within a few percent of the A.L.S.-1 estimate of β_1.

Multiple Minima

In the previous section, two equations were mentioned in which multiple minima were encountered. In nonlinear regression problems, which is what we have in the case of autoregressive errors, this possibility of multiple minima exists. The existence of multiple minima means that there are two (or more) local minima in the residual sum of squares (two or more local maxima in the likelihood function).

In the 17 equations re-estimated by Hildreth and Lu (23), no examples of multiple minima were encountered. In our applications of A.L.S., 21 separate equations were selected at random for investigation for the existence of multiple minima. These were selected at random, not in the sense of random sampling, but in the sense that there was no *a priori* reason for expecting multiple minima to be more or less likely in these than in other equations. The procedure was to select two different start vectors for the initiation of A.L.S. The fact that two different start vectors converge to the same solution is, of course, no assurance that a third start vector would have converged to the same solution. We

would expect that most cases of multiple minima would be found by the use of two sufficiently different start vectors, however.

Of the 21 equations, 4 had dual minima. The L.S. estimates of two of these equations had only 13 degrees of freedom (15). The dual minima might have disappeared with more degrees of freedom. The L.S. estimates of the other two equations, however, had about 35 degrees of freedom. In these two equations, the dependent variable was quarterly seasonally adjusted department-store inventories; the independent variables were also seasonally adjusted. One equation contained a time trend; the other did not.

Of a total of 38 equations (Hildreth and Lu's 17, plus 21 A.L.S.) 15 contained y_{t-1}, 23 did not. Of the 15 containing y_{t-1}, 4 possessed dual minima; of the 23 not containing y_{t-1}, none possessed dual minima. Evidently multiple minima are rare in equations not containing y_{t-1} and not so rare in equations containing y_{t-1}.

SUGGESTIONS FOR FURTHER WORK

To evaluate the adequacy of the work reported here and to consider possible future work, it is useful to conceive of a population of economic equations, all acceptable on the grounds of prior knowledge. In this study, interest centered on the temporal dependence properties of the errors in equations from this population. It may be more realistic to conceive of various populations of *a priori* acceptable equations. This report then covers samples from five such populations: (1) the population of annual and quarterly national aggregate food demand equations; (2) the population of monthly and quarterly Michigan consumer panel food demand equations; (3) population of national consumers' durable goods demand equations; (4) population of farmers' factor demand equations; and (5) population of farmers' product supply equations. Samples of 40, 50, 15, 15 and 15 equations, respectively, were drawn from these populations. (The remainder of the equations are from a variety of other populations.) These cannot be considered as random samples of independent observations since in many cases the results from one equation suggested additional equations.

It may, however, be useful to assume these to be random samples of independent drawings. The last two columns in table 22 are computed on this assumption. On this assumption the values of p from the first two populations are barely significantly different from each other at the 5-percent level. The results are consistent with the hypothesis that the true value of p is

Table 22. Statistics computed on assumption equations represent random samples of independent items.

Population of equations	p = Proportion of sample equations with significant values of β	Standard error of p	95-percent confidence interval for p
	(percent)	(percent)	(percent)
National aggregate food demand	62	8	46 to 78
Consumer panel food demand	40	7	26 to 54
Consumer durables demand	67	12	41 to 93
Farm factor demand	67	12	41 to 93
Farm supply	67	12	41 to 93

more than 0.5 in each population; i.e., that the errors in more than half of the equations from these populations do possess significant autocorrelation when tested by A.L.S. This is not the same thing as saying that half or more of the equations from these populations do possess autocorrelated errors. When these results are combined with the findings of Cochrane and Orcutt (6), Hildreth and Lu (23), Orcutt (36), and Wold (59), however, we do have sufficient evidence for concluding that autocorrelated errors are common.

Further work on autocorrelated errors is needed. It would be desirable to investigate possible modifications of the Theil-Nagar d test for application to equations not containing y_{t-1} to reduce the frequency of Type I errors. Research, perhaps using the Monte Carlo technique, is needed to study the small sample properties of A.L.S., Hildreth and Lu (23), Durbin (10), and Klein (25, pp. 85-89) estimates of equations containing autocorrelated errors. Similar work is needed on A.T.S. and Sargan (38) estimates of systems of equations containing autocorrelated errors.

A third problem which seems to merit further work arises from the existence of multiple minima in equations containing y_{t-1}. In 4 out of 15 such equations examined, multiple minima were encountered. The question of multiple minima in equations not containing y_{t-1} seems less serious. No cases of multiple minima were encountered in the examination of 23 such equations. If we assume that these represent independent random drawings from a binomial population, we can derive certain limits. Let p represent the probability of occurence of multiple minima and let success represent a case of multiple minima. What is the largest value of p such that, in a sample of 23 items, the probability of zero successes will be greater than or equal to 5 percent? Application of the binomial formula yields a maximum value of p of 0.12. The value of p which makes the probability of zero successes greater than or equal to 20 percent is 0.07.

APPENDIX: ESTIMATION PROCEDURES

Adequate discussions of T.S.L.S. and A.L.S. can be found elsewhere (2, 15, 16, 24, 30). They will be briefly summarized here only to lay the groundwork for presenting the A.T.S. method, which is a synthesis of the two.

The T.S.L.S. procedure is as follows, where Y_t is a $T \times M$ matrix of M endogenous variables, A is an $M \times M$ matrix of coefficients, Z_t is a $T \times N$ vector of N predetermined variables, Γ is an $N \times M$ matrix of coefficients, and ϵ_t is a $T \times M$ matrix of disturbances.

The system of equations is

(A.1) $\quad Y_t A = Z_t \Gamma + \epsilon_t.$

Suppose the equation in which we are interested is the first equation,

(A.2) $\quad y_{1t} = Y_{*t} A_1 + Z_{*t} \Gamma_1 + \epsilon_{1t} = X_{1t} A_{1*} + \epsilon_{1t}.$

The first step is to compute the least squares estimates,

(A.3) $\quad P = (Z_t' Z_t)^{-1} Z_t' Y_{*t}$

and

(A.4) $\quad \text{est } Y_{*t} = Z_t P.$

Estimates of A_{1*} are obtained from

(A.5) $\quad \text{est } A_{1*} = \begin{bmatrix} P'Z_t'Z_t P & P'Z_t'Z_{*t} \\ Z_{*t}'Z_t P & Z_{*t}'Z_{*t} \end{bmatrix}^{-1}$

$\begin{bmatrix} P'Z_t'y_{1t} \\ Z_{*t}'y_{1t} \end{bmatrix} = (X_{*t}'X_{*t})^{-1} X_{*t}'y_{1t}.$

Standard errors are computed from

(A.6) $\quad V(A_{1*}) = (X_{*t}'X_{*t})^{-1} \dfrac{e_t'e_t}{T\text{-}N_*\text{-}M_*}$

where

(A.7) $\quad e_t = y_{1t} - Y_{*t}(\text{est } A_1) - Z_{*t}(\text{est } \Gamma_1)$

and N_* and M_* are the number of predetermined and endogenous variables in equation A.2.

Let the equation we want to estimate by A.L.S.-1 be

(A.8) $\quad y_{1t} = Z_{*t}\Gamma_1 + \epsilon_{1t}$ where

(A.9) $\quad \epsilon_{1t} = \beta_1 \epsilon_{1t-1} + u_{1t} =$
$\qquad \beta_1 y_{1t-1} - \beta_1 Z_{*t-1}\Gamma_t + u_{1t}.$

Then u_{1t} can be written

(A.10) $\quad u_{1t} = y_{1t} - (y_{1t-1}, Z_{*t}, Z_{*t-1})(\beta, \Gamma_1, -\beta\Gamma_1)'$
$\qquad = y_{1t} - X_t C.$

Expand u_{1t} in a Taylor's series about a set of initial estimates of the coefficients, $\pi_i = (\beta_i, \Gamma_{1i})$, ignoring all terms of higher order than the first.

(A.11) $\quad f_1 = u_{1it} - (y_{t-1}, Z_{*t}, Z_{*t-1})$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\Gamma_{1i} & -\beta_i \end{pmatrix} \begin{bmatrix} \triangle\beta_i \\ \triangle\Gamma_{1i} \end{bmatrix}$

$= u_{1it} - X_t A_i \triangle\pi_i$

where u_{1it} is obtained from (A.10) by substituting the elements of π_i into C to obtain C_i.

Taking the partial derivatives of $f_{1t}'f_{1t}$ with respect to $\triangle\pi_i'$ and equating to zero, we obtain

(A.12) $\quad \triangle\pi_i = (A_i'X_t'X_t A_i)^{-1} A_i'X_t'(y_{1t} - X_t C_i).$

$\triangle\pi_i$ is the least squares solution of the regression,

(A.13) $\quad \text{est } u_{1it} = X_t A_i \triangle\pi_i + w_t.$

From $\triangle\pi_i$ compute

(A.14) $\quad \pi_{i+1} = \pi_i + k_i \triangle\pi_i$

where the value of k_i may be selected in various ways to assure convergence. In the IBM program used in this study, k_i is selected as the largest value of 0.5^j, $j = 0, 1, \ldots$, which yields a reduction in the residual sum of squares. This process is continued until the difference between successive estimates is satisfactorily small. Here, it was continued until every coefficient in the equation met the t-square test,

(A.15) $\quad t^2 = \dfrac{(\triangle P_j)^2}{V(P_j)} < 0.001$ for all j,

where $V(P_j)$ is the estimated variance of the j-th coefficient. The matrix of variances and covariances of the coefficients is obtained from

(A.16) $\quad (A_i'X_t'X_i A_i)^{-1} \dfrac{(\text{est } u_{1t})'(\text{est } u_{1t})}{T\text{-}N\text{-}1},$

where $N + 1$ is the total number of variables in the equation.

In our system of equations A.1, suppose ϵ_t follows the first-order autoregressive scheme.

(A.17) $\quad \epsilon_t = \epsilon_{t-1}\beta + u_t = (Y_{t-1}A - Z_{t-1}\Gamma)\beta + u_t,$

where β is a diagonal matrix.

A.T.S. proceeds as follows. Obtain L.S estimates of p_1, p_2 and p_3 in

(A.18) $\quad \text{est } Y_t = (\text{est } Y_{*t} \text{ est } Y_{**t})$
$\qquad = Z_t p_1 + Y_{t-1} p_2 + Z_{t-1} p_3.$

Substituting for ϵ_{1t}, the first equation A.2 can be written

(A.19) $\quad y_{1t} = \beta_1 y_{1t-1} + (Y_{*t} - Y_{*t-1}\beta_1)A_1 +$
$\qquad (Z_{*t} - Z_{*t-1}\beta_1)\Gamma_1 + u_{1t}.$

Table A-I. Number of iterations required for convergence.

Number of iterations required for convergence	Number of equations
2	1
3	10
4	16
5	13
6	6
7	10
8	6
9	4
10	4
11-18	7
21	1
24	1
25	1
39	1
Total	81

Mean, 7.4

Median, 6

Estimate by A.L.S. the coefficients in

$$(A.20) \quad y_{1t} = \beta_1 y_{1t-1} + (\text{est } Y_{*t} - Y_{*t-1}\beta_1) A_1$$
$$+ (Z_{*t} - Z_{*t-1}\beta_1)\Gamma_1 + u_{1t}.$$

At the end of each iteration, compute the variances and covariances as the product of the elements of the inverse matrix and $(\text{est } u_{1it})'(\text{est } u_{1it})/\text{T-N-1}$ where

$$(A.21) \quad \text{est } u_{1it} = y_{1t} - \beta_{1i}y_{t-1} - (Y_{*t} - Y_{*t-1}\beta_{1i})$$
$$A_{1i} - (Z_{*t} - Z_{*t-1}\beta_{1i})\Gamma_{1i}.$$

A.L.S. is an iterative technique, and the extra cost of using it over using L.S. is determined by the number of iterations required for convergence to a solution.

Table A-1 presents the number of iterations required for convergence in 81 equations estimated by A.L.S. The number of iterations required was not affected by the number of independent variables or by the presence or absence of the lagged dependent variable. In every case in which an equation was estimated by A.L.S.-1 and A.L.S.-2, A.L.S.-2 required fewer iterations.

Commonly, when a large number of iterations was required for a stable solution, changes in the coefficients were alternately positive and negative and declining in absolute magnitude from one iteration to the next. This type of oscillation could usually be stopped and convergence obtained rather quickly (usually in one or two iterations) by taking the averages of the solutions from two successive iterations as an estimate of the coefficients.

The number of iterations required is affected by how close the initial set of estimates is to the final solution. For equations not containing the lagged dependent variable, the initial estimate of β_1 was almost always computed as $(2-d)/2$, where d was obtained from the L.S. estimate of the equation. The L.S. estimates of the coefficients were almost always used as the initial estimates of the other coefficients. For equations containing the lagged dependent variable, different procedures were used. Sometimes the initial estimates of β_1 and the other coefficients were taken directly from the L.S. equation. Other times, when d was highly significant, the initial estimate of the lagged dependent variable was taken from the L.S. estimate of the equation, and initial estimates of β_1 and other coefficients were taken from L.S. or A.L.S. estimates of the corresponding equation which did not contain the lagged dependent variable.

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