Incomplete Block Designs With Blocks of Two Plots

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CONTENTS

SUMMARY

Various workers have shown that considerable economies in experimentation can be achieved by the use of identical twins, halves of leaves and halves of plants. The basic statistical principle enabling these economies is that of incomplete block designs. These make possible the comparison of treatments within pairs of identical twins, pairs of half-leaves or whatever grouping is used, so that experimental error arises only because of differences between individuals within the pairs.

Extensive use of these natural groupings is possible only when the possible incomplete block designs have been enumerated. The purpose of this bulletin is to present what is essentially a complete array of incomplete block designs with blocks of two plots.

The balanced incomplete block design requires $(n\text{-}1)$ replications, where n is the number of treatments. Any design using a lesser number of replications must necessarily be unbalanced. Designs are presented which require $(n\text{-}2)$, $(n\text{-}3)$, . . . , three replications, whenever such is possible, for any number of treatments between 6 and 12.

These designs have been extracted from a pool of designs formed by (a) the class of partially balanced incomplete block designs with two classes of associates and (b) the class of circulant designs. The design chosen to cover each situation is the design which has the highest efficiency factor among all the designs in the pool applicable to the given situation.

The tabulation of material necessary for the analysis of the respective designs is made to facilitate computations.

The computations are exemplified by the inclusion of a worked example covering both the intrablock and interblock analysis.

Incomplete Block Designs With Blocks of Two Plots[1]

By J. A. Zoellner[2] and O. Kempthorne[3]

A basic principle of statistical experimentation is the utilization of blocks or groups of experimental units within which the treatments are applied at random. Treatment differences are then estimated by comparisons within the groups and are subject to an error variance depending on the variance of the units within the groups. The experimenter in desiring the greatest possible accuracy in the estimation of treatment differences attempts to secure those groups or blocks of experimental units which possess the smallest internal variance. Accumulated evidence in recent years has indicated sharply the importance of considering natural grouping such as litters of pigs for this purpose.

Randomized complete block designs represent the earliest statistically valid method of utilizing blocks or groups of experimental units. These designs require that the size of the block be equal to the number of treatments being compared. As long as naturally occurring groups fit the structure of these designs, i.e., the size of the naturally occurring group is greater than or equal to the required block size, the problem is solved.

However, the experimenter is frequently confronted with naturally occurring groups which are smaller in size than the number of treatments under investigation. While we may find natural groups in a wide variety of sizes, one of the most common group sizes is two. Examples are numerous and varied. The value of twins in psychometry has long been realized and attention has been directed lately at their use in biology (8). The botanist has demonstrated the use of halves of leaves and more recently has employed halves of plants in experimentation (3). Paired organs such as eyes and kidneys are examples of this large class of groups.

The purpose of this bulletin is to present to the experimenter the best available designs for blocks of two plots.

The statistical utilization of naturally occurring groups started with the introduction of the balanced incomplete block designs in which each treatment occurs with every other treatment in a constant number of blocks and every treatment is repeated r times. A balanced design in blocks of two plots for testing n treatments requires $n(n-1)/2$ blocks, since each treatment is to occur with every other treatment once in a block. As n becomes large, the number of blocks required increases rapidly so that in many cases the experimenter may not be able to obtain the necessary number of blocks. Nor may he desire to have as extensive replication as balanced incomplete blocks require.

When this situation arises, the only alternative left to the experimenter, apart from disregarding the occurrence of the groups altogether, is the use of an unbalanced incomplete block design. The unbalance arises from the fact that if a design in blocks of two plots has less than $n(n-1)/2$ blocks some treatments will not occur together in a block. As a whole, the designs are both difficult and their general analysis is tedious. However, within this large class of unbalanced incomplete block designs there are subclasses which possess both ease of analysis and construction. Two of these subclasses were considered in the present case. They are the class of partially balanced incomplete block designs (2) and the class of circulant designs (4,9).

The enumeration covers designs for 6 to 12 treatments for which the number of degrees of freedom for estimating the error is at least five.

REVIEW OF LITERATURE

EFFECTIVENESS OF THE USE OF BLOCKS OF TWO PLOTS

Denote by σ^2_w the within pair variance and by σ^2_b the between pair variance on a per-plot basis. Then the analysis of variance in table 1 is applicable to blocks of two plots.

TABLE 1. ANALYSIS OF VARIANCE.

Due to	d. f.	M. S.	E. M. S.
Between blocks	b-1	B	$\sigma^2_w + 2\sigma^2_b$
Within blocks	b	W	σ^2_w

[1] Project 890 of the Iowa Agricultural Experiment Station. Based partially on a thesis submitted to the graduate faculty for the M. S. degree by J. A. Zoellner. The authors are indebted to Mr. R. F. White for checking the whole of the computations.

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If blocks are made up at random from unrelated units, the variance of a treatment comparison will be proportional to

$$\sigma_w^2 + \sigma_b^2$$

as compared to σ^2_w with blocks of related units. Hence the error variance of the design relative to blocks of unrelated units is

$$\frac{\sigma_w^2 + \sigma_b^2}{\sigma_w^2} = 1 + \frac{\sigma_b^2}{\sigma_w^2}$$

which is estimated by

$$\frac{B+W}{2W}.$$

The reduction in error variance does not, of course, give the relative efficiency of the incomplete block design, because the incomplete block design necessitates confounding. The effect of confounding is to make the relative efficiency equal to the relative error variance multiplied by the efficiency factor (E. F.). In the case of designs with blocks of two plots, the efficiency factor will be close to 0.5, so that as a rough guide to the value of an incomplete block design in blocks of two plots we may use the quantity R, where

$$R = \frac{B+W}{4W}.$$

Paul (7) reporting on a taste testing experiment using blocks of two plots gave data leading to an R value of 1.05. Using halves of plants physically disected, James and Bancroft (3) have presented data yielding an estimated R of 1.58.

Stormont (8) reports data which lead to the estimates of R given in table 2 for the case of monozygotic twins in cattle.

As a result of these findings, heavy emphasis has been placed in New Zealand on the utilization of monozygotic twins. In investigations concerning milk production, for example, the information obtainable through the use of an incomplete block design having an E. F. of 0.5 would be roughly equivalent at least to the information obtainable from a randomized complete block design utilizing as many as 11 times as many unrelated animals.

The effect of confounding in reducing relative efficiency is partly offset by the utilization or recovery of interblock information. The interblock information can be utilized or recovered when it is possible to obtain a reliable estimate of

$$\left(\sigma_w^2 + 2\sigma_b^2\right),$$

in other words, when the number of degrees of freedom for blocks eliminating treatments is not small, say 15 or more.

TABLE 2. ESTIMATES OF R VALUES FOR MONOZYGOTIC TWINS IN CATTLE.

Characteristic studies	R	No. of animals
Milk production	11.5	7
Butterfat production	27.5	7
Casein production	25.5	7
Butterfat percentage	8	7
Casein percentage	5.5	7
Grazing time	36.5	6
Loafing time	3.5	6
Body weight	13.5	10

LITERATURE ON DESIGNS

The class of partially balanced incomplete block designs was first introduced by R. C. Bose and K. R. Nair (1) in 1939. Later, Nair and Rao (6) and finally Bose and Shimamoto (2) expanded the class to its present form.

The definition of partially balanced incomplete block designs is rather abstract from the point of view of the experimenter. The essential feature is that the treatments are partitioned into m classes, called associate classes, with respect to each treatment. For example, treatments 2 and 3 may be the first associates of treatment 1, treatments 4 and 5 may be second associates of treatment 1, and so on. The classification of treatments in this manner must satisfy several conditions which are specified, for example, by Bose and Shimamoto (2). In general, the number of associate classes may be any number up to $(n-1)$, but enumerative work in the literature has been concentrated towards finding designs with two associate classes (2).

In 1952, Kempthorne (4) introduced a class of designs for use with blocks of two plots. A design in this class is possible if the quantity $(n-r+1)/2$, which we may denote by s, is integral, where n is the number of treatments and r is the number of times each treatment is represented. The structure of the design is such that treatments are numbered from 1 to n and treatment given the number j occurs with treatments with numbers $j+s$, $j+s+1$, . . . , $j+s+r-1$ once in a block, where any number greater than n is reduced to a number less than or equal to n by subtracting an appropriate multiple of n. The characteristic property these designs possess is that the coefficient matrix of the reduced normal equations (5) is a circulant.

A circulant is a square array or matrix of elements of such a nature that given any row of the array, the next row below can be obtained by shifting the given row one space to the right and placing the last element of the row in the first position, e.g.

a	b	c	d
d	a	b	c
c	d	a	b
b	c	d	a

is a circulant.

The computational methods developed with the introduction of these designs make for ease of analysis. Zoellner (9) later extended this class of designs to include all designs in which the coefficient matrix of the reduced normal equations is a circulant.

A design is said to be an element of the class of circulant designs if it satisfies the following requirements:

(a) Each treatment is repeated the same number

of times and occurs only once in any block.

(b) The number of times treatment j occurs with j' is $\lambda_{jj'}$ and the matrix of $\lambda_{jj'}$'s with $j \neq j'$, is a circulant.

The circulant designs are a subclass of the class of partially balanced incomplete block designs, which is in turn a subclass of the class of all incomplete block designs.

SCOPE OF ENUMERATION

A balanced incomplete block design for blocks of two plots with n treatments requires $n(n-1)/2$ blocks and every treatment occurs once with every other treatment. The construction and analysis of these designs are straightforward and can be found in standard texts (see, for example, reference 5). We are concerned with the enumeration of the best designs for the case where less than $n(n-1)/2$ blocks are available.

A complete enumeration of all the partially balanced incomplete block designs with two classes of associates was carried out for the case when $\lambda_{jj'}$ takes either the value $\lambda_1=1$ or the value $\lambda_2=0$, as well as a complete enumeration of all the possible circulant designs for which $\lambda_{jj'}$ is zero or unity. From this pool of designs, the design with the highest efficiency factor was chosen and tabulated. Whenever a partially balanced and a circulant design possessed the same efficiency factor, the circulant design was tablated.

DESCRIPTION OF THE TABLES

Every design except one, a design for 10 treatments in three replications for which table 4a has been prepared, is tabulated as a circulant design. The circulant designs and their characteristics are presented in three tables at the end of the bulletin. Table 4 indicates the plan for the design, table 5 the coefficients c_{ij} needed to obtain the intrablock estimates of the treatment effects and table 6 supplemental data useful in obtaining the combined intrablock and interblock estimates of the treatment effects.

BLOCK PLAN

The pairs of treatments which make up the blocks are specified by the fact that if treatment 1 occurs with treatments j, k, etc. then treatment 2 occurs with treatments $j+1$, $k+1$, etc., treatment 3 occurs with treatments $j+2$, $k+2$, etc. and so on. Hence it is necessary to specify only those treatments which occur with treatment 1, and this is done in table 4. For example, the design for seven treatments with four replications is specified by

* 1 1 0 0 1 1

so that treatment 1 occurs with treatments 2, 3, 6 and 7. Hence we have

Treatment	Occurs with treatments
1	2, 3, 6, 7
2	3, 4, 7, 1
3	4, 5, 1, 2
4	5, 6, 2, 3
5	6, 7, 3, 4
6	7, 1, 4, 5
7	1, 2, 5, 6

The blocks are therefore

1	1	1	1	2	2	2	3	3	4	4	5	5	6
2	3	6	7	3	4	7	4	5	5	6	6	7	7

INTRABLOCK ESTIMATES

Intrablock estimates of the treatment effects are given by the n equations

$$\hat{\tau}_j = \sum_k c_{jk} Q_k, \quad j = 1, 2, \ldots n$$

for example,

$$\hat{\tau}_3 = c_{31} Q_1 + c_{32} Q_2 + \ldots + c_{3n} Q_n,$$

where the c_{jk} are derivable from the form of the reduced normal equations and,

$$Q_k = V_k - \frac{T_k}{2}$$

where V_k = sum of those plots receiving treatment t_k, and T_k = sum of block totals for blocks containing t_k. The sum of the Q_k is zero, so that an arbitrary constant can be subtracted from every element in any row. The equations can be written in the matrix form;

$$\begin{bmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\tau}_n \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ \cdot \\ Q_n \end{bmatrix}$$

or symbolically as

$$\hat{\tau} = C\,Q.$$

If a design is a circulant design, the matrix C is also circulant so that it is specified completely by its first row or column. By use of the reference numbers in table 4, the matrices corresponding to each design can be found in table 5, which contains the first row, written vertically, for each design. In certain cases, moderately simple estimating equations in terms of associate classes could be given, but the present form is simpler to present and use for the designs given.

The variance associated with a treatment difference based on intrablock information is given by:

$$V(\hat{\tau}_j - \hat{\tau}_{j'}) = 2\sigma^2 (c_{jj} - c_{jj'}), \text{ for all } j \text{ and } j'$$

and the efficiency factor of these designs is

$$\frac{(n-1)}{nr\ c_{11}}.$$

COMBINED ESTIMATES

Table 6 is for use when estimates based on both intrablock and interblock information are desired. By use of the reference numbers, the characteristic roots $\Lambda_j (j=2, 3, \ldots n)$ of the coefficient matrix of the reduced normal equations can be found. To obtain the combined estimates, it is necessary to make the following computations based on the estimated weights W and W' corresponding to intrablock and interblock estimates respectively. The detailed procedure for the estimation of W and W' by means of an analysis of variance is given below. A new matrix with elements c^*_{11}, c^*_{12}, etc. is required; this matrix will be called the c^* matrix.

Compute

$$\Lambda^*_j = (W-W')\Lambda_j + rW',\ j=2, 3, \ldots, n,$$

where the Λ_j are the roots given in table 6. The coefficients of the c^* matrix are given by

$$c^*_{11} = \frac{1}{n}\sum_{k=2}^{n}\frac{1}{\Lambda^*_k} = \frac{1}{n}\left(\frac{1}{\Lambda^*_2}+\frac{1}{\Lambda^*_3}+\ \ldots\ +\frac{1}{\Lambda^*_n}\right)$$

and

$$c^*_{1j} = \frac{1}{n}\sum_{k=2}^{n}\frac{\cos(j-1)(k-1)\frac{2\pi}{n}}{\Lambda^*_k},\ j=2, 3, \ldots, n;$$

for example,

$$c^*_{13} = \frac{1}{n}\left[\frac{\cos\frac{4\pi}{n}}{\Lambda^*_2}+\frac{\cos\frac{8\pi}{n}}{\Lambda^*_3}+\ \ldots\ +\frac{\cos(n-1)\frac{4\pi}{n}}{\Lambda^*_n}\right].$$

Actually $\Lambda_1=0$, so $\Lambda^*_1=rW'$, and the true c^*_{ij}'s are equal to

$$c^*_{ij}+\frac{1}{n\,\Lambda^*_1}.$$

However, since ΣQ^*_k equals zero, the part $1/n\Lambda^*_1$ in the true c^*_{ij}'s may be omitted.

As before the c^* matrix is a circulant.

The combined estimates are

$$\tilde{\tau}_j = \sum_k c^*_{jk} Q^*_k,\ j=1,\ 2,\ \ldots,\ n$$

where

$$Q^*_k = WQ_k + \frac{W'}{2}\left(T_k - \frac{r\,Y..}{b}\right)$$

where $Y..$ = total of all observations

r = number of replications

b = number of blocks.

The estimated variance of a treatment difference with the recovery of interblock information is

$$V(\tilde{\tau}_j - \tilde{\tau}_{j'}) = 2(c^*_{jj} - c^*_{jj'}).$$

ANALYSIS OF VARIANCE AND ESTIMATION OF THE WEIGHTS

The form of the analysis of variance applicable to the designs together with the alternative breakdown necessary for the recovery of interblock information is given in table 3

TABLE 3. ANALYSIS OF VARIANCE.

Due to	d.f.	S. Sq.	S. Sq.	Due to
Blocks ignoring treatments	b−1	B	B′	Blocks eliminating treatments
Treatments eliminating blocks	n−1	T	T′	Treatments ignoring blocks
Intrablock error	b−n+1	S_E	S_E	Intrablock error
Total	2b−1	S	S	Total

where

B = ½{sum of squares of block totals}−correction

$T = \sum_j \hat{\tau}_j Q_j$

$\hat{\tau}_j$ = intrablock estimates of the treatment effects

T' = {sum of squares of treatment totals}/r −correction

S = Total sum of squares−correction

Correction = [sum of all observations]^2/2b

$S_E = S-B-T$

$B' = S-S_E-T'$

$$\hat{W} = \frac{b-n+1}{S_E},\ \text{and}\ \hat{W}' = \frac{2b-n}{2B'-\left(\frac{n-2}{b-n+1}\right)S_E}.$$

The mean variance of treatment differences as given by the intrablock estimates is equal to

$$2\,\frac{n}{n-1}\,c_{11}\,(\sigma^2)$$

which is estimated by

$$2\,\frac{n}{n-1}\,c_{11}\left(\frac{S_E}{b-n+1}\right).$$

The mean variance of treatment differences as given by the combined estimates is equal to

$$2\,\frac{n}{n-1}\,c^*_{11}.$$

If these average variances differ little, say by less than 5 percent, the experimenter need only obtain the intrablock estimates.

A WORKED EXAMPLE

A fictitious example will serve to illustrate the computational procedures. The design considered is the one for testing seven treatments in four replications which appears in the tables under reference number 2.

The data are:

Block	Treatments (X)		Block totals
1	54(1)	56(2)	110
2	35(1)	36(3)	71
3	48(1)	42(6)	90
4	46(1)	56(7)	102
5	61(2)	61(3)	122
6	52(2)	53(4)	105
7	54(2)	59(7)	113
8	45(3)	46(4)	91
9	31(3)	28(5)	59
10	56(4)	53(5)	109
11	36(4)	40(6)	76
12	42(5)	43(6)	85
13	56(5)	59(7)	115
14	61(6)	54(7)	115
			1,363

The following table computed:

V_j	T_j	$Q_j = V_j - T_j/2$
183(1)	373	−3.5
223(2)	450	−2.0
173(3)	343	1.5
191(4)	381	0.5
179(5)	368	−5.0
186(6)	366	3.0
228(7)	445	5.5
1,363	2,726	0.0

As a check on the computations, we should have

$$\Sigma V_j = \tfrac{1}{2} \Sigma T_j$$

and

$$\Sigma Q_j = 0.$$

The coefficient matrix C is taken from table 5. Rounding to four places we have

$$\begin{aligned} c_{11} &= 0.3956 \\ c_{12} &= -0.0220 \\ c_{13} &= -0.0440 \\ c_{14} &= -0.1319 \\ c_{15} &= -0.1319 \\ c_{16} &= -0.0440 \\ c_{17} &= -0.0220 \end{aligned}$$

the matrix C can be written out:

$$\begin{array}{ccccccc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} \\ c_{12} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{13} & c_{12} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{14} & c_{13} & c_{12} & c_{11} & c_{12} & c_{13} & c_{14} \\ c_{15} & c_{14} & c_{13} & c_{12} & c_{11} & c_{12} & c_{13} \\ c_{16} & c_{15} & c_{14} & c_{13} & c_{12} & c_{11} & c_{12} \\ c_{17} & c_{16} & c_{15} & c_{14} & c_{13} & c_{12} & c_{11} \end{array}$$

Retaining the position of the various elements in the array, we renumber the elements of the matrix C in the standard way, i.e. as follows:

$$\begin{array}{ccccccc} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} \\ c_{21} & c_{22} & c_{23} & . & . & . & c_{27} \\ c_{31} & c_{32} & c_{33} & . & . & . & c_{37} \\ c_{41} & . & . & . & . & . & . \\ c_{51} & . & . & . & . & . & . \\ c_{61} & . & . & . & . & . & . \\ c_{71} & c_{72} & c_{73} & . & . & . & c_{77} \end{array}$$

so that the proper elements are chosen for the estimating equations

$$\hat{\tau}_j = \sum_k c_{jk} Q_k, \quad j = 1, 2, \ldots, n.$$

The computed estimates in this case are:

$$\begin{aligned} \hat{\tau}_1 &= -1.0660 \\ \hat{\tau}_2 &= -0.7474 \\ \hat{\tau}_3 &= -0.1208 \\ \hat{\tau}_4 &= -0.0330 \\ \hat{\tau}_5 &= -1.6376 \\ \hat{\tau}_6 &= 1.3738 \\ \hat{\tau}_7 &= 2.2310 \end{aligned}$$

where $\Sigma \hat{\tau}_j = 0$ provides a final check on calculations.

To compute the analysis of variance we need:

Correction $= (1{,}363)^2/28 = 66{,}348.89$

Total sum of squares $= 68{,}839 -$ correction $= 2{,}490.11$

$$\text{Treatments ignoring blocks} = \frac{(183)^2 + \ldots + (228)^2}{4} - \text{correction} = 713.36$$

$$\text{Blocks ignoring treatments} = \frac{(110)^2 + \ldots + (115)^2}{2} - \text{correction} = 2{,}359.61$$

Treatments eliminating blocks $= \Sigma \hat{\tau}_j Q_j = 29.61$

Intrablock error $= 2{,}490.11 - 2{,}359.61 - 29.61 = 100.89$

Blocks eliminating treatments $= 2{,}490.11 - 100.89 - 713.36 = 1{,}675.86.$

The analysis of variance is:

Due to	d. f.	S. Sq.	S. Sq.	Due to
Blocks ignoring treatments	13	2,359.61	1,675.86	Blocks eliminating treatments
Treatments eliminating blocks	6	29.61	713.36	Treatments ignoring blocks
Intrablock error	8	100.89	100.89	Intrablock error
Total	27	2,490.11	2,490.11	Total

The estimated weights are:

$$\hat{W} = \frac{8}{100.89} = 0.0792943$$

and

$$\hat{W}' = \frac{21}{2(1{,}675.86) - (5/8)(100.89)} = 0.0063856.$$

For estimates based on both intrablock and interblock information we compute

$$\Lambda^*_j = (W - W')\Lambda_j + rW',$$

where the Λ_j's are tabulated in table 6. The quantities

$$Q^*_j = WQ_j + \frac{W'}{2}\left(T_j - \frac{r\,Y..}{b}\right)$$

are also needed. We find that

$\Lambda^*_2 = 0.1421257$ $\quad Q^*_1 = -0.3300$ $\quad c^*_{11} = 4.6297$

$\Lambda^*_3 = 0.2532720$ $\quad Q^*_2 = 0.0351$ $\quad c^*_{12} = -0.3412$

$\Lambda^*_4 = 0.1915905$ $\quad Q^*_3 = -0.0295$ $\quad c^*_{13} = -0.5339$

$\Lambda^*_5 = 0.1915905$ $\quad Q^*_4 = 0.0127$ $\quad c^*_{14} = -1.4397$

$\Lambda^*_6 = 0.2532720$ $\quad Q^*_5 = -0.4650$ $\quad c^*_{15} = -1.4397$

$\Lambda^*_7 = 0.1421257$ $\quad Q^*_6 = 0.1629$ $\quad c^*_{16} = -0.5339$

$\quad\quad\quad\quad\quad\quad Q^*_7 = 0.6138$ $\quad c^*_{17} = -0.3412$

Writing out the matrix of c^*_{ij} in the same manner as the matrix c_{ij} we find the following estimates of treatment effects.

$$\tilde{\tau}_1 = -1.1692$$
$$\tilde{\tau}_2 = 0.3856$$
$$\tilde{\tau}_3 = -0.8466$$
$$\tilde{\tau}_4 = -0.2868$$
$$\tilde{\tau}_5 = -2.1001$$
$$\tilde{\tau}_6 = 0.8648$$
$$\tilde{\tau}_7 = 3.1524.$$

The average variance of a treatment difference without recovery of interblock information is

$$2\sigma^2\, c_{11}\left(\frac{n}{n-1}\right) = 2\frac{(100.89)(0.3956)}{8}\left(\frac{7}{6}\right) = 11.64$$

while with recovery of interblock information the average variance is

$$2\, c^*_{11}\left(\frac{n}{n-1}\right) = 2(4.6297)\left(\frac{7}{6}\right) = 10.80.$$

A test of significance based on the combined estimates is given by the statistic

$$X^2 = \Sigma\hat{\tau}_j Q^*_j$$

which is distributed as Chi-square with $(n-1)$ degrees of freedom when the true weights W and W' are known. The Chi-square criterion may be used with estimated weights $\hat{W}$ and $\hat{W}'$ as an approximation if the degrees of freedom for blocks is large, say about 15. For small degrees of freedom, no exact test is known.

TABLE 4. DESIGNS.

Reference number	No. of treatments	No. of replications	Degrees of freedom for error	Efficiency factor	Plan of the design
1	6	4	7	0.576	*11011
2	7	4	8	0.542	*110011
3	8	6	17	0.560	*1110111
4	8	5	13	0.543	*1101011
5	8	4	9	0.538	*1010101
6	8	3	5	0.488	*1001001
7	9	6	19	0.545	*11011011
8	9	4	10	0.509	*10100101
9	10	8	31	0.549	*111101111
10	10	7	26	0.540	*111010111
11	10	6	21	0.525	*111000111
12	10	5	16	0.529	*101010101
13	10	4	11	0.500	*100101001
See table 4a	10	3			
14	11	8	34	0.538	*1111001111
15	11	6	23	0.521	*1101001011
16	11	4	12	0.487	*1010000101
17	12	10	49	0.541	*11111011111
18	12	9	43	0.537	*11101110111
19	12	8	37	0.532	*11011011011
20	12	7	31	0.524	*10101110101
21	12	6	25	0.524	*10101010101
22	12	5	19	0.502	*10010101001
23	12	4	13	0.479	*01100000110
24	12	3	7	0.394	*10000100001

TABLE 4a. PLAN AND FORMULA FOR DESIGN WITH 10 TREATMENTS AND 3 REPLICATES.

Plan 15 blocks:

1, 8; 1, 9; 1, 10; 2, 6; 2, 7; 2, 10; 3, 5; 3, 7; 3, 9; 4, 5; 4,6; 4, 8; 5, 10; 6, 10; 7, 8.

Formulas

Q_j: as in text

$\Sigma(Q_j)$ = sum of $Q_{j'}$'s for treatments j' which occur in a block with treatment j.

Similarly for Q^*_j and $\Sigma(Q^*_j)$

Intrablock estimates:

$$\hat{\tau}_j = \frac{1}{10}\left[8Q_j + 2\Sigma(Q_j)\right]$$

$$V(\hat{\tau}_j - \hat{\tau}_{j'}) = \frac{6\sigma^2}{5} \quad \text{if } j \text{ and } j' \text{ occur together in a block,}$$

$$= \frac{8\sigma^2}{5}, \quad \text{otherwise.}$$

Mean variance of differences: $7\sigma^2/5$

Combined estimates:

$$\hat{\tau}_j = \left[(8W+4W')\,Q^*_j + 2(W-W')\Sigma(Q^*_j)\right]/\Delta$$

where

$$\Delta = 3(W+W')(4W+2W') - 2(W-W')^2$$

$V(\hat{\tau}_j - \hat{\tau}_{j'}) = 12(W+W')/\Delta$ if j and j' occur together in a block

$= 4(4Q+2W')/\Delta$ otherwise.

Mean variance of treatment difference:

$(14W+10W')/\Delta$.

TABLE 5. COEFFICIENTS FOR INTRABLOCK ESTIMATES.

Reference Number				
1	2	3	4	5
0.3611111	0.3956044	0.2604167	0.3223039	0.4062500
—0.0555555	—0.0219780	—0.0312500	—0.0330882	—0.0312500
—0.0555555	—0.0439560	—0.0312500	—0.0208333	—0.0937500
—0.1388889	—0.1318681	—0.0312500	—0.0919118	—0.0312500
—0.0555555	—0.1318681	—0.0729167	—0.0306372	—0.0937500
—0.0555555	—0.0439560	—0.0312500	—0.0919118	—0.0312500
	—0.0219780	—0.0312500	—0.0208333	—0.0937500
		—0.0312500	—0.0330882	—0.0312500
6	7	8	9	10
0.5982143	0.2716049	0.4364364	0.2050000	0.2381818
0.0089285	—0.0246914	—0.0200200	—0.0200000	—0.0200000
—0.1875000	—0.0246914	—0.0920921	—0.0200000	—0.0163636
—0.1339286	—0.0617284	—0.0040040	—0.0200000	—0.0200000
0.0267857	—0.0246914	—0.1101101	—0.0200000	—0.0527272
—0.1339286	—0.0246914	—0.1101101	—0.0450000	—0.0200000
—0.1875000	—0.0617284	—0.0040040	—0.0200000	—0.0527272
0.0089285	—0.0246914	—0.0920921	—0.0200000	—0.0200000
	—0.0246914	—0.0200200	—0.0200000	—0.0163636
			—0.0200000	—0.0200000
11	12	13	14	
0.2858150	0.3400000	0.4500000	0.2113922	
—0.0061912	—0.0200000	0.0000000	—0.0136434	
—0.0138715	—0.0600000	—0.1000000	—0.0164925	
—0.0224922	—0.0200000	—0.1000000	—0.0165326	
—0.0665360	—0.0600000	0.0000000	—0.0168536	
—0.0676332	—0.0200000	—0.0500000	—0.0421741	
—0.0665360	—0.0600000	0.0000000	—0.0421741	
—0.0224922	—0.0200000	—0.1000000	—0.0168536	
—0.0138715	—0.0600000	—0.1000000	—0.0165326	
—0.0061912	—0.0200000	0.0000000	—0.0164925	
			—0.0136434	
15	16	17	18	
0.2908355	0.4670559	0.1694444	0.1898148	
—0.0113981	0.0183487	—0.0138889	—0.0138889	
—0.0060668	—0.0617181	—0.0138889	—0.0138889	
—0.0511076	0.0066722	—0.0138889	—0.0138889	
—0.0191194	—0.0884070	—0.0138889	—0.0324074	
—0.0577259	—0.1084237	—0.0138889	—0.0138889	
—0.0577259	—0.1084237	—0.0305556	—0.0138889	
—0.0191194	—0.0884070	—0.0138889	—0.0138889	
—0.0511076	0.0066722	—0.0138889	—0.0324074	
—0.0060668	—0.0617181	—0.0138889	—0.0138889	
—0.0113981	0.0183487	—0.0138889	—0.0138889	
		—0.0138889	—0.0138889	
19	20	21	22	
0.2152778	0.2500000	0.2916667	0.3653382	
—0.0138889	—0.0138889	—0.0138889	—0.0060386	
—0.0138889	—0.0416667	—0.0416667	—0.0603865	
—0.0347222	—0.0138889	—0.0138889	—0.0694444	
—0.0138889	—0.0416667	—0.0416667	0.0048309	
—0.0138889	—0.0138889	—0.0138889	—0.0495169	
—0.0347222	0.0000000	—0.0416667	—0.0042271	
—0.0138889	—0.0138889	—0.0138889	—0.0495169	
—0.0138889	—0.0416667	—0.0416667	0.0048309	
—0.0347222	—0.0138889	—0.0138889	—0.0694444	
—0.0138889	—0.0416667	—0.0416667	—0.0603865	
—0.0138889	—0.0138889	—0.0138889	—0.0060386	
23	24			
0.4781746	0.7745726			
—0.0734127	0.1463675			
0.0178571	—0.1613248			
0.0218254	—0.2638889			
—0.0932540	—0.1997863			
—0.0734127	—0.0074786			
—0.0773810	0.1976496			
—0.0734127	—0.0074786			
—0.0932540	—0.1997863			
0.0218254	—0.2638889			
0.0178571	—0.1613248			
—0.0734127	0.1463675			

TABLE 6. SCHEDULE OF CHARACTERISTIC ROOTS CORRESPONDING TO THE REDUCED NORMAL EQUATION FOR INTRABLOCK ESTIMATES, TABULATED VERTICALLY IN THE ORDER $\Lambda_2, \Lambda_3, \ldots \Lambda_n$, with $\Lambda_1 = 0$ IN ALL CASES

Reference Number				
1	2	3	4	5
2.0	1.599031	3.0	2.292893	2.0
3.0	3.123490	4.0	3.000000	2.0
2.0	2.277479	3.0	3.707107	2.0
3.0	2.277479	4.0	2.000000	4.0
2.0	3.123490	3.0	3.707107	2.0
	1.599031	4.0	3.000000	2.0
		3.0	2.292893	2.0
6	7	8	9	10
1.292893	3.0	1.733956	4.0	3.190983
1.0	3.0	2.326352	5.0	4.309017
2.707107	4.5	1.500000	4.0	4.309017
2.0	3.0	3.439693	5.0	3.190983
2.707107	3.0	3.439693	4.0	5.000000
1.0	4.5	1.500000	5.0	3.190983
1.292893	3.0	2.326352	4.0	4.309017
	3.0	1.733956	5.0	4.309017
			4.0	3.190983
11	12	13		14
2.190983	2.5	2.000000		3.540507
4.309017	2.5	1.381966		5.341254
3.309017	2.5	2.000000		3.845139
3.190983	2.5	3.618034		4.915415
4.000000	5.0	2.000000		4.357685
3.190983	2.5	3.618034		4.357685
3.309017	2.5	2.000000		4.915415
4.309017	2.5	1.381966		3.845139
2.190983	2.5	2.000000		5.341254
				3.540507
15	16	17	18	19
2.398192	1.301061	5.0	4.5	4.0
3.381761	2.544078	6.0	4.5	4.0
3.260554	1.726900	5.0	6.0	4.0
4.756669	1.813607	6.0	4.5	6.0
2.702824	3.614354	5.0	4.5	4.0
2.702824	3.614354	6.0	6.0	4.0
4.756669	1.813607	5.0	4.5	4.0
3.260554	1.726900	6.0	4.5	6.0
3.381761	2.544078	5.0	6.0	4.0
2.398192	1.301061	6.0	4.5	4.0
		5.0	4.5	4.0
20	21	22		23
4.0	3.0	2.633957		1.5
3.0	3.0	2.0		3.5
4.0	3.0	2.0		3.0
3.0	3.0	3.0		1.5
4.0	3.0	3.366025		1.5
6.0	6.0	2.0		2.0
4.0	3.0	3.366025		1.5
3.0	3.0	3.0		1.5
4.0	3.0	2.0		3.0
3.0	3.0	2.0		3.5
4.0	3.0	2.633957		1.5
24				
1.133975				
0.500000				
2.000000				
1.500000				
2.866025				
2.000000				
2.866025				
1.500000				
2.000000				
0.500000				
1.133975				

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