Crop Response Surfaces and Economic Optima in Fertilizer Use

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SUMMARY

The experiments upon which this study is based were designed to allow (1) estimation of the fertilizer-crop production surface and (2) specification of economic optima in level of fertilization and combination of nutrients. Two nutrients were varied on each experiment.

The corn experiment, on calcareous Ida silt loam soil, included nine rates each of N and P_2O_5. Red clover and alfalfa were on Nicollet and Webster loam soils with P_2O_5 and K_2O as the variable nutrients. Each experiment included two replicates of 57 different nutrient combinations — 114 completely randomized observations.

Production functions fitted to the yield observations included logarithmic, exponential, quadratic crossproduct and quadratic square root equations. When all observations were used, a full-term square root function allowed the best predictions for corn and alfalfa; a four-term square root function was used for red clover. The production function equations used for the three crops were:

Corn: $$Y = -5.68 - 0.3161N - 0.4174P + 6.3512\sqrt{N} + 8.5155\sqrt{P} + 0.3410\sqrt{PN}$$

Alfalfa: $$Y = 1.87 - 0.0014K - 0.0050P + 0.06173\sqrt{K} + 0.1735\sqrt{P} - 0.000001\sqrt{KP}$$

Red clover: $$Y = 2.47 - 0.0040P + 0.02683\sqrt{K} + 0.1279\sqrt{P} - 0.000978\sqrt{KP}$$

The production function equations were then used in deriving (1) single-nutrient input-output or response curves, (2) marginal response coefficients, (3) yield isoquants, (4) marginal replacement coefficients and (5) nutrient isoclines. As examples, the marginal response curve for P_2O_5 on alfalfa is:

$$\frac{dY}{dP} = -0.0050 + \frac{0.08376}{P^{0.5}} - 0.0000605\left(\frac{K^{0.5}}{P^{0.5}}\right)$$

The symbol d is used to denote partial derivatives throughout this bulletin.

The isoquant equation for 2.5 tons alfalfa is:*

Similar equations were derived for corn and red clover. A set of these equations will determine the optimum level of fertilization and the optimum nutrient combination for all prices of crops and nutrients. These optima change with each shift in price relationship because (1) marginal yield response is at a diminishing rate and (2) the marginal replacement or substitution rate between nutrients diminishes as the nutrient combinations change. The nutrient isoclines (which show equal replacement ratios of nutrients at different yields) are curved rather than linear. Their curvature indicates that the optimum combination of nutrients varies with yield level. The nutrient isoclines converge at the point of maximum yield, denoting no substitution of nutrients at the maximum.

As an example of how the optimum level of fertilization, the most profitable yield and the least-cost nutrient combination change with prices, the following summary data are presented for alfalfa. Similar data are presented in the text for corn and red clover. Of course, the empirical results obtained in these experiments would directly apply only to soils of the same type and fertility level as the experimental plots. Also, weather conditions, which vary from year to year, alter crop response to fertilizer.

Prices			Amounts applied and yield			
Price alfalfa per ton	Price P_2O_5 per lb.	Price K_2O per lb.	Total lb. nutrients	Lb. P_2O_5	Lb. K_2O	Yield (ton)
\$16	0.09	0.12	71.4	63.4	8.0	3.07
\$10	0.09	0.12	41.0	37.1	3.9	2.84
\$28	0.09	0.12	98.0	85.5	12.5	3.20
\$10	0.12	0.09	31.8	24.9	6.9	2.75
\$28	0.12	0.09	107.5	79.4	28.1	3.24
\$10	0.08	0.08	50.2	42.4	7.8	2.93

$$* P = \left[\frac{17.28905 - 0.143483\sqrt{K} \pm \sqrt{-0.0000259K - 0.0007392\sqrt{K} - 0.017538}}{-0.010036}\right]^2$$

Crop Response Surfaces and Economic Optima in Fertilizer Use*

BY EARL O. HEADY, JOHN T. PESEK AND WILLIAM G. BROWN

Greater use of fertilizer has been one of the important innovations in Iowa agriculture over the past decade. Total tonnage of fertilizer used increased by about 2,000 percent in the period, 1941-51. The trend in fertilizer use is still upward. Further increases can be made in the state's total production as fertilizer use is tied in with management of the farm and integrated with seeding rates, soil conservation and water management, and other resources of the farm.

OBJECTIVES OF STUDY

Initial research in fertilizer deals with the presence or absence of response in crop yield with the application of fertilizer. However, once responses have been found to exist, the farmer needs to consider fertilizer along with other resources and practices in his farm management decisions. First, he must decide whether or not to use any fertilizer. While crop responses may be certain, he must decide whether or not 1 dollar put into fertilizer will return more than the same dollar put into livestock, seed, machinery or other investment alternatives. If he has decided to use fertilizer, he must then decide (1) where to use fertilizer in terms of which crops and soils will return the greatest amount for each 1 dollar invested, (2) how and when to use fertilizer on a particular crop, (3) what combinations of fertilizer nutrients to use and (4) how much fertilizer of a given nutrient combination or grade to apply on a given crop. These decisions can be made most efficiently if fertilizer information is provided in the form of incremental response data. Incremental response data show the successive additions to yield resulting from successive fertilizer applications. Accordingly, once research has shown that crop yields do respond to fertilizer, the next steps in research and education are investigations to show (1) the incremental yields forthcoming from different rates of fertilizer application under specified crop and soil conditions and (2) the economic optimum quantity of fertilizer, considering crop and fertilizer prices and production costs.

This study has been designed specifically to investigate (1) rate of fertilizer application and (2) combination of fertilizer nutrients in a manner to maximize profits from fertilizer use. Many studies have been designed to analyze rates of application but most of these have dealt with only a few rates. Hence, these experiments have not been satisfactory for estimating the complete fertilizer production surface. The current study was designed specifically for this purpose and for computing marginal quantities to be used in specifying economic optima of fertilizer application and combination of nutrients. Two variable nutrients were applied in each corn, alfalfa and red clover experiment. These data show that the productivity of one nutrient depends on the amount of the other with which it is combined; the most profitable amount of one nutrient cannot be determined apart from the level of the other. Similarly, returns from one nutrient are affected by the amount of a third nutrient, the seeding rate, the amount of water applied or even by the amount of labor used on the farm. Additional studies are needed to analyze these facets of fertilizer productivity and returns and, hence, to determine the full economic potential in use of fertilizer.

This study is divided into four major parts: (1) a discussion of the fundamental logic basic to the design of experiments of this nature, (2) an explanation of the experimental procedure, (3) a discussion of the empirical procedures employed in deriving the production functions or response equations and an analysis of the findings, and (4) an economic analysis of the derived coefficients as they relate to level of fertilization and combination of nutrients.

BASIC LOGIC OF FERTILIZER INVESTIGATION

As methodological background for the empirical results which follow, we present possible alternatives of the manner in which fertilizer elements can (1) be transformed into crop products and (2) combine with or exchange for each other in production of a given amount of crop. Implications of these quantities in the economy of fertilizer use are discussed. Perhaps all of the alternatives presented have, at some time or other,

* Projects 1135 and 1189, Iowa Agricultural Experiment Station, Tennessee Valley Authority, cooperating.

served as hypotheses of the fertilizer production function or as a basis for fertilizer recommendations.

Crop production is a complex process involving many resources of which fertilizer nutrients represent but one class. The crop production function is of the general form

$$C = f(L, S, D, M, T, E, F_1, F_2, F_3, X_1, \ldots X_n)$$

where C refers to crop production, L refers to labor input, S refers to land input, D refers to seed input, M refers to machinery input, T refers to moisture and E refers to tractor fuel; F_1, F_2 and F_3 refer to three fertility elements while X_1 through X_n refer to other unspecified resources. If all of the resources included in crop production were variable, and were increased in the same proportion, it is very likely that each resource such as L, S or F_1 might have constant productivity over some range of inputs.[1] A linear homogeneous production function of degree 1 thus would mean: If 10 hours of labor, 1 acre of land, 100 pounds of fertilizer element F_1 and specified amounts of other resources yield 65 bushels of corn, then 20 hours of labor, 2 acres of land, 200 pounds of F_1 and double quantities of other resources would yield 130 bushels. However, most decisions on fertilizer are made in the framework of a crop production function such as

$$C = f(F_1, F_2, F_3, L, E, M \mid S, D, T, X_1, \ldots X_n).$$

Here only the resources to the left of the vertical bar are variable in quantity. Land is held fixed at 1 acre (or more) along with given seeding rates, moisture and other resources specified in the production function. More often the function is analyzed in the manner of

$$C = f(F_1, L, E \mid F_2, F_3, S, D, M, T, X_1 \ldots X_n)$$

where only one fertility element, or one particular element combination, is variable along with labor and fuel while other fertility elements are held fixed at some specified level with land and other resources. In other cases, of either farm decisions or fertilizer research, D (seed) and T (moisture through irrigation) are varied along with a fertilizer element to examine crop response. The productivity of each fertilizer increment ordinarily differs greatly depending on the number of other resources which are varied along with it (i.e., the number of resources which are transferred from the "fixed category" to the right of the perpendicular line to the "variable category" on the left side). Limits in economic use of fertilizer cannot be established for the multitude of possible resource combinations until research has established the crop production function and the marginal fertilizer response in the manner of the "generalized" production functions outlined above.

This study is a first step in this direction. It considers two fertilizer elements as simultaneously variable in the crop production function while other resources are held fixed. Production functions of the general form

$$C = f(F_1, F_2 \mid F_3, L, S, D, M, T, X_1, \ldots X_n)$$

are examined where F_1 represents nitrogen and F_2 represents P_2O_5 on corn, while for clover and alfalfa, F_1 represents K_2O and F_2 represents P_2O_5. The fertilizer production function is thus considered as $C = f(F_1, F_2)$ where variation in only two elements is considered.[2]

GEOMETRIC FORM OF FERTILIZER-CROP RESPONSE RELATIONSHIP

Geometric models can be used to illustrate crop response from two variable nutrients. One extreme possibility is shown in the production surface of fig. 1. While it likely has little application to two different elements such as N and P_2O_5, it has great application to N from two sources such as ammonium nitrate and ammonium sulfate or to P_2O_5 from two sources such as superphosphate or calcium metaphosphate. The production surface shown supposes that if either element is increased alone, or if the two are increased in constant proportions, smaller and smaller yields are attained with each increment of fertilizer. How-

[1] This is simply one alternative out of three, i.e., constant, increasing or decreasing productivity, with all resources increased to scale in crop production.

[2] Actually, labor and machine or equipment services were also varied to apply different amounts of fertilizer and harvest yields. However, it was impossible to measure these resource inputs successively, and, even had it been possible to do so, the results from the small plots and experimental machine techniques would not have served satisfactorily for inferences to farm decisions.

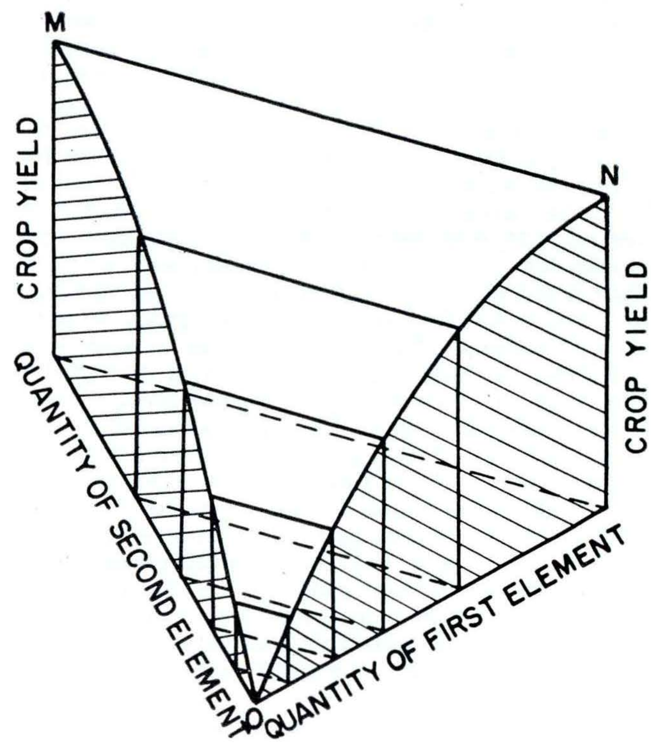

Fig. 1. Hypothetical production surface with perfect substitution between nutrients.

ever, it also assumes that each element is a perfect substitute (substitutes at a constant rate) for the other in producing a given level of crop yield.[3] If we reduce this three-dimensional surface to two-dimensional diagrams, we obtain the geometric models of fig. 2. Figure 2A refers to the transformation function between the input of fertilizer (one nutrient variable and the other fixed, or both varied by the same proportions) and output of crop.[4] Its convex curvature indicates diminishing marginal productivity of the fertilizer. This curve, one of two major relationships in fertilizer use, simply represents a vertical profile of a slice through the surface of fig. 1 and passing through the origin. Its slope at any one point (the point of tangency of a straight line and the curve shown) indicates the marginal product of fertilizer (i.e., the amount added to total yield, at the particular fertilizer input, as fertilizer use is increased by a small added amount). This relationship is important in determination of the most profitable level of fertilizer application.

Figure 2B is a "contour map" of fig. 1; each of the straight-line contours (indicated as 5, 10, 15 and 20) represents a horizontal slice of the surface of fig. 1. The number represents the yield level; the lines of fig. 2B are isoproducts (equal products) or yield isoquants (equal quantities) since they indicate all of the possible combinations of the two fertility elements which will produce a given yield. Since these isoproducts or "contour" lines are linear, the two fertilizer elements substitute for each other at constant rates in production of a given crop yield; using all of one or all of the other fertilizer element would produce the same crop product. The elements would always substitute at a fixed rate (i.e., 1 pound of one element would always replace the same quantity of the other regardless of the combination of elements used). As stated previously, it appears unlikely that two distinct nutrients ever substitute at constant rates, although this situation likely holds true for the same nutrient from different sources (i.e., the case where N from ammonium nitrate is represented by the horizontal axis while N from ammonium sulfate or anhydrous ammonia is represented on the vertical axis). Hence, this particular model of fertilizer relationships can be used to specify the most profitable source of a particular element, under a unique situation to be outlined later.

Diminishing returns (or a decreasing rate of transforming fertilizer into crop product) also are expressed in fig. 2B. The fact that isoproduct lines representing equal increments of yield (5, 10, 15 and 20) move farther apart along any straight line through the origin (such as OF) indicates that increasingly larger quantities of a fixed fertilizer mixture are necessary to attain equal increments in crop yield (or conversely, equal in-

[3] For further details on the nature and significance of a production surface, see: Heady, Earl O. Economics of agricultural production and resource use. Ch. 3 and 4. Prentice-Hall, Inc., New York, 1952.

[4] This figure, like all of the other geometrical presentations of this section, refers only to crop yield attributable to fertilizer, i.e., response to fertilizer beyond the production level attained without fertilization.

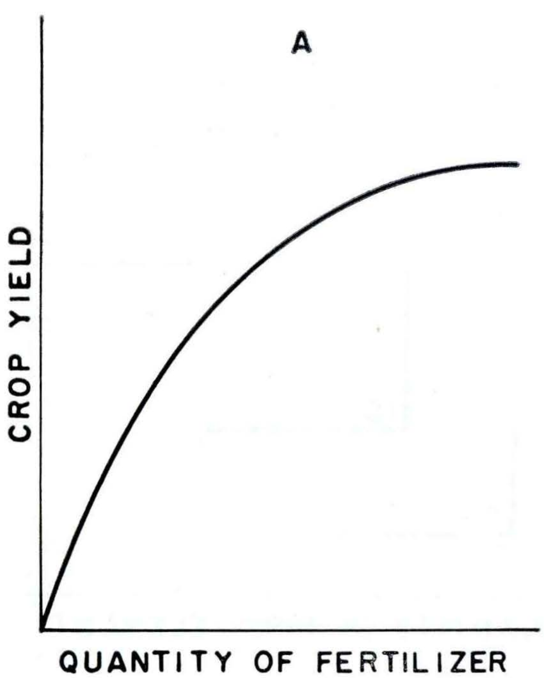

Fig. 2A. Response curve representing a vertical slice through surface of fig. 1.

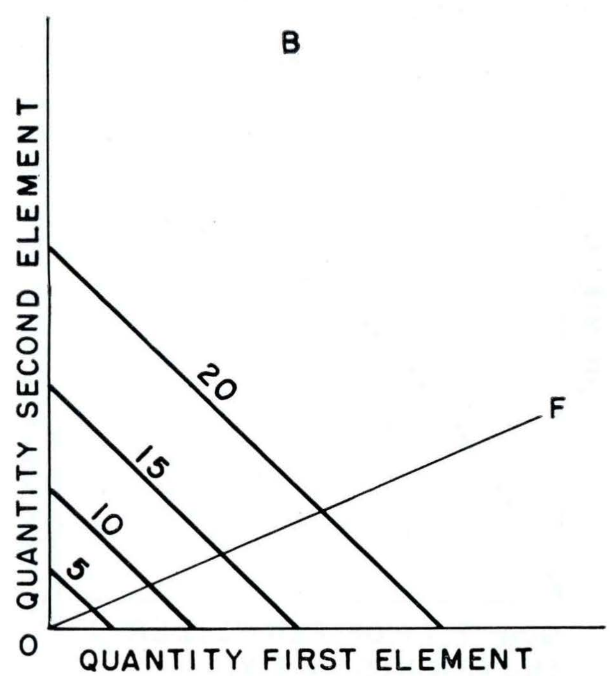

Fig. 2B. Contour map of fig. 1.

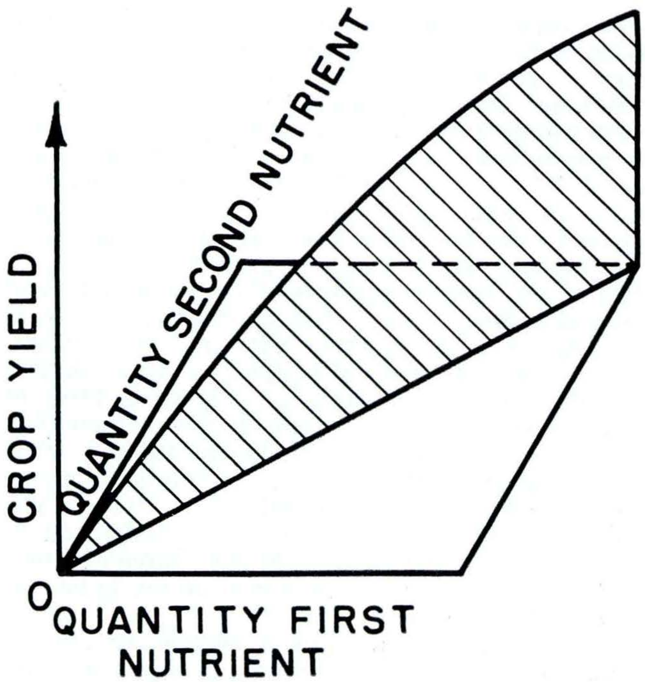

Fig. 3. Hypothetical production surface representing perfect complementarity between nutrients.

crements in a fixed fertilizer mixture add increasingly smaller quantities to total yield.)[5]

[5] Any straight line through the origin (including one identical with the horizontal or vertical axis) is a "fixed fertilizer mixture" line: It indicates that the two elements are held in fixed proportions as larger amounts are applied. While many recommendations on fertilizer are made in terms of this "scale" or "fixed fertilizer mixture" line, it is an appropriate basis for fertilizer recommendations only if it is a true isocline, a point to be explained later.

Figure 3 represents an "opposite extreme" in possible fertilizer relationships. It is somewhat representative of the "law of the soil" or the "law of the minimum" hypothesis advanced by early soil chemists such as von Liebig, Meyer and Wollney.[6] This model supposes that fertility elements must be combined in fixed proportions; one element does not substitute for the other and a given crop yield cannot be maintained as we shift to more of one and less of another nutrient; the surface in this case narrows to a "knife's edge." The "contour lines," representing given levels of crop yields, reduce to points at the "ridge" of the yield or production surface. The two-dimensional input-output curve (with both nutrients increased in fixed proportions, since it is assumed that yield increments are not forthcoming from one nutrient increased by itself) is shown in fig. 4A; it has the same implications as explained for fig. 2A. The isoproduct lines representing this "extreme hypothesis" are shown in fig. 4B. The form of these isoyield curves illustrates the supposition of zero substitution between the two nutrients. If a given yield is to be attained, only the single combination, represented by the corner of the contour "angle" (i.e., the lines indicated by 5, 10 and 15 bushel yields) will allow attainment of this yield. Addition of more of one element, the quantity of the other held constant, (1) will add nothing to production and (2) will not replace any of the other element, if the given yield is to be main-

[6] Cf. Spillman, W. J. Law of diminishing returns. World Book Company, New York, 1924. The "law of minimum" supposed, however, that the "ridge line" or "knife's edge" production surface was linear rather than curved indicating constant productivity of "fixed fertilizer combinations," up to a maximum per-acre yield.

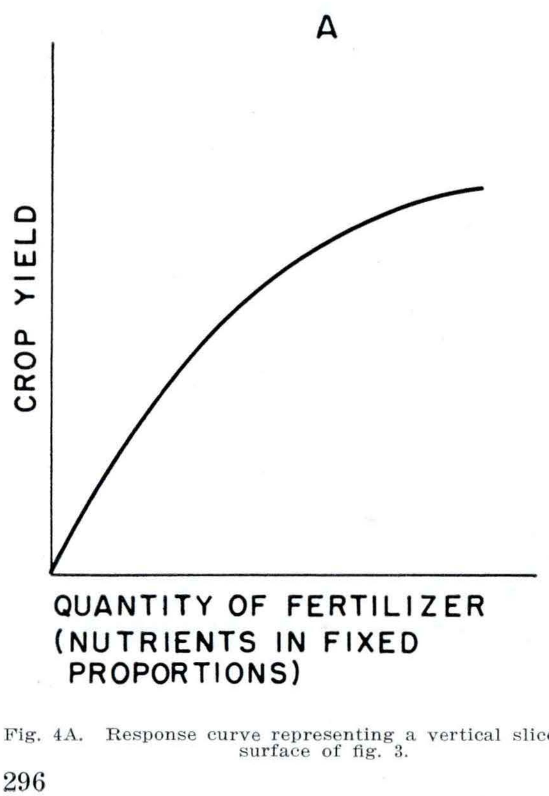

Fig. 4A. Response curve representing a vertical slice through surface of fig. 3.

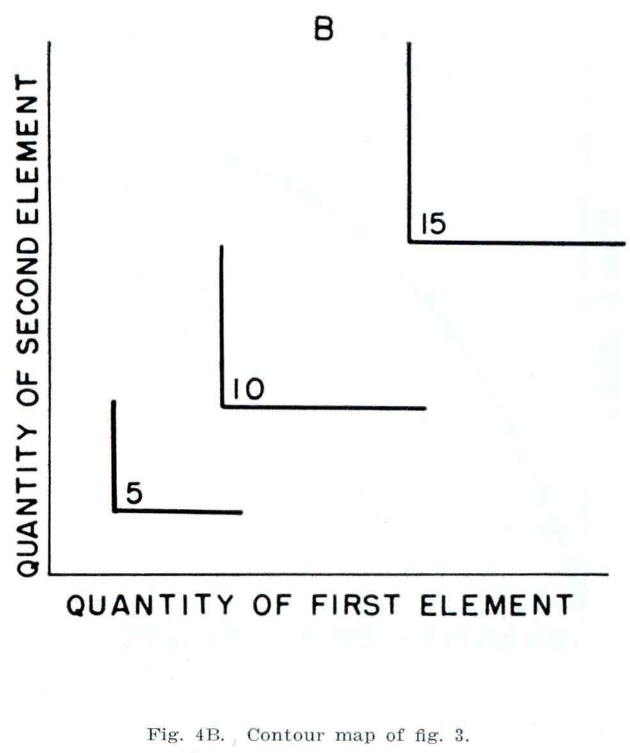

Fig. 4B. Contour map of fig. 3.

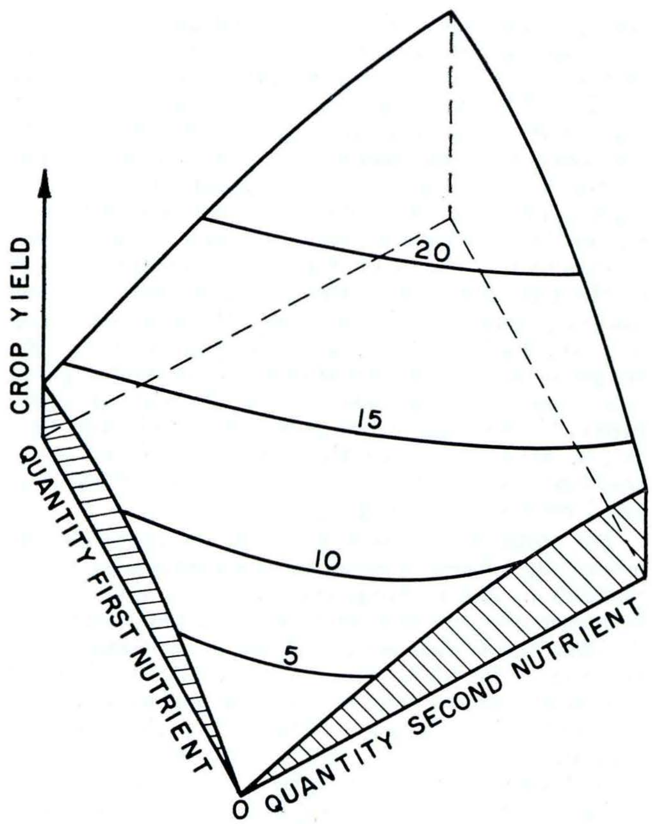

Fig. 5. Hypothetical production surface showing both substitution and complementarity between nutrients.

tained. This proposition supposes, then, that a given level of yield can be attained only by use of a single combination of elements. The two elements are technical complements, and, if they are to be used at all, they should be used in this single combination.[7]

While the surface of fig. 3 perhaps has some applications, its "pure form" existence is probably less widespread than other models. A third model of the two-element fertilizer production is shown in fig. 5. At one extreme, it approaches the situation in fig. 1, and, at the other, it approaches that of fig. 3. While many adaptations of it exist, in general form it probably has wider application than the other models. The convex surface indicates diminishing returns to each element alone or to two elements in fixed combinations; the curved contour lines, suggesting the possible combinations of the two elements which will produce the same yield, suggest that the elements (1) do not replace each other at a constant rate as in figs. 1 and 2B and (2) do not require use in fixed proportions as in figs. 3 and 4B, but replace each other at a diminishing rate in producing a given yield. In other words, the same yield can be attained by replacing some of one nutrient in a fertilizer with more of another. However, less and less of the first will be replaced by each successive 1-pound increase in the second, the yield remaining at a specified level. This surface also supposes that increases in yields can be obtained when both nutrients are increased in combination. This is commonly observed in fertilizer practice.

Figures 6A and 6B represent alternative contour maps which may serve as the two-dimensional counterpart of the surface from fig. 5. (The single-line, input-output curve such as 2A and 4A

[7] As in the other isoproduct maps, increasing distances between the contours in fig. 4B indicate diminishing productivity of fertilizer for elements in fixed proportions.

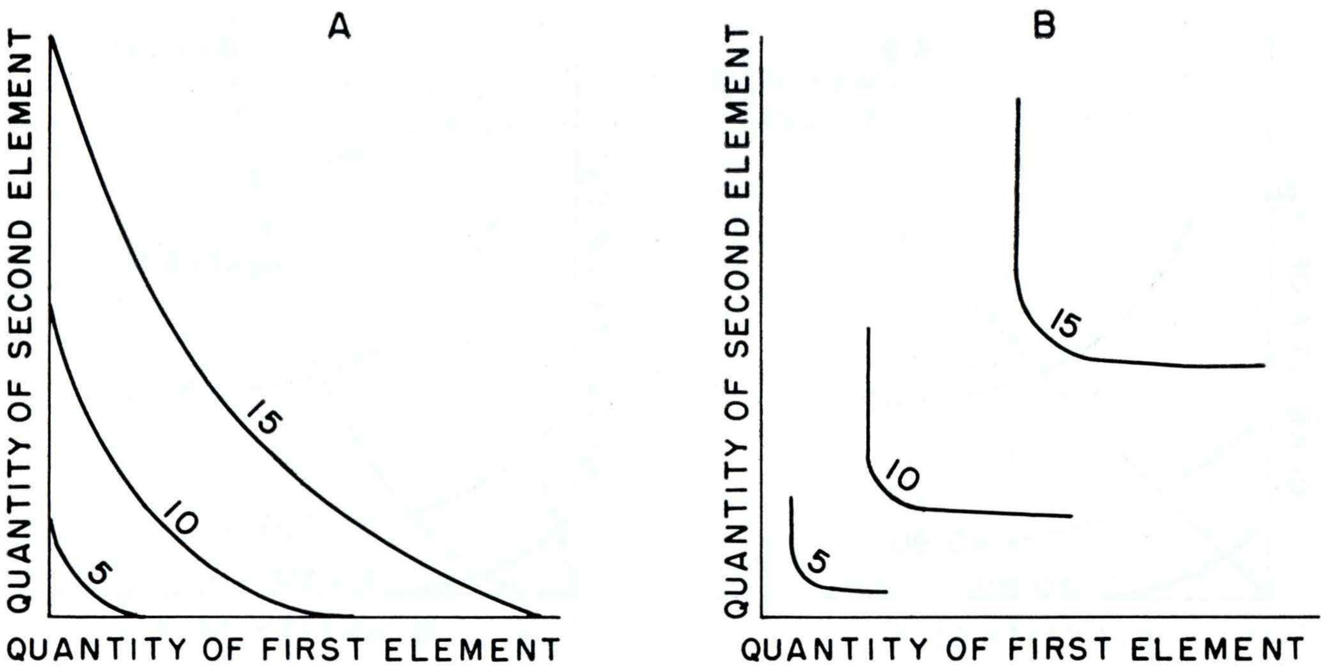

Fig. 6. Alternative contour maps of a surface such as fig. 5.

is not presented since it is of the same general form as those presented previously.[8]) Figure 6A represents the case when a given yield can be attained by complete displacement of one element by the other, but, since the lines are curved, replacement is at a diminishing rate; smaller and smaller quantities of one element are replaced by each successive 1-pound increment of the second. Figure 6B illustrates the case in which the two elements substitute at diminishing rates over a limited range but have zero substitution possibilities outside of this range; just as they are from the outset in fig. 4B.[9] The fact that the contour lines become vertical and horizontal suggests complementarity. (The algebraic function derived to conform with this relationship actually is an ellipse with single points, near the ordinate and abscissa, which have infinite and zero slopes respectively.) A more likely situation for most crops, where a limited amount of two elements is present in the soil, is a combination of the contour maps in figs. 6A and 6B. For small increases attributable to fertilizer, the given yield level may be attained entirely by one element or the other or by some combination of the two as shown in fig. 6A. At higher yield levels, however, the isoquants may take the form of those in fig. 6B indicating that substitution possibilities are more limited as higher yield levels are attained; the maximum yield may be attained by a single combination of elements (see fig. 7). This is a logical contour map for certain conditions (presence of a small amount of both elements but insufficient for high yields). The contours may take entirely different slopes as higher and higher yield isoquants are attained. Under this situation, the contours or isoquants representing low yields may intersect one or both of the nutrient axes. The isoquants representing higher yields may not intersect the axes and may become shorter in length, indicating that the range of nutrient ratios which will produce a given yield becomes narrower with increasing yield levels. Under these conditions, the maximum yield can be attained with only a single combination of nutrients (i.e., the isoquant for the maximum yield reduces to a single point).

The slope of the yield isoquants along a line representing a fixed ratio of the nutrients also is important in determining the economic optimum of nutrient combination and fertilization rates. In fig. 7A, line L represents a fixed ratio of nutrients. However, the yield isoquants change in slope at the points where they are intersected by the fixed nutrient ratio line, L. Therefore, the nutrient combination which is most economic for a 20-bushel yield is not the same as the optimum for a higher yield level. The same nutrient combination will be optimum for all yield levels only if the successive yield isoquants have the same slope at their point of intersection with the fixed ratio line. When the replacement rate between nutrients (the

[8] To keep the drawings simple, the negative productivity or diminishing total yield phase has not been illustrated in any of the figures. If one element or a combination of elements is applied at a sufficiently great rate, it will often cause total yield to decline, if other resources are held fixed at a sufficiently low level.

[9] Just as in fig. 4B, a sufficiently large quantity of one element, added through the range of technical complementarity, with the other element fixed, will eventually cause yield to decline from the stated level.

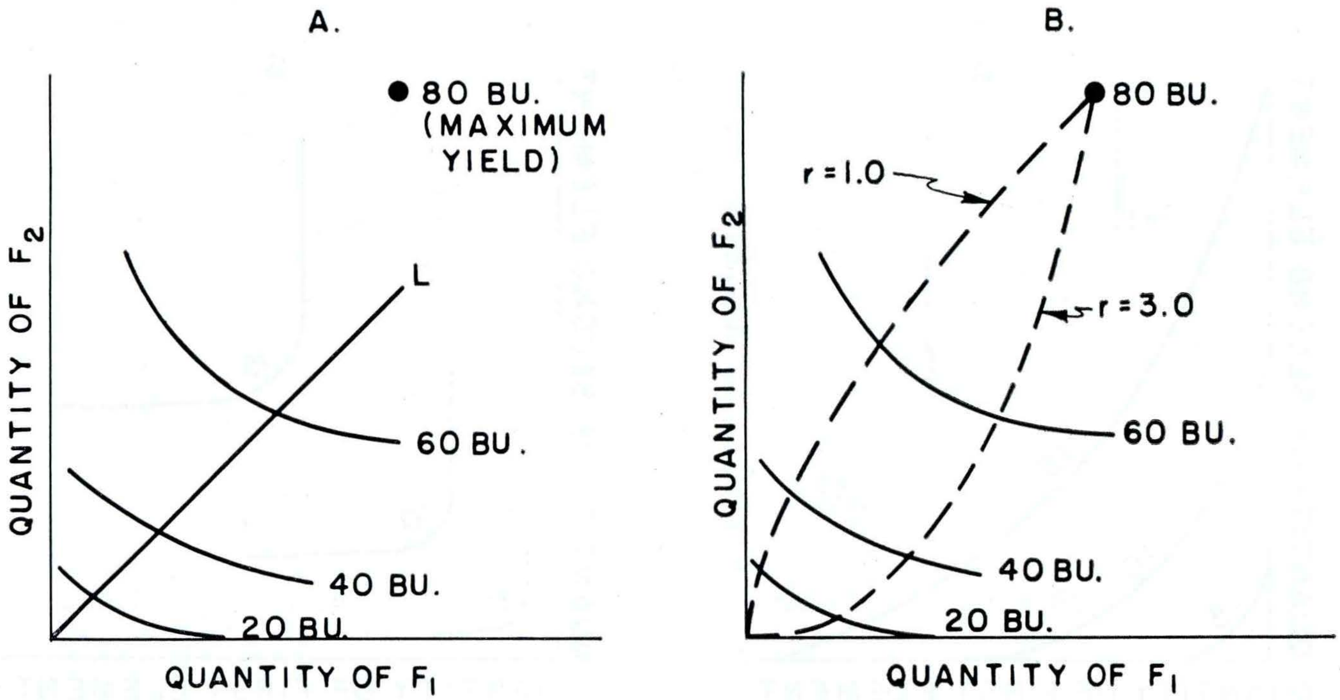

Fig. 7. Contour maps of a hypothetical production surface comparing a fixed ratio line and isoclines.

slope of the yield isoquants) changes along a fixed nutrient ratio line, it is of the nature indicated by the dotted lines in fig. 7B. These are called isoclines, since they trace out the points on the yield isoquants which have the same slope or "incline." The curve labeled $r = 1.0$ indicates all points on the yield isoquants with a slope of 1.0 (i.e., 1 pound of F_2 replaces 1 pound of F_1 along this line). If the price of the nutrients is the same (e.g., if the ratio of prices is 1.0), this line shows the optimum combination of nutrients for each yield level. The curve labeled $r = 3.0$ indicates all points where the curves have this slope; 1 pound of F_2 replaces 3 pounds of F_1 along this line. It indicates the optimum combination of nutrients for different yields when the price of F_2 is three times greater than the price of F_1 (i.e., the price ratio is 3.0). Since the isoclines are not straight lines, the proportions of the fertilizer nutrients should change with yield level. The same combination of nutrients is optimum for all yield levels only if the isoclines are straight lines. However, this condition is usually impossible because they must converge at the point of maximum yield.

PROFIT MAXIMIZATION, FERTILIZATION LEVELS AND ELEMENT COMBINATION

Given the production relationships outlined above, statements can be made about the conditions of fertilizer use including (1) the rate of application and (2) the combination of elements which will maximize farm profits. In addition to consideration of other resources which also may be varied with fertilizer and the optimum timing and method of application, the questions of (1) the optimum rate and (2) the optimum combination of elements are major ones in respect to optimum usage of fertilizer. To answer these questions, we need the following information: (1) the price per unit of the crop product being produced, (2) the price per unit of fertilizer and other resources necessary to produce it, (3) the marginal rate of replacement between nutrients and (4) the marginal rate of transformation (i.e., the marginal product) of each increment of fertilizer. Hence, we see that information on fertilizer designed to be of maximum use in farmers' decisions especially needs to be in the form of marginal or incremental quantities. This fact is illustrated even further with the conditions of profit maximization explained below.

THE OPTIMUM LEVEL OF A GIVEN ELEMENT OR A GIVEN COMBINATION OF ELEMENTS

For a single element, or a given combination of elements, the input-output or response curve usually is of the form indicated in figs. 2A and 4A. For a farmer with unlimited capital, the optimum level of fertilization is attained under the condition of equation (1a) below where P_c refers to the price per unit of the crop, P_f refers to the price per unit of the fertilizer (or the cost of the fertilizer and labor, fuel and other resources used in applying it and harvesting a larger yield), $\triangle C$ refers to the change in yield (i.e., the increment in yield) and $\triangle F$ refers to the increment in fertilizer. The ratio $\frac{\triangle C}{\triangle F}$ is the transformation ratio or the marginal product of fertilizer; it is the slope, for any designated quantity of fertilizer, for the input-output curve such as fig. 2A or 4A. Hence, an optimum fertilization level has been attained when the transformation ratio or marginal product of fertilizer is equal to the fertilizer/crop price ratio; the optimum rate of fertilization changes with each change in the price ratio.

Under this condition, the value of the increment in crop production exactly equals the value of the fertilizer increment, a condition expressed in equation (1b) which has been derived from (1a) by arithmetic. However, as equation (2a) and (2b) show, the value added to crop production will be

$$\text{(1a)} \quad \frac{\triangle C}{\triangle F} = \frac{P_f}{P_c} \qquad \text{(1b)} \quad (\triangle C)(P_c) = (\triangle F)(P_f)$$

$$\text{(2a)} \quad \frac{\triangle C}{\triangle F} > \frac{P_f}{P_c} \qquad \text{(2b)} \quad (\triangle C)(P_c) > (\triangle F)(P_f)$$

$$\text{(3a)} \quad \frac{\triangle C}{\triangle F} < \frac{P_f}{P_c} \qquad \text{(3b)} \quad (\triangle C)(P_c) < (\triangle F)(P_f)$$

greater than the value added to fertilizer cost (2b) if the transformation ratio or marginal product of fertilizer is greater than the price ratio (2a). If the transformation ratio (the marginal product of fertilizer) is less than the price ratio (3a), the value added to crop production will be less than the value added to fertilizer costs (3b). These statements are identical with this condition: The optimum fertilization rate is attained and profits are at a maximum, when the marginal (added) cost of the fertilizer is just equal to the marginal (added) return from the crop. This is evident since, from equation (1a), we can derive the equation $P_c = \frac{(\triangle F)\ (P_f)}{\triangle C}$. Here the right hand member represents the marginal or added cost per added bushel of crop while P_c represents the marginal or added return per bushel.[10] Under equation (2a), the marginal or added cost is less than the marginal

[10] In addition to the price or cost of the nutrients, application of fertilizer may require outlays for labor, machine services, etc. Where these inputs or expenses vary directly with the pounds of fertilizer, they can be added to the price (cost) of fertilizer and the ratios of equation (1a) again specify the optimum rate of fertilizer use. In some cases, a fixed amount of expense is involved in applying fertilizer; it is not proportional to the quantity of fertilizer but is the same regardless of the rate of fertilizer application. However, the conditions of equation (1a) still hold true. This fixed cost (K) gives a total cost (C) function which can be defined as $C = K + p_1F$, where p_1 is the price (variable cost) of fertilizer and F is the quantity of fertilizer. The revenue or gross returns (R) figure then is $R = p_2Y$ where p_2 is price of the crop and Y is total yield. Profit (gross revenue minus cost) is at a maximum when marginal or additional revenue is equal to marginal or additional cost of using fertilizer. These two marginal or additional quantities, therefore, are those shown in (a) and (b) below:

$$\text{(a)} \quad \triangle C = p_1 \frac{\triangle F}{\triangle Y} \qquad \text{(b)} \quad \triangle R = p_2$$

by equating these two quantities we have the condition of equation (1a) in the text, and the constant cost, as long as it is covered by income, need not be used in defining the optimum level of fertilization.

or added return while under (3a) the marginal cost is greater than the marginal return.

The same condition can be represented geometrically as in fig. 8. Here the curve OP is the same as the response or input-output curves of figs. 2A and 4A. The slope of the curve at any point defines the marginal product for the particular quantity of fertilizer (i.e., the slope of the curve is the same as the $\frac{\triangle Y}{\triangle F}$ indicated above; it is the amount added to yield by one more unit of fertilizer). The curve OC can be plotted in the graph; it shows the total quantity of the crop which is required to purchase the amount of fertilizer (and accompanying labor) represented on the horizontal axis. For example, if corn is $1.50 per bushel and fertilizer is 15 cents per pound, the curve OC will pass through a point such as *a* indicating "one bushel" on the vertical axis and "10 pounds of fertilizer" on the horizontal axis; its slope represents the fertilizer/corn price ratio (the $\frac{P_f}{P_c}$ ratio of the equations above) since it shows the exchange value between different amounts of crop and fertilizer. It also represents the physical cost, in crop units, of obtaining the total product represented by OP. Thus, the slope of OC is the marginal cost, in crop units, of using fertilizer. Since the slope of OP represents the marginal crop return of using more fertilizer, we find its tangent line, TL, which has the same slope as the cost line OC; the marginal cost of fertilizer is then equal to the marginal return. As this condition is attained, the distance between the tangent line, TL, and the cost line, OC, is at a maximum defining a maximum difference between return and cost. The farmer with ample capital is interested in this point of maximum profits. In our example, OF units of fertilizer are used and total crop yield from fertilizer is DG. The amount of fertilizer in terms of crop yield necessary to get this yield is DE. Hence returns exceed costs by EG. If we multiply EG by the price of the crop, net profit from using fertilizer can be determined. Any tangent line other than TL will not denote maximum profits (i.e., the point of tangency will lie a shorter distance from OC than does TL).[11]

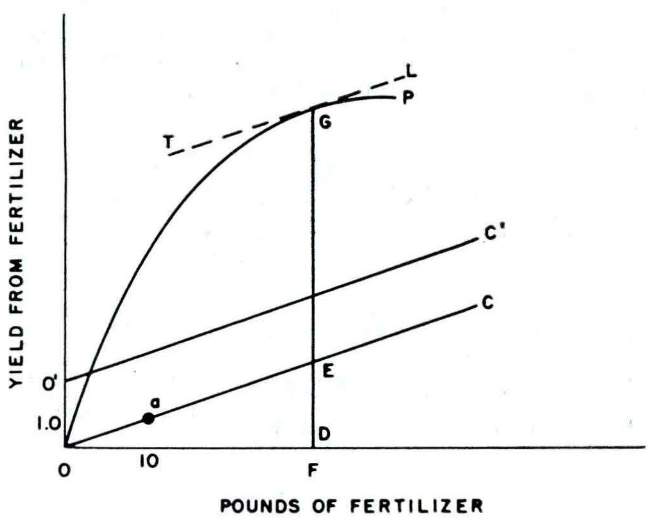

Fig. 8. Geometric representation of optimum conditions for input of fertilizer.

Even if a "first" or "fixed" cost (in the form of labor for application, etc.) is necessary for applying the fertilizer, the principle is the same. The cost line then moves up the vertical axis in the manner of O′C′ in fig. 8; the point of origin represents the value, in units of crop, of the fixed cost. The task is still to find the maximum distance between the two lines — TL, the tangent line and O′C′, the cost line. It will be the same as previously, if the cost of fertilizer (and the labor to go along with it) has the same relationship to the price of corn as before (again at OF of fertilizer since the slope of OC and O′C′ are the same even though the latter is higher than the former).

These figures illustrate the type of basic information needed for determining the optimum rate of fertilization; incremental or marginal quantities are necessary for determining the most profitable level of fertilization, a quantity which does not remain fixed between years but varies with price ratios. The marginal yield information also is necessary for determining, for the farmer with limited capital, how far fertilizer investment can be extended before the return on capital falls below other opportunities within the farm.

OPTIMUM COMBINATIONS AND EXPANSION PATH OVER YIELD LEVELS

Where opportunity exists or soil situations encourage the use of more than one element, the farmer must decide the combination of elements which will minimize the cost for any given level of yield, and then he needs to determine how far yield should be extended to maximize profits. Under a situation where elements can be used effectively only in fixed combination (fig. 4B), there is no choice. However, where different combinations of elements can be used to produce the same yield, choice is possible. The optimum fertilizer mixture (the element combination) for attaining this yield is the one which minimizes the cost for the given yield level. In terms of element combination, fertilizer cost is at a minimum for a given yield when the condition of equation (4a) is attained. Here P_1 and P_2 refer to the price of the first and second nutrient, $\triangle F_1$ refers to the amount of the first nutrient replaced and $\triangle F_2$ refers to the amount of the second nutrient added to produce a given yield; the marginal replacement ratio is then $\frac{\triangle F_1}{\triangle F_2}$. The quantity $\frac{\triangle F_1}{\triangle F_2}$, the replacement ratio of nutrients, also represents

[11] We suppose variable labor and harvesting costs to be included with OC. In conventional presentations, total revenue and total costs are presented in monetary terms, with yield or output measured along the horizontal axis and the vertical axis representing dollars. The principle is the same as that represented here, however. We express the principle in physical terms to lessen the number of steps in presentation.

the slope at any one point on yield isoquants. Except for those in fig. 2B, the replacement ratio $\frac{\triangle F_1}{\triangle F_2}$ changes at each point on the yield isoquant. This replacement ratio must equal the inverse price ratio, $\frac{P_2}{P_1}$.[12] From equation (4a), we can derive (4b) which shows that the value of the added F_1 nutrient replaced (units of replaced F_1 multiplied by price per unit) is just equal to the value

$$(4a)\quad \frac{\triangle F_1}{\triangle F_2} = \frac{P_2}{P_1} \qquad (4b)\quad (\triangle F_1)(P_1) = (\triangle F_2)(P_2)$$

$$(5a)\quad \frac{\triangle F_1}{\triangle F_2} > \frac{P_2}{P_1} \qquad (5b)\quad (\triangle F_1)(P_1) > (\triangle F_2)(P_2)$$

$$(6a)\quad \frac{\triangle F_1}{\triangle F_2} < \frac{P_2}{P_1} \qquad (6b)\quad (\triangle F_1)(P_1) < (\triangle F_2)(P_2)$$

of F_2 replaced (units of added F_2 multiplied by price per unit). While the value of the one element added just equals the value of the one replaced, a substitution ratio greater than the inverse price ratio (5a) indicates that cost of the added quantity of F_2 is less than the value of F_1 replaced (5b). Conversely, if the replacement ratio $\frac{\triangle F_1}{\triangle F_2}$ is less than the price ratio (6a), F_1 can be added at a lower cost than the value of F_2 replaced (6b). Again, it is obvious that the optimum combination of elements, aside from the "fixed proportions" case illustrated in fig. 4B, vary with the cost of the different elements. It will also change with yield level if slope of the yield isoquants change as in fig. 7A, or if the isoclines are curved as in fig. 7B. In the case of one element from two different sources which substitute at constant rates (fig. 2B), the least-cost combination of nutrients for any one yield is always attained with use of all of the nutrient from one source and none from another source. This is true since (as the linear isoyield lines suggest) the two elements replace each other at constant rates.

BIOLOGICAL LIMITS IN NUTRIENT COMBINATIONS

In conformity with accepted economic terminology, nutrient combinations have been expressed in terms of their substitution or replacement rates. In the chemical processes of the plant one element may not substitute for another; however, it is true that moderate yield increases may be attained with several combinations of elements. A farmer may obtain a 5-bushel increase in corn from use of ammonium nitrate alone, from phosphate alone or from a mixed fertilizer such as 20-20-0 or 8-8-8. If all of the mixtures give the 5-bushel increase, they can be looked upon as substitutes for each other in attaining the given yield even though physiological substitution does not actually take place. Elements Na and K may be real substitutes over wide ranges in the chemical processes of some plants. However, even though plant nutrients such as N, P or K do not directly serve as substitutes in the chemical functions of the plant, the fact that similar yield increases can be attained with different combinations of nutrients causes them to serve as substitutes in the decision-making framework of the farmer. Within limits, he can use more of one nutrient and less of another in attaining yield increases under many soil situations. While the terms "substitution" or "replacement rates" thus may not represent an entirely accurate physiological concept, they are employed in the remainder of this study in the absence of more appropriate terms. While substitution is discussed in forthcoming sections, the biological exceptions mentioned above should be kept in mind. From the standpoint of fertilizer ratios, the problem is perhaps as much one of finding "optimum combinations of nutrients" (least cost combinations for a given yield) as in determining "substitution" rates.

RIDGELINES AND PARTICULAR RECOMMENDATIONS

Since an isocline connects all points of the same slope (i.e., equal substitution rates) on successive isoquants, the isocline conforming to a particular price ratio also is an expansion path. It traces all combinations of nutrients which give least-cost yields. If an isocline conforming to a particular price ratio is nearly straight, an increase in nutrients by a fixed proportion is "nearly consistent" with the least-cost use of nutrients. If the isocline "bends sharply," a fixed-ratio fertilizer increase will not give the most economic nutrient combinations. While little is known about them, isocline maps can take on many distinct forms. They can be established only by basic experiments. In a family of isoclines, one denoting a substitution ratio of 1.0 may be "bent"; one for a 0.5 substitution ratio may be linear. Hence, with a nutrient price ratio of 1.0; least-cost fertilizer mixes will not include a 1:1 — or even a fixed ratio; with a 0.5 price ratio, the fertilizer mix should follow a fixed ratio line, although no particular ratio can be specified without knowledge of the function. One of a family of isoclines may be straight (although it need not be one along a 1:1 ratio); none may be straight.

Two isoclines can be called *ridge lines*. They denote zero substitution or replacement rates. If (1) the ridge lines are not far apart, (2) the isoclines within their boundary are fairly straight and (3) the yield isoquants for a particular yield have only a slight curvature with a slope not far different from the nutrient price ratio, several nutrient ratios, within the boundaries of the ridge lines, will give costs which are only slightly different (although only one isocline will denote *ex-*

[12] We have considered only the price of the elements here. If costs of application are the same, or nearly so, only these quantities need be considered. If proportional costs of application differ with the elements, labor and other costs per unit of element must be included in P_1 and P_2. The substitution ratio is always negative since the change in one nutrient is always negative and the change in the other is positive. For the sake of simplicity, however, the negative signs are not included with the ratios of this section.

actly the least-cost nutrient combination). If (1) the ridge lines are "sprung far apart," (2) isoclines "bend sharply" and (3) yield isoquants "curve sharply" away from price ratios, savings from changing nutrient ratios along an isocline will be considerable. We do not know whether the first or last situation will generally hold true for fertilization. We suspect that the range between the two situations will vary between crops, soils and years. Our method and principle are useful for any yield level. We later illustrate it with rates which will maximize profits above fertilizer cost in the sense of (1) the least-cost ratio for a given yield and (2) the optimum fertilization level.

However, most farmers are limited on capital and seldom go to fertilization rates where the added return of the last unit of fertilizer just exceeds or equals its added cost. They still need, however, knowledge of the least-cost ratio for about the yield level they can attain, considering the opportunity returns their capital will yield in hog feed, cattle or tractor fuel. In other words, the fertilizer recommendation needs to vary with the capital level of the farmer, as well as the soil. While the economic optima specified later are for conditions of unlimited capital, the data derived are of the kind useful for farmers regardless of their capital position. Perhaps the data are more useful for farmers with limited capital than for those with unlimited capital. For example, "rules of thumb" can be used for high yield levels and the amount of fertilizer specified without any great loss in profits. The yield isoquants for high yields fall "near" the convergence of the ridge lines. Specification of numerous possible nutrient combinations for yields in the range 120-125 bushels of corn (shown later) give somewhat similar costs. However, for lower yields, the ridge lines are further apart and the isoquants have greater curvature. Use of the "exact" principles outlined here then give considerable gain over "rule of thumb" principles or procedures which lead to nutrient ratios near the ends of the isoquants.

We need to emphasize this: The loss or gain from "rules of thumb" or "economic principle" depends on the yield level within the boundaries of the ridge line. If the yield to be attained is relatively near the convergence point, as is the yield for the prices used under an "unlimited capital situation," the isoquant is short because the ridge lines are close together; a relatively few combinations will produce a given yield and they may have only slightly different costs. However, as one moves to lower yields, the ridge lines spring farther apart, and the isoquants within their boundaries have much greater curvature. So the "correct principle for a given yield" can give much greater profit than a "rule of thumb," which takes one near the ends of isoquants falling low in the isocline map.

SOURCE OF DATA AND EMPIRICAL PROCEDURES

The preceding section provided basic principles which serve as a guide (1) in providing marginal or incremental quantities for determining economic rates of fertilizer application and (2) recommendations of economic combinations of nutrients. Using these models as a basis for empirical and statistical procedures, experiments were set up to allow derivation of the relevant production relationships. Experiments were conducted in 1952 with corn on calcareous Ida silt loam soil in western Iowa and with alfalfa and red clover on Webster and Nicollet loam in north-central Iowa. Two variable nutrients were used on each experiment. Nitrogen in the form of ammonium nitrate and P_2O_5 in the form of concentrated superphosphate were applied to corn while K_2O in the form of potassium chloride and P_2O_5 in the form of concentrated superphosphate were applied to both alfalfa and red clover. Observations were obtained from an incomplete factorial experimental design of the nature indicated by table 1.

The same design was used for alfalfa and red clover except that the second variable nutrient was K_2O. With replication, there were 114 observations for each of the three experiments.[13] Two cuttings were obtained from both the alfalfa and red clover. Yield measurements for hay were in terms of 12-percent moisture. This design, with randomized plots, allows continuous observations at the extremes of application rates with combinations of the various nutrients. It also provides sufficient observations over other points of the production surface for estimation of the two-variable nutrient function. In the experiments, all resources or inputs but fertilizer were held constant except for the variable quantities of labor and machine services for application and harvesting; seeding rates were constant.

WEATHER IN EXPERIMENTAL YEAR

The 1952 growing year was one favorable for use of fertilizer. The spring was fairly cool and wet. Rainfall was ample to mid-August when a 2-month drouth began. For these reasons, the experimental data do not necessarily serve as a basis for inference to average years. On corn the constant plant population of 18,000 plants per acre for all treatments may have limited the response obtained from heavy fertilization rates.

We again point out that fertility nutrients may

[13] The treatments were assigned at random (completely randomized block design).

TABLE 1. DESIGN OF EXPERIMENT FOR CORN. EACH "X" REPRESENTS AN EXPERIMENTAL PLOT.

Pounds P_2O_5 per acre	Pounds nitrogen per acre								
	0	40	80	120	160	200	240	280	320
0	xx	xx	xx	xx	xx	xx	xx	xx	xx
40	xx	xx			xx	xx			xx
80	xx		xx		xx		xx		xx
120	xx			xx	xx			xx	xx
160	xx	xx	xx	xx	xx	xx	xx	xx	xx
200	xx	xx			xx	xx			xx
240	xx		xx		xx		xx		xx
280	xx			xx	xx			xx	xx
320	xx	xx	xx	xx	xx	xx	xx	xx	xx

not substitute in the biological processes involved in producing a given amount of a specified part of the plant. However, they do serve as substitute means of attaining specified yield responses. With corn, for example, an average yield of 24.8 bushels per acre was obtained on the plots receiving 120 pounds of nitrogen and no P_2O_5. The plots receiving 40 pounds of P_2O_5 averaged 28.6 bushels. With slightly fewer pounds of phosphoric acid, equal increments in yield might have been attained with entirely different nutrient combinations of nitrogen and P_2O_5. With 160 pounds of N and 40 pounds of P_2O_5, the plots averaged 101.5 bushels; with 240 pounds of P_2O_5 and 80 pounds of N, the average was 102.5 bushels. Similarly, for clover, the plots receiving 120 pounds of P_2O_5 and 160 pounds of K_2O averaged 3.66 tons while those receiving 160 pounds of P_2O_5 and 80 pounds of K_2O averaged 3.68 tons. Thus, while the nutrients may not serve as substitutes in the chemical process of the plants, they do serve as substitute means of attaining given yield increases. These are the kind of data needed in farmer decision-making; it is the cost of producing a given yield, rather than the chemical process itself, which directly concerns him.

While nutrients may serve as substitutes over a limited range in attaining given levels of crop response, the data also show how they eventually serve as technical complements as one is increased alone. By technical complementarity, we refer to the situation where an increase in one element without an increase in the other either (1) does not add anything to total yield or (2) actually decreases total yield. On corn, for example, any increase in N alone (a path followed horizontally from left to right in table 1 with P_2O_5 held "fixed" at any level in the table) causes first an increase and then a decrease in total yield. The same situation holds true for P_2O_5. That is, yield increases and then decreases down the column of the table with N fixed at specified levels, and this decrease indicates that N also is a limitational nutrient with P_2O_5.[14] That the two nutrients serve as limitational resources or technical compliments to each other also was illustrated by the fact that yields were taken to successively higher levels with diagonal movements from northwest to southeast in the table; under this "movement" over the cells and columns of the table, the two elements are, in effect, increased simultaneously and in fixed proportions.

DERIVATION OF PRODUCTION OR YIELD FUNCTIONS

After collection of yield observations, the next step was that of deriving production functions, input-output or response coefficients. This step is itself complex. Only meager attention has been devoted to forms of algebraic equations best suited to estimating the fertilizer yield surface. While Mitscherlich and Spillman advocated or tried application of an exponential function for experiments with single variable nutrients, there is not sufficient evidence that this type of function adequately describes the fertilizer-input crop-output relationship under all situations. It does not allow diminishing total returns (a negative marginal product) and hence can be rejected on logical grounds for experiments with high fertilization.[15] Since all three experiments included some treatments denoting diminishing total yields, application of the exponential function to the data required discarding these observations. In the opinion of the writers, functions which allow use of all experimental observations are more efficient and more objective than those which necessitate dropping part of the data.

Another production function equation used for many situations has been the Cobb-Douglas or logarithmic function. It is similar to the Spillman function in the sense that it cannot be applied to diminishing total yield. Also, it assumes a constant elasticity of response over the entire surface. Finally, while it allows the isoquants to approach technical complementarity, it does not allow the range of substitution or combination ratios to narrow as higher yields are attained (i.e., they do not allow the marginal rate of substitution to change along a fixed ratio line as higher yields are attained).

Because of these difficulties in finding one appropriate algebraic function and since little previous work has been done in deriving equations with two variable elements, several functions were fitted to the field observations. First, five functions with a single nutrient variable were fitted. Thirty-five of these single-variable equations were derived for each of the three crops. Five different single-variable functions were fitted to the observations in (1) each complete column and row of nine observations in table 1 for the three crops and (2) to the observations in the cells along the northwest-southeast diagonal of this table. The five equations fitted to each of these seven different sets of single-variable observations for each crop were as follows:

(7) $Y = a + bF + CF^2$,

(8) $Y = m - ar^F$,

(9) $Y' = aF^b$,

(10) $Y = a + bF + c\sqrt{F}$ and

(11) $Y = a + bF + cF^2 + d\sqrt{F}$,

[14] For further details on these terms and situations, see: Heady, Earl O. Economics of agricultural production and resource use. Chs. 2-5. Prentice-Hall, Inc., New York. 1952.

[15] See: Baule, B. Zu Mitscherlichs Gesetz der Physiologischen Beziehungen. Landw. Jahrb. 51:363-385, 1918. The function developed and employed by these individuals is of the form $Y = m - ar^x$ where m is the maximum yield which can be attained with the use of fertilizer, a is the difference between m and yield with no fertilizer, r is the ratio by which one yield increment exceeds the previous increment, x is the rate of fertilizer application and Y is the predicted yield. The ratio of successive increments is constant under this equation, a situation which may or may not be unreal. In other words, if the first 20-pound increment adds 10 bushels to the yield and the second adds 8 bushels, the value of r (the ratio of increments) is 0.8 and, therefore, the third 20-pound increment would be expected to yield 0.8 of 8 bushels, or 6.4 bushels.

where Y is total yield, Y′ is yield above check plots and F refers to total quantity of the particular variable nutrient.

After the correlation coefficients and error terms for these functions were derived, the equations which were statistically acceptable were plotted against a scatter diagram of the observations for each set of data. This was done as a first step in determining which of the following functions, with two variable nutrients, would best describe the phenomena at hand. In describing the input-output relationship for any particular set of observations, the response curve for a single-variable function is affected by this set of observations alone, and not by those relating to other portions of the response surface. The single-variable response curves derived from a two-variable equation are affected by all observations on the surface. Comparisons of single-variable response curves derived from single-variable equations with those derived from two-variable equations thus helped suggest which of the latter are best in overall prediction. Statistics for the single-variable functions are given in the Appendix. No one algebraic form of equation (of the single-variable ones tried) was best for each separate set of observations of a single-variable nature.

TWO-VARIABLE FUNCTIONS

Three functions with two variable nutrients were fitted to the observations for each crop. One was a logarithmic equation, one was a quadratic equation with a simple cross-product term and the third was a function with square root terms. The logarithmic function "forces" a restraint on the production surface which parallels the agronomic assumption often used in fertilizer recommendations; namely, that the yield isoquants have a constant slope along a fixed nutrient line in the nutrient plane, and, therefore, the same nutrient combination should be used for all yield levels. The other two functions allow yield isoquants to change in slope along a fixed nutrient line. Hence, use of the several functions allows the testing of these alternative hypotheses. The central mathematical prediction problem is one of finding a two-variable function which best fits the observations. Isolation of this best fit was attempted by (1) examining the statistics for each function, (2) comparing single-nutrient response curves derived by the two-variable functions with a similar curve predicted from the best fitting single-variable response curves and (3) comparing the response curves and yield isoquants predicted from the two-variable functions with scatter diagrams of the observations. One two-variable function was then selected for prediction for each crop.

CORN

Two-variable functions derived from the corn experiment are presented below. Since substitution ratios do not change along a scale line for the logarithmic function, different equations were fitted separately to the "lower" and "upper" portions of the observations over the production surface; the same function also was fitted to the pooled observations. In the equations, P refers to P_2O_5 in pounds per acre and N refers to nitrogen in pounds per acre. For equations (13) and (14), Y refers to total yield in bushels per acre; in equation (12, a-c), Y′ refers to total yield above the check plot level.

(12) Logarithmic
(a) "lower" observations
$$Y' = 2.5198P^{0.4393} N^{0.3159}$$
(b) "upper" observations
$$Y' = 34.405P^{0.1353} N^{0.0770}$$
(c) pooled data
$$Y' = 2.7649P^{0.4090} N^{0.2877}$$

(13) Crossproduct
$$Y = -7.51 + 0.584N + 0.664P - 0.00158N^2 - 0.00180P^2 + 0.00081NP$$

(14) Square root
$$Y = -5.682 - 0.316N - 0.417P + 6.3512\sqrt{N} + 8.5155\sqrt{P} + 0.3410\sqrt{NP}$$

BASIC STATISTICS AND PRODUCTION SURFACE ESTIMATES

The basic statistics relating to the three types of functions are given in table 2. The coefficients of determination (R^2) show the following percentages of variance in yield explained by quantities of the two nutrients: Logarithmic overall (12c), 86 percent, crossproduct (13), 83 percent and square root (14), 91 percent. The t values show each individual regression coefficient, for the overall functions,[16] to be significant at the 1-percent level of probability. After examining the multiple correlation coefficients, the residual mean squares (see Appendix) and comparing predictions from these equations with (1) a scatter diagram of the observations and (2) the same quantities predicted from single variable equations, the square root function was selected as being most efficient for predicting the production surface, input-output or response curves, yield isoquants and marginal quantities for corn.

PRODUCTION SURFACE

The two-variable equation (14) was employed to predict the two total yield quantities shown in table 3. These quantities are the counterpart of a production surface, except that they represent dis-

[16] Overall functions refer to the functions fitted from all the observations over the entire range of treatments.

TABLE 2. VALUES OF R FOR TWO-VARIABLE NUTRIENTS AND VALUES OF t FOR INDIVIDUAL REGRESSION COEFFICIENTS. CORN.

Equation	Value of R	Value of t for coefficients in order listed in equations				
12a	0.8952*	11.94*	9.56*	....	....	....
12b	0.3882*	2.85*	1.62†	....	....	....
12c	0.9255*	18.62*	15.23*	....	....	....
13	0.9122*	9.21*	10.46*	5.24*	8.96*	10.19*
14	0.9582*	7.91*	10.44*	7.32*	9.81*	8.85*

* $0 < P < 0.01$
† $0.10 < P < 0.20$

TABLE 3. PREDICTED TOTAL YIELDS FOR SPECIFIED NUTRIENT COMBINATIONS ON CORN.

Lbs. P_2O_5 per acre	Pounds nitrogen per acre								
	0	40	80	120	160	200	240	280	320
0	—5.7	21.8	25.8	25.9	24.0	20.9	16.8	12.1	6.8
40	31.5	72.6	82.3	88.7	88.5	88.6	87.4	85.3	82.5
80	37.1	83.9	95.9	102.1	105.4	106.8	106.9	105.9	**104.1**
120	37.5	88.7	102.4	110.1	114.5	116.9	117.9	117.8	116.8
160	35.3	90.1	105.4	114.2	119.6	122.9	124.6	125.2	124.9
200	31.6	89.3	105.9	115.7	122.0	126.1	128.5	129.7	**130.0**
240	26.1	87.0	104.8	115.6	122.6	127.4	130.4	132.2	133.0
280	19.9	83.6	102.5	114.1	121.9	127.2	130.8	133.2	134.5
320	13.1	79.2	99.2	111.5	120.0	126.0	130.1	132.9	**134.7**

tinct points on it corresponding to the P_2O_5 and nitrogen quantities in the rows and columns.

Since both nutrients were present in the soil in limited amounts, yields were not high for either nutrient used alone. With no P_2O_5, 120 pounds of nitrogen gives a maximum yield of 25.9 bushels in table 3; with no nitrogen, 120 pounds of P_2O_5 gives a maximum of 37.5 bushels.[17] However, with the addition of 40 pounds of P_2O_5, a large yield increase takes place across the nitrogen columns; a similar change takes place for P_2O_5 down the first column. In other words, the productivity of one nutrient is highly limited by the amount of the other with which it is combined. With both nutrients variable, the predicted maximum yield is 135.8 bushels with 397.6 pounds of nitrogen and 336.6 pounds of P_2O_5.[18]

Diminishing total yields for nitrogen, as the variable nutrient, are indicated up to 200 pounds of P_2O_5, as the fixed nutrient. Similarly, negative marginal products hold true for P_2O_5, as the variable nutrient, for up to 240 pounds of nitrogen, as the fixed nutrient. Diminishing total yields are not predicted, within the range of the observation, when both nutrients are variable in a 1:1 ratio. Just as these two nutrients interact to affect the productivity of each other, another variable resource, such as stand, might well have caused different productivity coefficients for either nitrogen or P_2O_5.

Figure 9 is the response surface showing these productivity relationships more vividly. A vertical slice through this surface perpendicular with the P_2O_5 axis is the counterpart of a single-nutrient response curve with nitrogen as the variable nutrient; a slice perpendicular to the nitrogen axis represents P_2O_5 as the variable resource and nitrogen as the fixed nutrient. A vertical slice intersecting the origin is the counterpart of a response curve with both nutrients variable in fixed proportions. Horizontal slices through the surface provide yield isoquants showing all possible combinations of the two nutrients which will produce a given yield; these quantities are provided in later paragraphs.

Table 4 indicates the marginal products or yields corresponding to the total yields of table 3; they are the counterparts of the slopes of vertical slices through fig. 9, at the yield levels of table 3. These figures again illustrate that the quantity of one nutrient affects the productivity of the other. For example, movement down any column represents an increase in the ratio $\frac{P}{N}$; movement across a row represents a decrease in the $\frac{P}{N}$ ratio. Down any column, the marginal product of P_2O_5 decreases while the marginal product of nitrogen increases; across rows, the opposite holds true. Marginal yields per pound of nutrient are equal for the two nutrients when the quantity of each is 120 pounds. The negative marginal products represent diminishing total yields; the small positive marginal products in much of the table correspond to the fact that the production surface is quite flat over a large section.

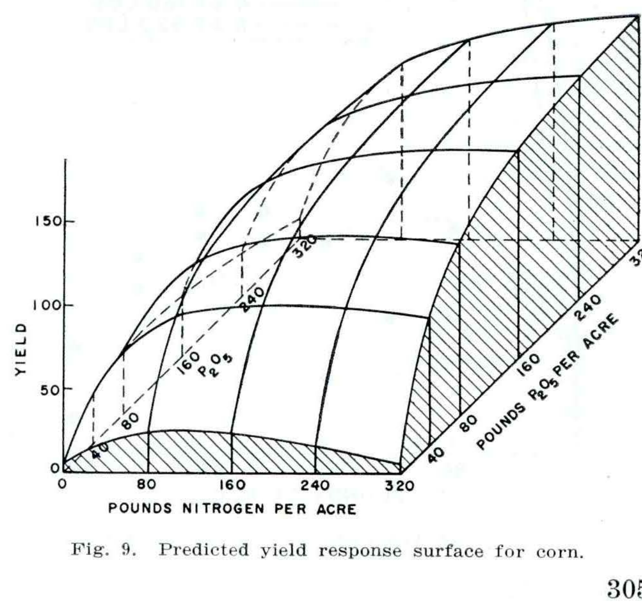

Fig. 9. Predicted yield response surface for corn.

[17] These fertilizer quantities do not represent the exact maximum yield. The maximum yields for nitrogen variable with P_2O_5 fixed at zero or P_2O_5 variable with nitrogen fixed at zero are determined by setting the derivatives for each variable nutrient equal to zero and solving for N or P respectively as in (a) and (b) below. The maximum for P_2O_5 is with 104.3 pounds; the maximum for N is with 101.0 pounds of this nutrient. The corresponding yields are 37.7 and 26.4 bushels, respectively.

(a) $0 = + 0.316 - 3.1756N^{-0.5}$ $N = 101.0$ lbs.

(b) $0 = + 0.417 - 4.2578P^{-0.5}$ $P = 104.3$ lbs.

[18] The predicted maximum yield, an extrapolation beyond the observations of the experiment, was obtained as follows. The partial derivatives (the marginal products) for each nutrient were set at zero; the quantity of each nutrient, to give a partial derivative of zero, was then computed. These are the quantities of nutrients which give a maximum yield. They were substituted back into the original function and the maximum yield was predicted accordingly.

TABLE 4. MARGINAL PRODUCT OR YIELD (BUSHEL PER POUND OF FERTILIZER NUTRIENT) FOR COMBINATIONS INDICATED IN ROWS AND COLUMNS. UPPER FIGURE FOR NITROGEN; LOWER FIGURE FOR P_2O_5.*

Lbs. P_2O_5	Pounds of nitrogen								
	0	40	80	120	160	200	240	280	320
0	— —	0.19 —	0.04 —	0.02 —	—0.07 —	—0.09 —	—0.11 —	—0.12 —	—0.14 —
40	— 0.26	0.36 0.43	0.16 0.49	0.07 0.55	0.02 0.60	—0.02 0.64	—0.04 0.67	—0.06 0.71	—0.08 0.74
80	— 0.06	0.43 0.17	0.21 0.23	0.11 0.27	0.06 0.30	0.01 0.33	—0.01 0.35	—0.04 0.38	—0.05 0.40
120	— —0.03	0.48 0.07	0.24 0.11	0.14 0.14	0.08 0.17	0.04 0.19	0.01 0.21	—0.01 0.23	—0.03 0.25
160	— —0.08	0.52 0.01	0.28 0.04	0.17 0.07	0.11 0.09	0.06 0.11	0.03 0.13	0.01 0.14	—0.02 0.16
200	— —0.11	0.57 —0.04	0.31 —0.01	0.19 0.02	0.13 0.03	0.08 0.05	0.04 0.07	0.02 0.09	—0.01 0.10
240	— —0.14	0.60 —0.07	0.33 —0.04	0.21 —0.02	0.14 —0.01	0.10 0.02	0.06 0.03	0.03 0.04	—0.01 0.05
280	— —0.16	0.63 —0.10	0.36 —0.07	0.23 —0.05	0.16 —0.03	0.11 —0.02	0.07 —0.01	0.04 0.01	0.02 0.02
320	— —0.18	0.67 —0.12	0.37 —0.09	0.25 —0.07	0.18 —0.06	0.12 —0.04	0.09 —0.03	0.06 —0.03	0.03 0.01

* These figures are the derivatives of yield in respect to the single-nutrient variable while the other is fixed. They are derived from equation (14), with the nitrogen and P_2O_5 quantities shown at the top of the columns and to the left of the rows. The 0.36 in the cell where both nutrients are 40 pounds is the derivative or marginal product for nitrogen as the variable nutrient while P_2O_5 is fixed at 40 pounds. The 0.43 is the marginal product for P_2O_5 as the variable nutrient while nitrogen is fixed at 40 pounds.

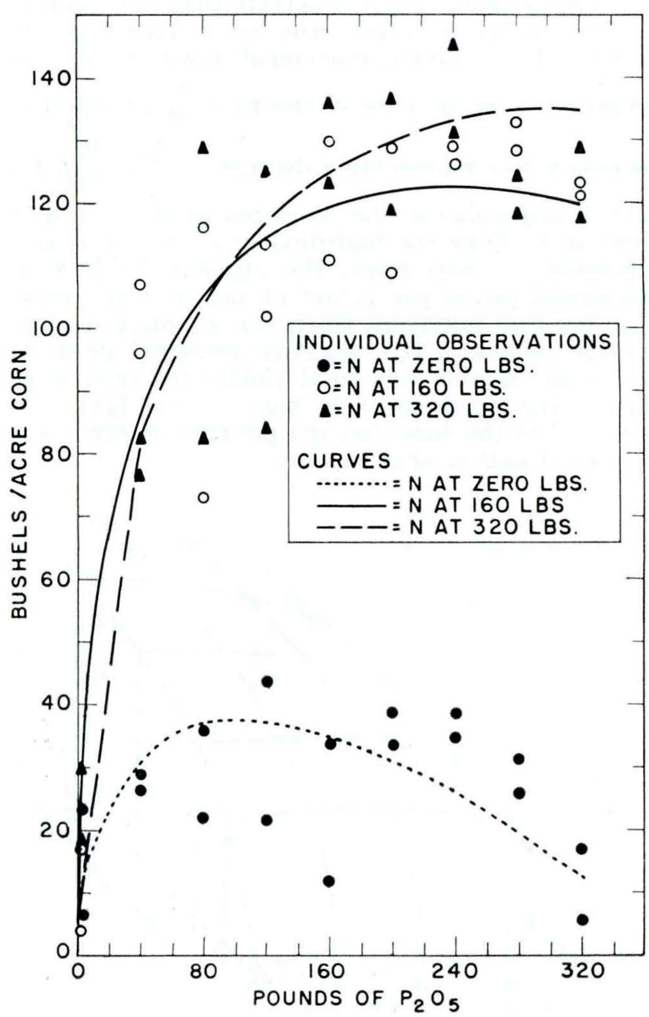

Fig. 10. Total yield with P_2O_5 variable and nitrogen fixed at three levels.

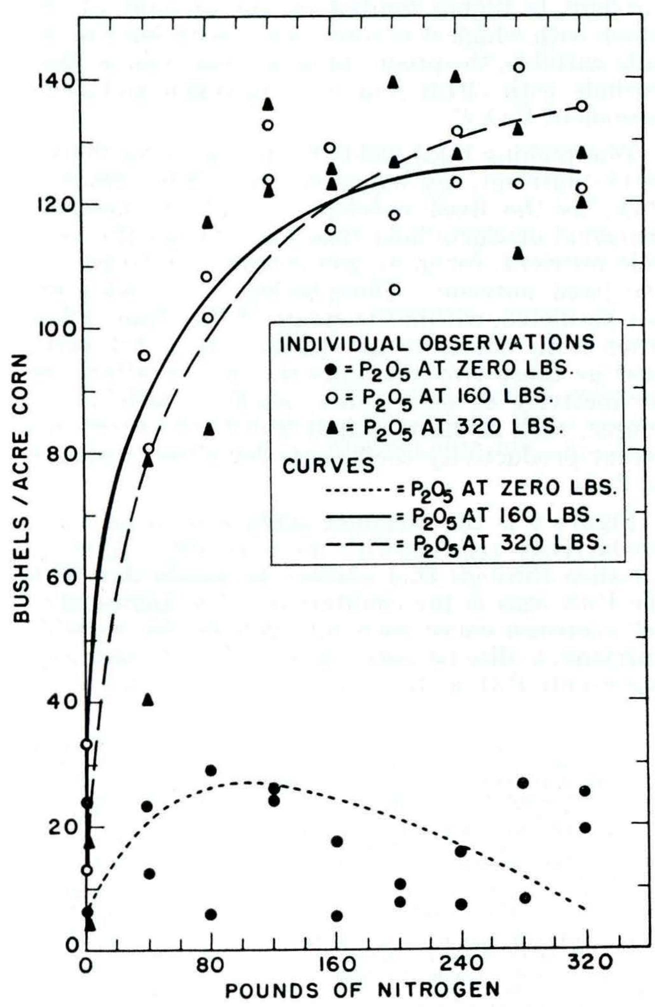

Fig. 11. Total yield with nitrogen variable and P_2O_5 fixed at three levels.

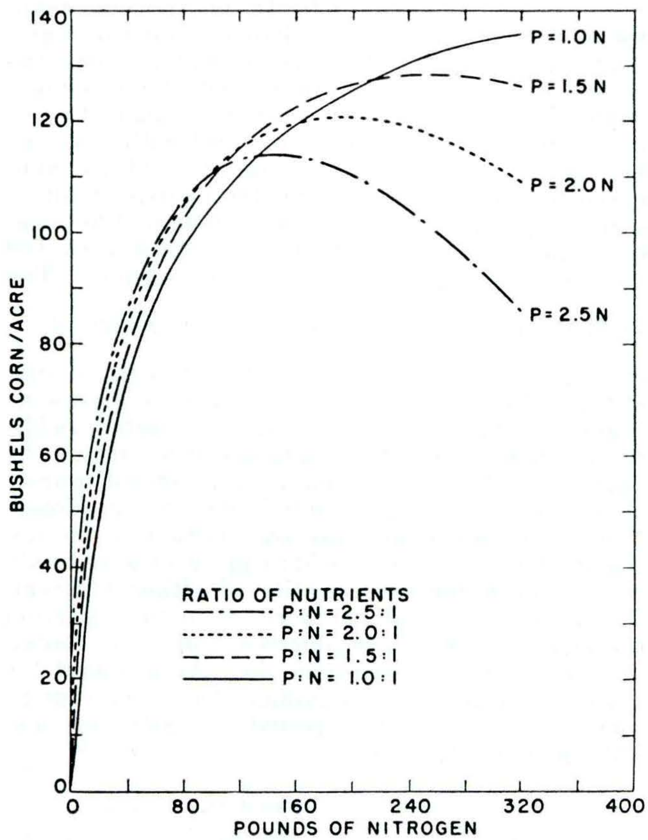

Fig. 12A. Yield of corn with nutrients increased in fixed proportions.

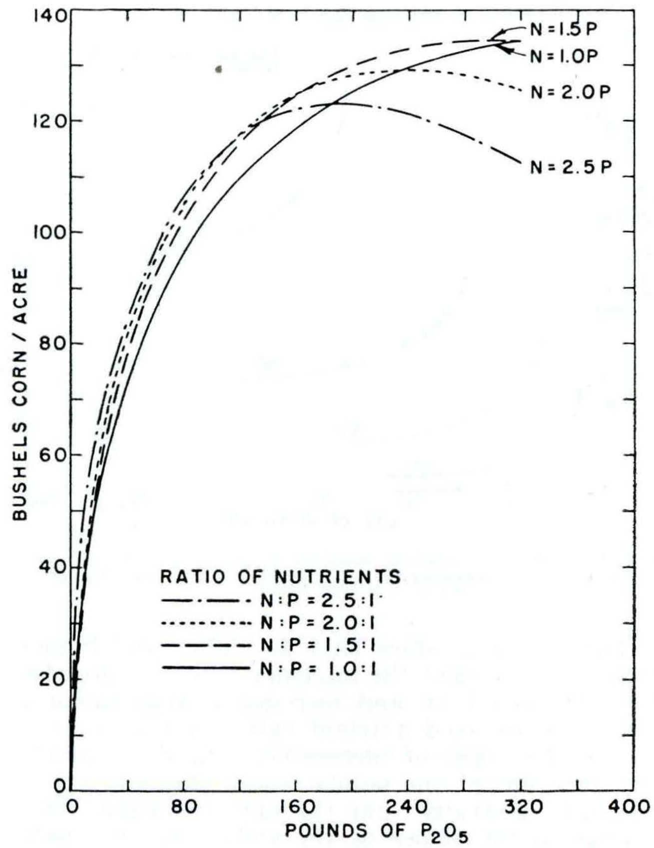

Fig. 12B. Yield of corn with nutrients increased in fixed proportions.

SINGLE VARIABLE INPUT-OUTPUT CURVES

Figures 10 and 11 provide total response as yield curves when one nutrient is fixed at specified levels and the other is variable. With a zero nitrogen input for fig. 10, the P_2O_5 curve falls low in the plane with diminishing total yield indicated for small inputs of P_2O_5. With nitrogen input at 160 and 320 pounds, the response curves for P_2O_5 cross each other. This is due to the fact that, with small quantities of P_2O_5, 320 pounds of nitrogen gives an excessive quantity of nitrogen; with larger quantities of P_2O_5, the two nutrients interact to give slightly higher yields for 320 than for 160 pounds of nitrogen. A similar situation exists for nitrogen as the variable nutrient. With P_2O_5 fixed at 160 and 320 pounds, the nitrogen response curves in fig. 11 again cross each other. An increase in P_2O_5 from 160 to 320 pounds adds nothing to yield if nitrogen inputs are small. The fact that the maximum yield from nitrogen, with no P_2O_5, is lower than the maximum of P_2O_5, with no nitrogen, suggests that the soil, while deficient in both nutrients, was lacking especially in P_2O_5.

"SCALE LINE" RESPONSE CURVE WITH BOTH NUTRIENTS VARIABLE

Figures 12A and 12B show predicted input-output or response curves when the two nutrients are increased in fixed ratios. The amount of one element is always in a fixed ratio to the amount of the other, as indicated on the bottom of the graphs. In fig. 12A, for example, the ratio line of $1P = 2.0N$ means that 2 pounds of P_2O_5 is used for each pound of nitrogen; with a nitrogen input of 160 pounds, input of P_2O_5 is 320 pounds; and with nitrogen at 320 pounds, input of P_2O_5 is 640 pounds. These two figures indicate that greatest yields can be obtained from use of the two nutrients in a 1:1 ratio. For light applications of fertilizer, greater response per pound may be obtained with nutrient ratios differing from 1:1.

YIELD ISOQUANTS

Yield isoquants derived from the same basic yield surface equation are shown in fig. 13A. The isoquant equations, derived from the production surface equation, are those shown below* for nitrogen (15) and P_2O_5 (16).

$$\text{* (15)} \quad N = \left[\frac{(10.05 + 0.539\sqrt{P}) \pm \sqrt{(-0.4115P) + (15.0996\sqrt{P}) - 1.2645Y + 33.153)}}{-0.6323} \right]^2$$

$$\text{(16)} \quad P_2O_5 = \left[\frac{(10.20 + 0.408\sqrt{N}) \pm \sqrt{-0.4115N + 16.4115\sqrt{N} - 1.6696Y + 63.027}}{-0.8348} \right]^2$$

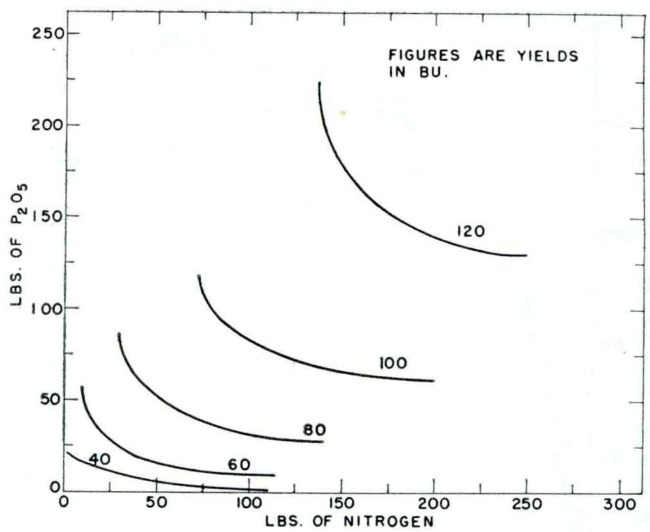

Fig. 13A. Yield isoquants showing all possible nutrient combinations in producing specified yield (ends of curves give limits in nutrient substitution).

The isoquants show that as higher and higher yields are attained, the marginal rates of substitution between P_2O_5 and nitrogen change along a scale line (a fixed nutrient ratio line). In other words, the slopes of successively higher isoquants are different at the points where they are intersected by a straight line through the origin. This change in the slopes of the yield isoquants indicates that the combinations of nutrients (the fertilizer ratio) which gives lowest cost for one yield level is not the same mixture which gives lowest cost for another yield level. In other words, the least-cost combination is not the same for yields of 60 and 120 bushels. This same point is illustrated in table 5 which shows several predicted combinations of the two nutrients which will produce the same yield and the marginal rates of nutrient substitution for the indicated combinations.

Figures for isoquants indicate, on the one hand, the minimum quantity of nitrogen and the maximum quantities of P_2O_5 which will produce the stated yield and, on the other hand, the maximum quantities of P_2O_5 and the minimum quantities of nitrogen. More P_2O_5 must be used with a stated amount of nitrogen for a higher yield as compared to a lower yield. With 160 pounds of nitrogen, 165 pounds of P_2O_5 allows a yield of 120 bushels; only 64 pounds of P_2O_5 is required with 160 pounds of nitrogen to produce 100 bushels. The yield isoquants also indicate that the range of $\frac{N}{P}$ ratios, over which the two nutrients can be substituted in obtaining a given yield, narrows as higher yield levels are attained. For higher yields, the nutrients become limitational in nature as the "upper ends" of the isoquants take on an infinite slope and as the "lower ends" take on zero slopes. Low yields can be attained by addition of one nutrient alone, but high yields can be attained only with some minimum quantity of either nutrient. The maximum yield per acre, as predicted from the equation, can be produced by only one combination of P_2O_5 and nitrogen (i.e., the isoquant for a yield of 135.8 bushels reduces to a single point corresponding to 397.6 pounds of nitrogen and 336.6 pounds of P_2O_5).

NATURE OF ISOCLINES FOR CORN

Figure 13B includes two isoclines for corn. As indicated previously, an isocline is a line indicating points of equal slope on successive yield isoquants. It, therefore, indicates points on all yield isoquants which denote the same replacement or substitution rate between nutrients.[19] The line RR = 1.5 shows all points in the nutrient plane where 1

[19] The dotted axes in fig. 13B indicate the limits in levels of nutrients used in the study. Hence, the portion of the isoclines falling outside the dotted lines represents predictions outside experimental observations.

TABLE 5. ISOQUANT COMBINATIONS OF NUTRIENTS FOR PRODUCING SPECIFIED YIELDS AND CORRESPONDING MARGINAL RATES OF SUBSTITUTION.

60-bushel yield			120-bushel yield		
Lbs. of N*	Lbs. of P_2O_5*	Marginal rate of substitution ($\Delta P/\Delta N$) showing lbs. P_2O_5 replaced by 1 lb. nitrogen†	Lbs. of N*	Lbs. of P_2O_5*	Marginal rate of substitution ($\Delta P/\Delta N$) showing lbs. P_2O_5 replaced by 1 lb. nitrogen†
10	57.60	—5.10	150	183.14	—2.55
20	32.30	—1.31	160	165.10	—1.32
30	23.30	—0.63	170	154.43	—0.86
40	18.50	—0.37	180	147.15	—0.61
50	15.50	—0.24	190	141.91	—0.45
60	13.60	—0.16	200	138.04	—0.24
70	12.30	—0.11	210	135.14	—0.19
80	11.30	—0.08	220	133.03	—0.14
90	10.60	—0.06	230	131.53	—0.10
100	10.20	—0.04	240	130.53	—0.06
110	9.90	—0.03	250	129.94	—0.03
120	9.70	—0.01	260	129.71	—0.01

* Derived from yield isoquant equations and show possible combinations of nutrients in producing a single, specified yield.

† From equations of marginal substitution rates which are derivatives from isoquant equations and show substitution or replacement rate at "exactly" the nutrient combination shown; they are not averages between nutrient combinations.

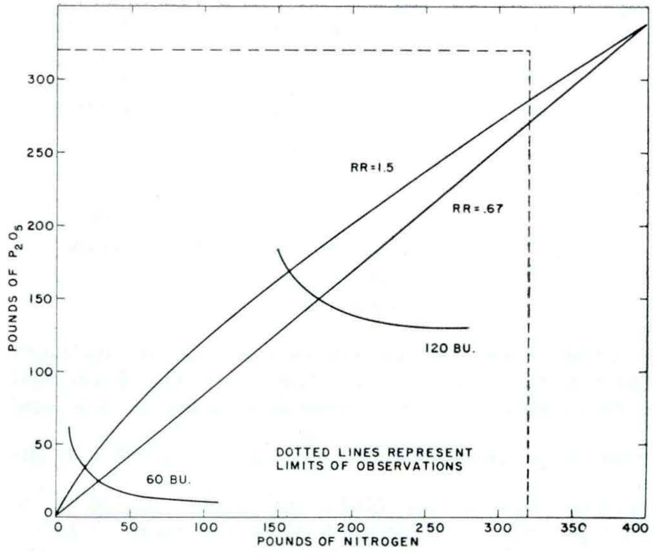

Fig. 13B. Yield isoclines showing points of equal slope and replacement rate for corn isoquants.

pound of nitrogen will replace 1.5 pounds of P_2O_5; the curve, RR = 0.67 indicates all points where 1 pound of nitrogen replaces 0.67 pound of P_2O_5. The 0.67 isocline is quite close to a straight line; isoclines denoting larger or smaller replacement ratios have greater curvature than those shown. The isoclines denote the path of optimum (least cost) nutrient combinations as higher yield levels are attained. If, for example, nitrogen has a price (or total cost of application) 1.5 times that of P_2O_5, the upper curve denotes the optimum nutrient combinations for all possible yield levels. All isoclines converge at the point of maximum yield. Since the isoclines have only slight curvature for intermediate replacement or substitution ratios, sacrifices in profits would be small if the same nutrient combination were used for all yield levels in this experiment. For price and substitution ratios at the extreme, however, sacrifices in returns increase as the same nutrient combination is used for all yield levels.

ECONOMIC OPTIMA

Quantities such as those derived in a previous section provide the basis for specifying (1) the optimum combination of nutrients for any yield level and (2) the optimum rate of fertilization. This section specifies these quantities under various price ratios for a farmer who might have unlimited capital.

SINGLE NUTRIENT VARIABLE

The optimum level of fertilization, whether one or both nutrients can be varied, depends on the fertilizer/crop price ratio, as well as the marginal yield rate. As explained in the first section, the quantities, $\frac{\triangle Y}{\triangle N}$ and $\frac{\triangle Y}{\triangle P}$ are determined from the yield equation as derivatives. Since the changes in N are very small, the partial derivatives are denoted as $\frac{dY}{dN}$ rather than as $\frac{\triangle Y}{\triangle N}$. With corn at \$1.40 per bushel and nitrogen at \$0.18 per pound (including nitrogen and the cost of application) the price ratio is $\frac{0.18}{1.40}$ or 0.129. Hence the derivative (of equation 14) for corn yield with respect to nitrogen is set at this quantity in equation (17) below. Solving equation (17) for N, 53.3 pounds of nitrogen equates the marginal product, and therefore is the most profitable quantity of this nutrient when no P_2O_5 is used. The corresponding yield (from equation 14) is 24.8 bushels.

$$(17) \quad \frac{dY}{dN} = -0.316 + 3.1756N^{-0.5} = 0.129$$

$$(18) \quad N = 53.3$$

With corn at 0.80 cents and nitrogen at 0.18, the price ratio is $\frac{0.18}{0.80}$ or 0.225 and, as the equations below show, 34.8 pounds of N is the level of fertilization to maximize profits.

$$(19) \quad \frac{dY}{dN} = -0.316 + 3.1756N^{-0.5} = \frac{0.18}{0.80} = 0.225$$

$$(20) \quad N = 34.8$$

With the price ratio at $\frac{0.10}{1.40}$ or 0.071, 67.4 pounds of N is most profitable. However, when 80 pounds of P_2O_5 are used and the price ratio is $\frac{0.18}{1.40}$, the derivative takes on the values shown in (21) below and 136.9 pounds of nitrogen represents the optimum.

$$(21) \quad \frac{dY}{dN} = -0.316 + 3.1756N^{-0.5} + 1.5653N^{-0.5} = 0.129$$

$$(22) \quad N = 113.5$$

Using the same price ratios for P_2O_5, we obtain the values below. For any one of the nutrient/crop price ratios shown, the optimum quantity of P_2O_5, with a stated amount of nitrogen, is slightly greater than the optimum quantity of nitrogen with the same amount of P_2O_5.

Zero input of nitrogen
- Price ratio of 0.129; P_2O_5 optimum is 60.8 pounds
- Price ratio of 0.225; P_2O_5 optimum is 44.0 pounds
- Price ratio of 0.071; P_2O_5 optimum is 76.1 pounds

160 pounds of nitrogen
- Price ratio of 0.129; P_2O_5 optimum is 140.5 pounds
- Price ratio of 0.225; P_2O_5 optimum is 101.6 pounds
- Price ratio of 0.071; P_2O_5 optimum is 175.9 pounds

MINIMUM COSTS FOR A SPECIFIED YIELD

Selection of the optimum quantity of a single nutrient is only a partial solution of the economic problem of fertilizer use. Still to be solved is (1) the optimum quantity of each nutrient or rate of application when both nutrients are variable and (2) the best combination of nutrients for any one yield level. The change in the slopes of the yield isoquants (along a scale line) suggests that the combination of the two nutrients which will give the lowest cost, for a stated yield, changes with the level of yield. The nutrient combination which is best for a 100-bushel yield is not also best for a 50-bushel yield. This point also is illustrated with the isoquant and substitution data of table 5. The least-cost resource combination for a given yield is attained when the marginal rate of substitution of the resources (i.e., the derivative of one nutrient in respect to the other, with yield at stated levels) is equal to the inverse price ratio. Hence, we can illustrate that the proportion of the two nutrients, to give the least cost, differs with yield level. First, we derive the equations of marginal rates of substitution (the first derivatives of change in one nutrient with respect to the other) as in equations (23) and (24). Second, we set these equal to the particular price ratio for the nutrients and solve for the nutrient combination which minimizes cost for the particular yield.[20] As was illustrated in an earlier section, the nutrient combination giving the minimum cost, for any one yield level, is attained when the marginal substitution ratio (the first derivative) is equal to the inverse price ratio. We now denote the substitution ratio as $\frac{dN}{dP}$, rather than $\frac{\triangle N}{\triangle P}$ as in the earlier equations.

$$(23)\quad \frac{dN}{dP} = \frac{-0.8348\sqrt{PN} + 8.5155\sqrt{N} + 0.3410\,N}{-0.6323\sqrt{PN} + 6.3512\sqrt{P} + 0.3410\,P} = -\frac{P_2O_5}{N}\text{ price ratio}$$

$$(24)\quad \frac{dP}{dN} = \frac{-0.6323\sqrt{PN} + 6.3512\sqrt{P} + 0.3410\,P}{-0.8348\sqrt{PN} + 8.5155\sqrt{N} + 0{\cdot}3410\,N} = -\frac{N}{P_2O_5}\text{ price ratio}$$

Using these procedures, we obtain the nutrient combinations in table 6; these are the least-cost combinations for the specified price ratios and yield isoquants. With a $\frac{N}{P}$ price ratio of 1.5, the combination for a 50-bushel yield should total 36.1 pounds; 32.7 percent of this should be nitrogen and 67.3 percent should be P_2O_5. For the same price ratio, a total of 180.9 pounds, composed of 43.8 percent nitrogen and 56.2 percent P_2O_5, should be used to minimize fertilizer cost for a 100-bushel yield. The mixture, to minimize cost for a given yield, should contain relatively more nitrogen for higher yield levels. Traditionally, this distinction has not been made in fertilizer recommendations; the same fertilizer mix has, for a given soil and productivity situation, usually been recommended for numerous yield levels. Similarly, with a change in the price ratio from 1.5 to 0.67, the percentage of nitrogen, for a 50-bushel yield, should change from 32.7 percent to 54.8 percent. For a 100-bushel yield, similar changes in the price ratio should cause the nutrient combination to change from 43.8 percent to 54.5 percent nitrogen.

SOLUTION FOR TWO-VARIABLE NUTRIENTS

In the analysis above, principles of profit maximization were used to independently specify (1) the optimum quantity of one variable nutrient, with yield as a variable and the second nutrient fixed and (2) the optimum combination for two variable nutrients for a given or fixed yield. However, these conditions need to be imposed simul-

TABLE 6. COMBINATIONS OF NITROGEN AND P_2O_5 TO MINIMIZE FERTILIZER COSTS PER SPECIFIED YIELD LEVEL FOR DIFFERENT PRICE RATIOS.

Yield level	Optimum pounds of N	Optimum pounds of P
	Price of \$0.18 per lb. for N and \$0.12 per lb. for P ($\frac{N}{P}$ ratio of 1.5)	
50 bu.	11.8	24.3
100 bu.	79.3	101.6
	Price of \$0.12 per lb. for N and \$0.18 per lb. for P ($\frac{N}{P}$ ratio of 0.67)	
50 bu.	19.8	16.3
100 bu.	99.1	82.7

[20] The particular function (equation 14) is somewhat difficult to handle in specifying derivatives (marginal rates of substitution) equal to a price ratio. One of the easiest procedures is to define the isocline equal to a particular price ratio and draw it on a contour map. Its point of intersection with each contour defines the marginal rate of substitution on the contour equal to the ratio defined by the isocline. Hence, we have used the following procedure where the particular price and substitution constant is defined as $\propto$: First, we start out with the original function (1) where a to f represent the regression coefficients.

$$(1)\quad bP + (d + f\sqrt{N})\sqrt{P} + (cN + e\sqrt{N} - Y + a) = 0$$

Differentiating (1) implicity, we get:

$$(2)\quad b + \sqrt{P}\left[\frac{f}{2\sqrt{N}}\frac{dN}{dP}\right] + \left[d + \frac{f\sqrt{N}}{2\sqrt{P}} + c\frac{dN}{dP} + \frac{e}{2\sqrt{N}}\frac{dN}{dP}\right] = 0$$

$$(3)\quad \frac{dN}{dP}\left[\frac{f\sqrt{P}}{2\sqrt{N}} + c + \frac{e}{2\sqrt{N}}\right] = -b - 0.5\left[\frac{d + f\sqrt{N}}{\sqrt{P}}\right]$$

Setting $\frac{dN}{dP} = -\propto$, to equal the price ratio, we obtain:

$$(4)\quad -(\propto)\left[\frac{f\sqrt{P}}{2\sqrt{N}} + c + \frac{e}{2\sqrt{N}}\right] = -b - 0.5\left[\frac{d + f\sqrt{N}}{\sqrt{P}}\right]$$

$$(5)\quad -(\propto)(0.05fP + c\sqrt{PN} + 0.05e\sqrt{P}) = -b\sqrt{PN} - 0.5d\sqrt{N} - 0.5fN$$

By letting $\sqrt{N} = u$ and $\sqrt{P} = v$, we obtain, from (1) and (5) as simultaneous equations, the following for $\frac{dN}{dP}$:

$$(i)\quad cu^2 + fuv + bv^2 + eu + dv - (y - a) = 0$$

$$(ii)\quad \tfrac{1}{2}fu^2 + (b - \propto c)uv - 0.5\propto fv^2 + \tfrac{1}{2}du - 0.5\propto ev = 0$$

From i and ii, the values of N and P can be solved, by "substituting in" the regression coefficients.

taneously if the economic optimum usage of fertilizer is to be determined. In other words, we must simultaneously determine the optimum (1) combination of nutrients and (2) level of application. It was explained in an earlier section that the combination of nutrients which gives lowest cost for one yield level does not similarly give the least cost for other yield levels. This is true since the slopes of the yield isoquants, and hence the marginal rates of substitution between nutrients, change with higher yield levels.

One approach to determining the dual solution outlined above is that of successive approximation. One can use the principle that application of more fertilizer is profitable (for a farmer with unlimited capital) as long as the marginal product of a fertilizer nutrient is greater than the nutrient/crop price ratio. Hence, with a price of \$1.40 per bushel for corn, \$0.18 per pound for nitrogen and \$0.12 for P_2O_5, we can obtain solutions by successive approximations using table 4. The $\frac{P_2O_5}{\text{corn}}$ price ratio is 0.085; we can move down the first column until we find a marginal product for P_2O_5 which is greater than 0.085. The marginal product of the 40th pound of P_2O_5 is 0.26 — hence, it is profitable. The 80th pound of P_2O_5 is not profitable since its marginal product of 0.06 is less than the price ratio of 0.85. Starting from zero nitrogen, we can then move across the second row to determine the amount of nitrogen which is profitable, with 40 pounds of P_2O_5 already applied. Since the nitrogen/corn price ratio is 0.125, the 80th pound of nitrogen is profitable; the 120th pound is not since the marginal product of 0.07 is less than the price ratio of $\frac{0.18}{1.40}$ or 0.125.

Now, with 40 pounds of P_2O_5 and 80 pounds of nitrogen, we move down the second column. With 80 pounds of nitrogen, the 120th pound of P_2O_5 becomes profitable since its marginal product of 0.11 is greater than the price ratio of 0.085. With 120 pounds of P_2O_5, the 120th pound of nitrogen also becomes profitable. From the data in table 4 and with the prices quoted, the method of "successive approximation" indicates that 120 pounds of each nutrient is profitable. As is brought out for red clover, however, the successive approximation may require added steps in arithmetic before the final solution is attained.

The successive approximation indicates only which of the combinations in the table are most profitable. It does not indicate the exact combinations which might be more profitable. The exact fertilizer combination can be solved by setting the marginal products or partial derivatives for both nutrients equal to the price ratios and simultaneously solving for the quantity of the nutrients to apply for maximum profits. These optima are attained when the partial derivatives (the marginal products) for both nutrients are equal to the nutrient/corn price ratio. Hence, with a price of \$1.40 for corn, \$0.18 for nitrogen and \$0.12 for P_2O_5, the equations become (25) and (26) below.

$$(25)\quad \frac{dC}{dN} = -0.316 + \frac{3.1756}{\sqrt{N}} + \frac{0.1705\sqrt{P}}{\sqrt{N}} = \frac{0.18}{1.40}$$

$$(26)\quad \frac{dC}{dP} = -0.417 + \frac{4.2578}{\sqrt{P}} + \frac{0.1705\sqrt{N}}{\sqrt{P}} = \frac{0.12}{1.40}$$

From simultaneous solution of these equations, we obtain the figures for situation A in table 7: 298.93 pounds of fertilizer should be used, including 156.45 pounds of P_2O_5 and 142.48 pounds of nitrogen.

The same procedure has been used for the other price situations in table 7. With a decline in corn price by 36 percent (from situation A to situation B), total usage of fertilizer should decline by 30 percent. Input of nitrogen should decline 34 percent and input of P_2O_5 should decline 26 percent, if profit is to be at a maximum; inputs should not be reduced by the same proportions. With a 43-percent increase in corn price (from situation A to situation C), total input of fertilizer should increase by 25 percent; input of nitrogen should increase by 30 percent and input of P_2O_5 should increase by only 21 percent. With corn at \$1.40 and a 1:1 price ratio for nutrients (situation D), input of nitrogen should be greater than input of P_2O_5. With a $\frac{N}{P}$ price ratio of 1.5 (situation C), input of phosphate should be about 5 pounds greater than input of nitrogen. However, with a $\frac{N}{P}$ price ratio of 0.667 (situation E), input of nitrogen should exceed input of phosphate by 33 pounds.

We have illustrated that simultaneous solution of (1) the optimum rate of fertilization and (2) the optimum combination of nutrients, is possible from appropriate experimental data. We also have illustrated some points ordinarily overlooked in both economic and agronomic recommendations; a reduction in product price not only may call for

TABLE 7. OPTIMUM QUANTITY OF FERTILIZER AND OPTIMUM COMBINATION OF NUTRIENTS FOR SPECIFIC PRICE RELATIONSHIPS.

Price situation	Optimum yield (bu.)	Optimum fertilizer use: Total pounds	Pounds N	Pounds P_2O_5
A: corn at \$1.40; N at 0.18; P at 0.12	117.21	298.93	142.48	156.45
B: corn at \$0.90; N at 0.18; P at 0.12	104.99	209.27	94.06	115.21
C: corn at \$2.00; N at 0.18; P at 0.12	124.22	374.84	185.04	189.80
D: corn at \$1.40; N at 0.12; P at 0.12	122.30	349.50	180.19	169.31
E: corn at \$2.00; N at 0.12; P at 0.18	124.91	384.18	208.72	175.46

a reduction in the total quantity of fertilizer used on corn; it also may specify a change in the fertilizer grade. These and many other basic principles can be applied when fertilizer experiments are designed to provide relevant marginal quantities and the corresponding economic analysis.

For high level yields and recent prices, the cost of the optimum nutrient combination (computed by both partial derivatives with their respective price ratios) for corn is only slightly less than numerous other nutrient combinations which will give yields in the neighborhood of 125 bushels. The reasons for this outcome are explained earlier in the section on *ridge lines and particular recommendations*. Also, as pointed out previously, use of the least-cost principle results in relatively greater savings for lower yield levels (i.e., for farmers who can afford enough fertilizer for only 60-, 70- or 80-bushel yields). While it is not illustrated here, the optimum nutrient combination for a 60-bushel yield is computed by equating the derivative, $\frac{dN}{dP}$, with the nutrient price ratio, $\frac{\text{price of } P_2O_5}{\text{price of N}}$.

RED CLOVER

The general empirical procedures for red clover were the same as for corn, namely the derivation of 35 single-variable functions for estimation of single-nutrient response curves for later comparisons with parallel predictions from two-variable functions. The first two-variable functions derived were the following, where Y refers to yield in tons, Y′ refers to yield above check plot, P refers to P_2O_5 and K refers to K_2O in pounds:

(27) Logarithmic

(a) "lower" observations

$$Y' = 0.35304\, K^{0.0488}\, P^{0.1997}$$

(b) "upper" observations

$$Y' = 0.84750\, K^{0.0587}\, P^{0.0039}$$

(c) pooled data

$$Y' = 0.36551\, K^{0.0384}\, P^{0.1864}$$

(28) Crossproduct

$$Y = 2.657 + 0.0019K + 0.0079P - 0.0000018K^2 - 0.0000167P^2 - 0.0000031KP$$

(29) Square root

$$Y = 2.46 - 0.000073K - 0.003952P + 0.028141\sqrt{K} + 0.128004\sqrt{P} - 0.000980\sqrt{KP}$$

Basic statistics of the first two-variable equations are given in table 8. The clover data were relatively more variable than were the corn data. For the preceding three algebraic functions, the largest portion of variance explained by fertilizer nutrients was the 64 percent for the square root functions. The t value for one regression coefficient in both the crossproduct and square root function was not significant at the 40-percent level of probability. Since the interval functions for the logarithmic equation did not provide two segment contours which were logically (or statistically) acceptable, and since the overall logarithmic function (1) does not allow slopes of yield isoquants to change on a scale line and (2) does not allow diminishing total yields as are present in the observations, another attempt was made to derive two variable functions. The K^2 term was dropped from the crossproduct equation and the K term from the square root equation since these terms were not significant. The new regression coefficients are shown in equations (30) and (31). (Yield is again measured in tons for these equations.)

TABLE 8. VALUES OF R FOR TWO-VARIABLE NUTRIENT EQUATIONS AND VALUES OF t FOR INDIVIDUAL REGRESSION COEFFICIENTS.

Equation	Value of R	Value of t for coefficients in order listed				
27a	0.7510*	8.14*	1.99†	....	....	....
27b	0.1060§	0.05§	0.77§	....	....	....
27c	0.7510*	15.16*	3.12*	....	....	....
28	0.7622*	2.17*	9.20*	0.76§	6.99*	1.48‡
29	0.8016*	0.10§	5.52*	1.81‡	8.23*	1.82‡

* $0 < P < 0.01$
† $0.05 < P < 0.10$
‡ $0.10 < P < 0.20$
§ $P > 0.40$

(30) $$Y = 2.68 + 0.0013K + 0.0079P - 0.00000017P^2 - 0.0000031KP$$

(31) $$Y = 2.468 - 0.003947P + 0.026834\sqrt{K} + 0.127892\sqrt{P} - 0.000979\sqrt{KP}$$

As table 9 indicates, dropping one term from each of the equations did not result in a significant increase in yield variance.[21] After comparing response curves and isoquants from the new two-variable functions with (1) individual observations from the experiment and (2) similar estimates from the single-variable function, it was decided to use the latter four-term square root function for the estimates which follow.

PRODUCTION SURFACE ESTIMATES

The first estimates made from the regression equation (31) above are for the production surface. Table 10 shows total yield for the discrete nutrient combinations shown. As these data, and the surface of fig. 14 show, the surface is represented by a relatively great slope for quite small quanti-

[21] The R values are 0.7670 and 0.8016 respectively for the new crossproduct and square root functions. In order of the coefficients in the regression equations, the t values are: crossproduct = 3.21, 9.20, 6.98 and 1.50; square root = 5.55, 3.08, 8.29 and 1.82.

TABLE 9. SUM OF SQUARES AND VALUE OF F IN ANALYSIS OF VARIANCE FOR ADDED REGRESSION TERM.

Item*	Crossproduct	Square root
Deviation from regression, four terms	15,026.07	12,751.18
Deviation from regression, five terms	14,946.95	12,749.88
Reduction due to added term	79.12	1.30
Value of F	0.572	0.011

* Degrees of freedom are 109 for four terms, 108 for five terms and 1 for regression term analyzed.

TABLE 10. PREDICTED TOTAL YIELDS FOR SPECIFIED NUTRIENT COMBINATIONS ON RED CLOVER (TONS PER ACRE).

Lbs. P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	2.468	2.638	2.708	2.762	2.807	2.847	2.884	2.917	2.948
40	3.119	3.249	3.303	3.345	3.380	3.411	3.439	3.464	3.488
80	3.296	3.410	3.458	3.494	3.525	3.552	3.576	3.598	3.619
120	3.395	3.497	3.539	3.572	3.599	3.623	3.645	3.665	3.683
160	3.454	3.454	3.583	3.612	3.637	3.658	3.678	3.696	3.712
200	3.487	3.569	3.603	3.629	3.651	3.671	3.688	3.704	3.719
240	3.502	3.576	3.606	3.630	3.649	3.667	3.682	3.697	3.711
280	3.501	3.569	3.596	3.617	3.635	3.650	3.665	3.678	3.690
320	3.493	3.552	3.576	3.595	3.610	3.624	3.637	3.649	3.659

ties of either or both nutrients; the surface is relatively flat for large inputs of either or both nutrients. Diminishing total yields are attained with extremely large quantities of P_2O_5. While the marginal products of K_2O decline for large inputs, negative marginal products do not exist, on the predicted surface, for this nutrient. The marginal products for small nutrient inputs are largest for P_2O_5. Hence, it is the most limiting of the two nutrients (table 11). However, with more than 160 lbs. of both nutrients, K_2O has higher marginal products than P_2O_5. The first 40 pounds of P_2O_5 have a greater effect in increasing total yields than for K_2O although increases from P_2O_5 are smaller as K_2O is increased. This is because P_2O_5 and K_2O substitute more for each other in red clover production than did P_2O_5 and N for corn (table 4).

SINGLE NUTRIENT RESPONSE CURVES

Figures 15 and 16 illustrate the nature of the predicted response or total product curve for one variable nutrient, while the other is fixed at three levels. All of these curves, as well as the surface and the curve in figs. 14 and 17 respectively, are derived from the same two-variable function (31). Both figures illustrate (1) a small amount of one variable nutrient, with or without a fixed amount of the other, causes a relatively large increase in yield and (2) large amounts of the same nutrient add only a small increment to yield. Figure 15 again illustrates that P_2O_5 by itself, although it has lower marginal products for large inputs, has a greater effect in increasing yields than does a parallel amount of K_2O by itself; the P_2O_5 curve, with no K_2O, is closer to the other curves than is the comparable curve for K_2O in fig. 16. The K_2O predicted curve, with 320 pounds of P_2O_5, crosses the predicted curve for 160 pounds of P_2O_5. For the curves of fig. 15, the maximum yields for P_2O_5, starting from top to bottom of the curves respectively, come with 195.5, 214.1 and 262.5 pounds of P_2O_5. Larger amounts of K_2O cause the maximum yield to come with smaller inputs of P_2O_5.

The input-output curves for red clover start at higher yield levels than do the alfalfa response curves in the next section (i.e., they intersect the yield axis at a higher level). However, maximum yields are only about as high as for alfalfa. Part of this difference arises because the soil at the alfalfa location was less fertile on the basis of the phosphorus soil test and because of physiological differences between the crops. Generally, the data for red clover were more variable than the data for alfalfa.

"SCALE LINE" RESPONSE CURVE

Response curves with both nutrients held in fixed ratios are shown in fig. 17. With large

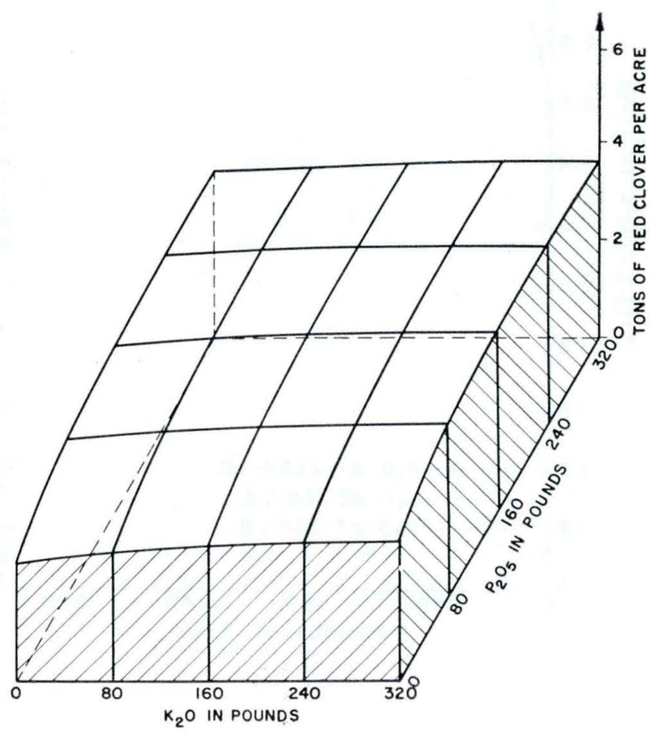

Fig. 14. Predicted yield response surface for red clover.

TABLE 11. MARGINAL PRODUCTS OR YIELDS (POUNDS HAY PER POUND OF FERTILIZER) FOR COMBINATIONS INDICATED IN ROWS AND COLUMNS. UPPER FIGURE FOR K_2O; LOWER FIGURE FOR P_2O_5.*

Lbs. P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	— —	4.24 —	3.00 —	2.45 —	2.12 —	1.90 —	1.73 —	1.60 —	1.50 —
40	— 12.32	3.26 11.94	2.31 10.94	1.88 10.63	1.63 10.37	1.46 10.14	1.33 9.93	1.23 9.74	1.15 9.56
80	— 6.41	2.86 5.71	2.02 5.43	1.65 5.21	1.43 5.02	1.28 4.86	1.17 4.71	1.08 4.57	1.01 4.45
120	— 3.78	2.55 3.22	1.80 2.98	1.47 2.30	1.27 2.65	1.14 2.52	1.04 2.40	0.96 2.29	0.90 2.18
160	— 2.22	2.29 1.73	1.62 1.52	1.32 1.37	1.14 1.24	1.02 1.12	0.93 1.02	0.86 0.92	0.81 0.83
200	— 1.16	2.05 0.71	1.45 0.53	1.19 0.39	1.03 0.27	0.92 0.17	0.84 0.08	0.78 —0.01	0.73 —0.09
240	— 0.37	1.85 —0.04	1.30 —0.20	1.07 —0.33	0.92 —0.44	0.83 —0.53	0.75 —0.62	0.70 —0.70	0.65 —0.77
280	— —0.25	1.65 —0.62	1.17 —0.77	0.95 —0.89	0.83 —0.99	0.74 —1.08	0.67 —1.16	0.62 —1.23	0.58 —1.30
320	— —0.74	1.47 —1.09	1.04 —1.23	0.85 —1.34	0.74 —1.44	0.66 —1.52	0.60 —1.59	0.56 —1.66	0.52 —1.72

* These figures are derivatives of yield in respect to the nutrients from equation 31, with K_2O and P_2O_5 fixed at the quantities shown at the top of the columns or to the left of the rows.

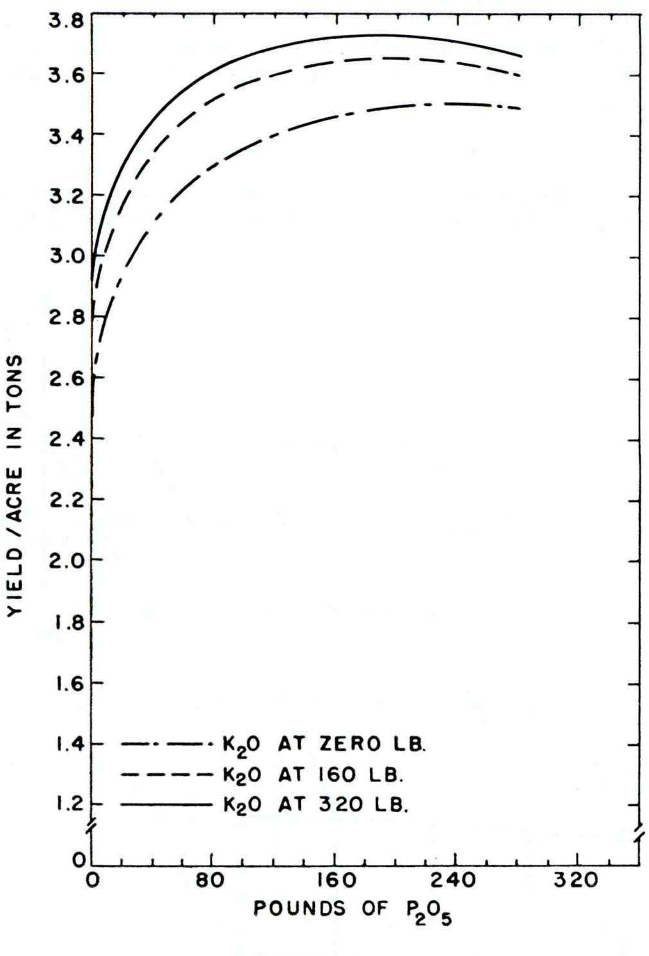

Fig. 15. Total yield with P_2O_5 variable and K_2O fixed at three levels.

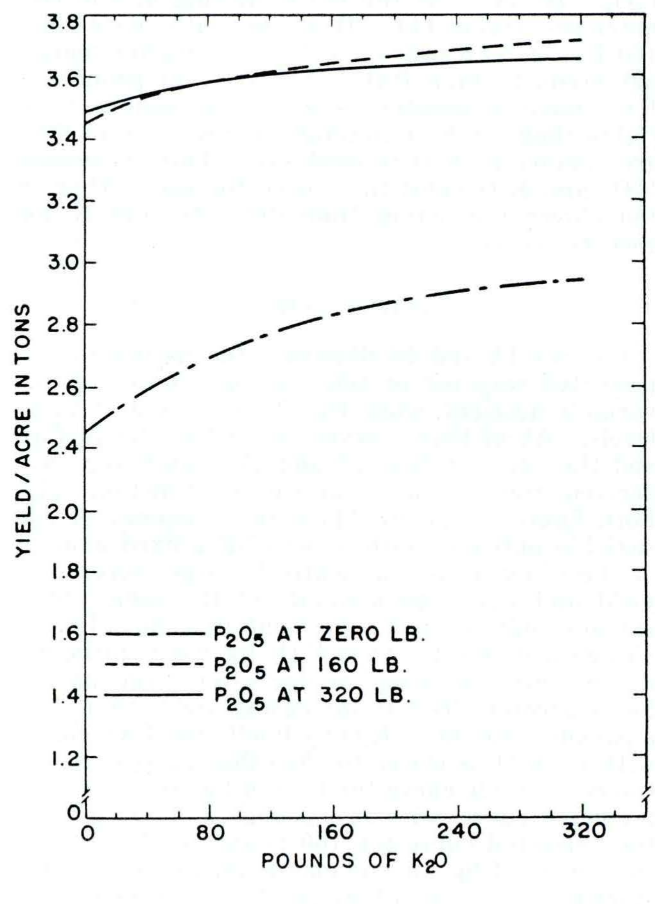

Fig. 16. Total yield with K_2O variable and P_2O_5 fixed at three levels.

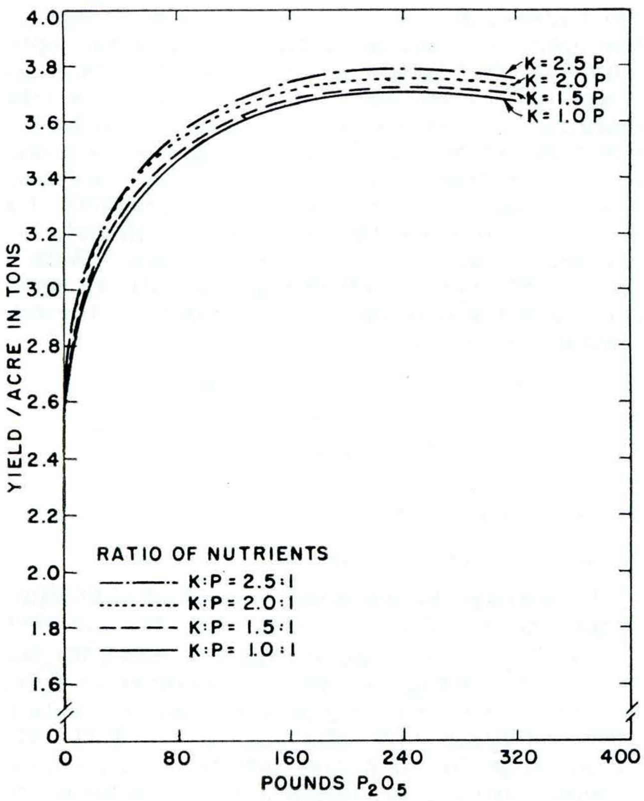

Fig. 17. Yield of red clover with nutrients increased in fixed proportions.

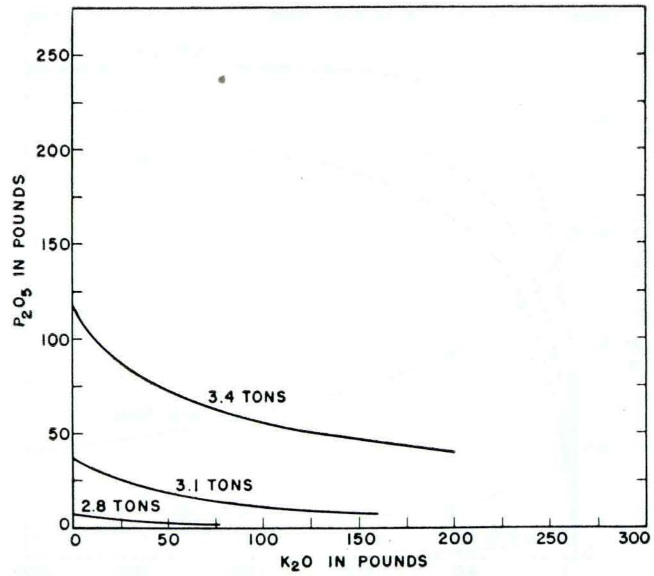

Fig. 18A. Yield isoquants showing all relevant combinations in producing specified yields (ends of curves give limits in nutrient substitution).

amounts of the nutrients, diminishing marginal productivity and negative marginal yields hold for each nutrient ratio shown. Ratios of 1 pound of K_2O to 1 pound of P_2O_5 and 1 pound of K_2O to 1.5 pounds of P_2O_5 give similar response curves. Differences are not great for any of the four ratios shown.

YIELD ISOQUANTS AND SUBSTITUTION RATIO

Figure 18A includes the product isoquants predicted for yields of 2.8, 3.1 and 3.4 tons per acre. These isoquants are derived from the equation below where yield is in pounds.*

Only slight quantities of P_2O_5 alone or P_2O_5 in combination with K_2O are predicted to produce a yield of 2.8 tons. This yield can be produced with about 8 pounds of P_2O_5 alone; for any quantity of P_2O_5, 1 pound of the K_2O replaces less than 1 pound of P_2O_5. Although the isoquant extends out as far as 80 pounds on the K_2O axis, this quantity of K_2O would never be profitable in producing a 2.8-ton yield. On this portion of the isoquant, 1 pound of K_2O substitutes for only a very small fraction of a pound of P_2O_5. Actually, economic combinations of nutrients for a 2.8-ton yield do not exist away from the P_2O_5 axis. Only this nutrient should ever be used for a 2.8-ton yield; for all practical purposes, the 2.8-ton isoquant does not exist. Somewhat the same situation holds true for a yield of 3.1 tons. A 3.1-ton yield is predicted to be attained with 37 pounds of P_2O_5 and none of K_2O, or any of the other combinations indicated on the middle isoquant. The replacement rate of K_2O for P_2O_5 becomes less than 1:1 with 36 pounds of P_2O_5 and 2 pounds of K_2O. While the isoquant is predicted to extend far out toward the K_2O axis, the substitution rates are extremely low at these extremes.

The isoquant for a 3.4-ton yield has greater slope than the 3.1-ton isoquant. Accordingly, K_2O replaces P_2O_5 at a higher rate over a wider range of K_2O inputs. One pound of K_2O replaces more than 1 pound of P_2O_5 up to 14 pounds of K_2O; the substitution rate becomes 1:1 with 14 pounds of K_2O and 98 pounds of P_2O_5. But only a very slight amount of K_2O will, according to the predictions of the production relationships, be profitable at the usual ratio of prices for the two nutrients. However, larger amounts of K_2O are predicted to be necessary for higher yields.

This point is illustrated in fig. 18B. The lines labeled RR again are isoclines indicating the path of a given replacement rate over the map of yield isoquants or contours. They indicate the point on any yield isoquant where the replacement rate, for the stated yield, is that indicated by the isocline. The curves denoted by tons are yield isoquants of the nature indicated previously. For yields of 3.4 tons or less, the three isoclines (which represent a range of price ratios which might be attained for the two nutrients) are close to the phosphate axis. For higher yields, they are pre-

$$* \ (32) \quad P = \left[\frac{(255.784 - 1.958\sqrt{K}) \pm \sqrt{3.8338K + 692.97\sqrt{K} + 221269 - 31.576Y}}{-15.788}\right]^2$$

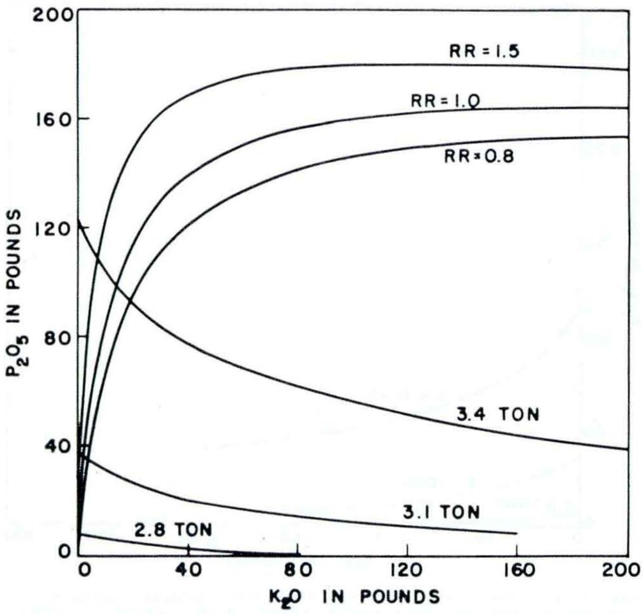

Fig. 18B. Yield isoclines showing points of equal slope and replacement rate for red clover isoquants.

dicted to veer in a direction specifying a larger proportion of K_2O. Yields as high or greater than 3.4 tons would never be profitable; the nutrient/hay price ratio has never been low enough and the marginal response for fertilizer is not high enough at this yield level. However, the isoclines do predict the least-cost nutrient combination for each possible yield level. With P_2O_5 and K_2O costing the same amount per pound, the price ratio is 1.0. Since the least-cost nutrient combination is attained when the price ratio is equal to the replacement or substitution ratio, each isocline traces out the path of least-cost nutrient combinations as higher yields of red clover are attained.[22]

The limited nature of nutrient replacement, and the predicted small contribution of K_2O for specified yields, is illustrated further in table 12. These figures again illustrate the small amount of P_2O_5 that is replaced per pound of K_2O. The replacement ratio is so low that, in terms of the predictions, only P_2O_5 should be used for either yield.

DETERMINING ECONOMIC OPTIMA

As in the case of corn, the optimum level of fertilization and the optimum combination of nutrients under specified prices can be determined either by "successive approximation" or exact methods. Table 11, showing marginal yield in pounds, can be used for the "successive approximation method."[23] Using the simultaneous solution illustrated for corn, we can determine "exactly" the optimum level of fertilization and the optimum combination of nutrients. Again, this is accomplished by setting the partial derivatives (the marginal products) of each nutrient equal to the hay/nutrient price ratio and solving for the quantities of nutrients necessary for this condition. The partial derivative (marginal product) for P_2O_5 is set to equal the price ratio as in (33) below, and equations (34) and (35) follow. With a price of \$16 per ton (\$0.008 per pound), 12 cents per pound for K_2O and 15 cents for P_2O_5, the following solutions are obtained:

$$(33)\quad \frac{dY}{dP} = -7.894 + \frac{127.892}{\sqrt{P}} - \frac{0.979\sqrt{K}}{\sqrt{P}} = \frac{0.15}{0.008} \text{ or } 18.75$$

$$(34)\quad 26.644\sqrt{P} = 127.82 - 0.979\sqrt{K}$$

$$(35)\quad \sqrt{P} = 4.8000 - 0.0367\sqrt{K}$$

By setting the marginal product of K_2O equal to the price ratio in (36), equations (37) and (38) can be derived. By substituting $\sqrt{K}$ into (35), we obtain (39) and hence (40), the quantity of P_2O_5. By substituting this value into (38), we obtain (41) and hence (42), the quantity of K_2O. By substituting the quantities into the original production function, we obtain a predicted optimum yield of 3.02 tons.

$$(36)\quad \frac{dY}{dK} = \frac{26.834}{\sqrt{K}} - \frac{0.979\sqrt{P}}{\sqrt{K}} = \frac{0.12}{0.008} \text{ or } 15.0$$

$$(37)\quad 15\sqrt{K} = 26.834 - 0.979\sqrt{P}$$

$$(38)\quad \sqrt{K} = 1.7889 - 0.0653\sqrt{P}$$

$$(39)\quad \sqrt{P} = 4.8000 - 0.03674\,(1.7889 - 0.653\sqrt{P})$$

$$(40)\quad P = 22.52$$

$$(41)\quad \sqrt{K} = 1.7889 - 0.0653\,(4.7459)$$

$$(42)\quad K = 2.19$$

TABLE 12. NUTRIENT COMBINATIONS AND MARGINAL REPLACEMENT RATES FOR TWO YIELD LEVELS OF RED CLOVER.

3.1 tons per acre			2.8 tons per acre		
Lb. K_2O*	Lb. P_2O_5*	Rate of substitution of K_2O for P_2O_5 (replaced by 1 lb. K_2O)†	Lb. K_2O*	Lb. P_2O_5*	Rate of substitution of K_2O for P_2O_5 (Lb. P_2O_5 replaced by 1 lb. K_2O)†
10	27.7	—0.433	10	4.52	—0.154
20	24.2	—0.286	20	3.33	—0.094
30	21.7	—0.221	30	2.54	—0.067
40	19.7	—0.182	40	1.96	—0.051
50	18.0	—0.156	50	1.51	—0.040
60	16.6	—0.136	60	1.15	—0.032
70	15.3	—0.121			
80	14.1	—0.109			
90	13.1	—0.099			
100	12.2	—0.090			
110	11.3	—0.083			
120	10.5	—0.076			
130	9.8	—0.071			
140	9.1	—0.066			
150	8.5	—0.061			

* Computed from the isoquant equation presented in the text.
† Marginal replacement rates are computed, as derivatives, for exactly the nutrient combinations shown; they are not averages between combinations.

[22] Since the isoclines are in terms of the rate at which K_2O substitutes for P_2O_5 $\left(\frac{dP}{dK}\right)$, the corresponding price ratios are found by dividing the price of K_2O by the price of P_2O_5.

[23] A problem in using the "successive approximation" method is this: Often there will be two or more locations where the marginal products of both nutrients will be greater than the price ratio· movement along a "southeast" diagonal to another cell may drive marginal products below the price ratio for both original cells. Which then is most profitable? Solutions can be determined only by determining total gross return and total costs, with the latter subtracted to indicate net return.

TABLE 13. OPTIMUM RATES AND COMBINATIONS OF FERTILIZER FOR SPECIFIED CROP AND NUTRIENT PRICES.

Price situation	Price per unit			Optimum quantity			
	Hay per ton	K_2O per lb.	P_2O_5 per lb.	Total lbs. nutrients	Lbs. P_2O_5	Lbs. K_2O	Yield (tons)
A	$16	0.12	0.15	24.7	22.5	2.2	3.02
B	10	0.08	0.08	29.9	28.1	1.8	3.06
C	16	0.15	0.10	39.9	38.7	1.2	3.13
D	16	0.09	0.12	33.9	30.3	3.6	3.09
E	22	0.09	0.12	50.7	44.5	6.2	3.20
F	10	0.09	0.12	17.4	15.8	1.6	2.94
G	28	0.09	0.12	66.7	57.6	9.1	3.27
H	16	0.15	0.12	32.0	30.7	1.3	3.08
I	16	0.10	0.15	35.6	22.4	3.2	3.02
J	16	0.12	0.09	45.5	43.7	1.8	3.17
K	22	0.12	0.09	64.7	61.6	3.1	3.26
L	10	0.12	0.09	25.0	24.1	0.9	3.02
M	28	0.12	0.09	81.7	77.2	4.5	3.33

Using the "exact" procedure we obtain the results shown in table 13 for several different price situations. For any of these situations, only a very small quantity of K_2O is predicted for the most profitable nutrient combination and fertilization rate. This nutrient is only 3.0 percent of total nutrient input for situation C; the maximum percentage of K_2O is 13.6 for situation G. A relatively greater proportion of K_2O would be needed for maximum profits under the last situation because (1) the yield is high and the marginal rate of replacement of P_2O_5 by K_2O increases with yield level and (2) the price of K_2O is low relative to the price of P_2O_5.

Since very small quantities of K_2O are predicted to be profitable for any price situation, an appropriate question is this: What are the income sacrifices if we were to eliminate the bother of applying any? This question can be examined by solving the optimum quantity of P_2O_5 in the absence of any K_2O. Setting the derivative or marginal product of P_2O_5 equal to the price ratio for price situation G, the optimum amount of P_2O_5 is 60.3 pounds; the corresponding yield is 3.22 tons.[24] The value of hay, above the cost of P_2O_5, is $83.00 for this condition; it is $83.82 for situation G in table 13 where K_2O is also used. Hence, the sacrifice would be only 82 cents if no K_2O were used under these favorable prices. Since the standard error of estimate is higher for clover than for other crops, use of no K_2O would still be consistent with the derived function for this particular location.

ALFALFA

Five two-variable functions were derived from the alfalfa yield data and are listed below.[25] Predictions from these were compared with (a) predictions from 35 single-variable functions and (b) a scatter diagram of observations. These comparisons, along with the statistics of table 14, suggested that the square root, two-variable

(43) Logarithmic

(a) "lower" observations

$$Y' = 0.8293\, K^{0.0791}\, P^{0.1561}$$

(b) "upper" observations

$$Y' = 1.5031\, K^{0.0293}\, P^{0.0407}$$

(c) pooled data

$$Y' = 0.87935\, K^{0.0542}\, P^{0.1310}$$

(44) Crossproduct

$$Y = 2.2514 + 0.0033K + 0.0097P - 0.000007K^2 - 0.000020P^2 - 0.000001KP$$

(45) Square root

$$Y = 1.8737 - 0.0014K - 0.0050P + 0.061731\sqrt{K} + 0.173513\sqrt{P} - 0.001440\sqrt{KP}$$

function (45) provided the best estimates of the production or yield surface and related quantities.

PRODUCTION SURFACE ESTIMATES

Predicted total yields are shown in table 15 for specified nutrient combinations. The two-variable, square root function was used in deriving these quantities and in providing the production surface of fig. 19. Diminishing total yields are indicated for either nutrient increased alone (with the other one fixed at the specified levels of the rows or columns) or for both nutrients increased in fixed proportion. The predicted maximum yield is forthcoming (i.e., the marginal products or first derivatives are zero) at a 3.64-ton yield with 232.2 pounds of P_2O_5 and 203.6 pounds of K_2O. (About the same total yield is shown for some bordering cells. The maximum yield in such cases falls between the nutrient combination shown.)

The marginal products of table 16 show that small inputs of either nutrient gave relatively high incremental yields. For red clover, P_2O_5 by itself gave higher marginal yields than K_2O by itself. Marginal products of the two nutrients came nearest to being equal with 120 pounds of P_2O_5 and 80 pounds of K_2O. Marginal products were negative for both nutrients with inputs greater than 200 pounds of K_2O and P_2O_5. While 320 pounds of K_2O alone does not cause marginal products for this nutrient alone to become negative, even 240 pounds of this nutrient causes negative marginal products when it is combined with 200 pounds of P_2O_5.

TABLE 14. VALUES OF R FOR TWO-VARIABLE NUTRIENT EQUATIONS AND VALUES OF t FOR INDIVIDUAL REGRESSION COEFFICIENTS.

Equation	Value of R	Value of t for regression coefficient in order listed				
43a	0.7197*	6.66*	3.66*	....	....	....
43b	0.1428‡	0.74‡	0.84‡	....	....	....
43c	0.7329*	9.01*	4.29*	....	....	....
44	0.8128*	3.31*	9.74*	2.50†	7.31*	0.42‡
45	0.8793*	1.99†	6.81*	3.85*	10.83*	2.03†

* $0 < P < 0.01$

† $0.01 < P < 0.05$

‡ $P > 0.05$

[24] The calculated optimum of P_2O_5 then is as follows with zero K_2O:

$$(1)\quad \frac{dY}{dP} = -7.894 + \frac{127.892}{\sqrt{P}} = \frac{0.12}{0.014}$$

$$(2)\quad \sqrt{P} = 7.7673$$

$$(3)\quad P = 60.32$$

[25] Y' is yield, above check plots, in tons while Y is total yield in tons. As for the other crops, the logarithmic functions were fitted after yields denoting negative marginal products were discarded.

TABLE 15. PREDICTED TOTAL YIELDS FOR SPECIFIED NUTRIENT COMBINATIONS ON ALFALFA (TONS PER ACRE).

Lbs. P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	1.874	2.208	2.314	2.383	2.432	2.470	2.496	2.516	2.532
40	2.770	3.047	3.129	3.179	3.213	3.235	3.251	3.260	3.265
80	3.024	3.277	3.349	3.392	3.419	3.436	3.446	3.451	3.452
120	3.172	3.407	3.472	3.508	3.530	3.543	3.549	3.551	3.548
160	3.266	3.485	3.544	3.575	3.593	3.603	3.607	3.604	3.598
200	3.324	3.530	3.582	3.610	3.624	3.630	3.630	3.626	3.617
240	3.357	3.551	3.598	3.622	3.633	3.636	3.633	3.626	3.616
280	3.372	3.554	3.597	3.617	3.625	3.625	3.620	3.611	3.599
320	3.372	3.544	3.582	3.599	3.604	3.602	3.595	3.584	3.569

PREDICTED INPUT-OUTPUT RELATIONSHIPS

Figures 20 and 21, along with 19 and 22, have been predicted from the two-variable equation (45). A small amount of the "fixed" nutrient again has a large effect on the yield of the variable nutrient; addition of still more of the fixed nutrient has a smaller effect.[26] In fig. 20, for example, the curve for P_2O_5 with K_2O fixed at 160 pounds is considerably above the curve with K_2O at zero. An increase of K_2O from 160 to 320 pounds does not have a similar effect. These same differences are apparent with K_2O as the "variable" nutrient in fig. 21. Figure 22 shows response curves for both nutrients increased together in fixed ratios. Diminishing total yields occur for any one of the nutrient ratios shown. The 1:1 ratio gives a lower marginal response and a lower total yield than the ratios which include a greater proportion of P_2O_5.

[26] As equations (a) and (b) below show, the maximum yield with P_2O_5 alone is predicted to come with 301.1 pounds of this nutrient. As (c) and (d) show, the predicted yield comes with 486.6 pounds of K_2O alone (a quantity outside the range of observations in the study).

(a) $\frac{dY}{dP} = -0.0050 + 0.086756P^{-0.5} = 0$

(b) $P = \left(\frac{0.086756}{0.0050}\right)^2 = 301.1$

(c) $\frac{dY}{dK} = 0.0014 + 0.0309K^{-0.5} = 0$

(d) $K = \left(\frac{0.0309}{0.0014}\right)^2 = 486.6$

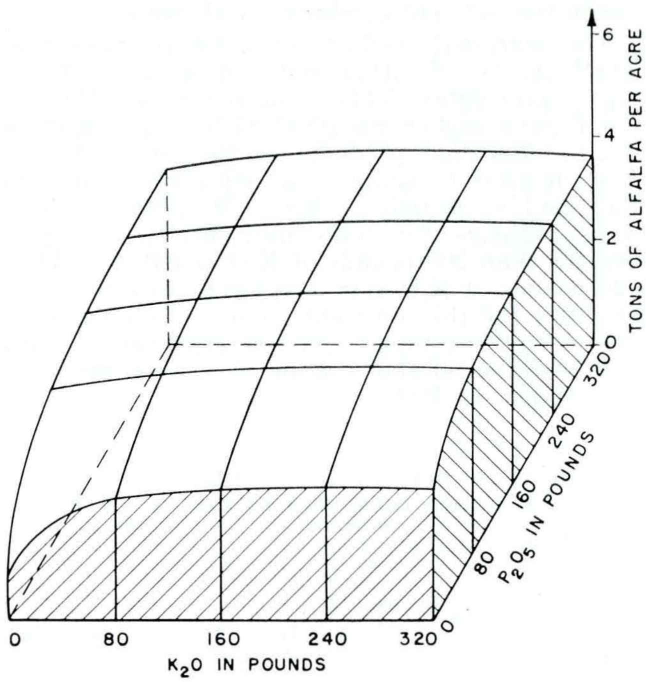

Fig. 19. Predicted yield response surface for alfalfa.

YIELD ISOQUANTS

Yield isoquants for alfalfa are shown in fig. 24. A 2.5- or 3.0-ton yield can be attained with P_2O_5 alone. However, the maximum P_2O_5 for a 3.5-ton yield is 225 pounds or 90 percent of the total nutrient application. Hence, the range of possible substitution again declines with increased yield. Table 17, with yield isoquants at 2.4 and 3.6 tons, shows the economic range of nutrient combinations. For 3.6 tons, the substitution ratio drops below 1.0 with less than 202.9 pounds of P_2O_5 and more than 100 pounds of K_2O. The range of replacement possibilities for the 2.4- ton yield ranges only from zero to 120 pounds of K_2O.

Table 17 illustrates again that marginal rates of substitution change between yield levels; the same ratio of nutrients result in different replacement rates. Therefore, the nutrient combination which gives the lowest cost for a 2.4-ton yield differs from the least cost combination for a 3.6-ton yield.

ISOCLINES AND LEAST-COST NUTRIENT COMBINATIONS

The slopes of the contour or isoquant lines change at higher yields in fig. 23. Hence, the least-cost fertilizer mixture differs somewhat for 2.5-, 3.0- or 3.5-ton yields. (Similarly, this point is suggested in the substitution ratios of table 17.) Also, the range of P_2O_5/K_2O ratios, indicated by the yield contours, is similar as higher yields are attained. This point is illustrated further in fig. 24. The contour or isoquant lines have the same mean-

TABLE 16. MARGINAL PRODUCTS OR YIELDS (POUNDS HAY PER POUND FERTILIZER) FOR COMBINATIONS INDICATED IN ROWS AND COLUMNS. LOWER FIGURE FOR K_2O; UPPER FIGURE FOR P_2O_5.*

Lbs. P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	— —	— 7.01	— 4.21	— 2.81	— 2.01	— 1.61	— 1.21	— 0.81	— 0.61
40	17.40 —	15.96 5.57	15.36 4.97	14.90 4.52	14.52 4.12	14.18 3.79	13.87 3.48	13.59 3.20	13.33 2.94
80	9.36 —	8.35 4.98	7.92 2.77	7.60 1.64	7.33 0.99	7.09 0.70	6.87 0.38	6.67 0.04	6.48 —0.11
120	5.80 —	4.97 4.51	4.63 4.48	4.36 1.37	4.14 0.76	3.95 0.50	3.77 0.19	3.60 —0.12	3.45 —0.27
160	3.68 —	2.96 4.12	2.66 2.18	2.43 1.15	2.24 0.57	2.07 0.32	1.92 0.04	1.79 —0.28	1.65 —0.41
200	2.23 —	1.59 3.79	1.32 1.94	1.12 0.95	0.95 0.40	0.79 0.17	0.66 —0.10	0.53 —0.40	0.41 —0.53
240	1.16 —	0.58 3.48	0.33 1.72	0.15 0.78	—0.01 0.25	—0.15 0.03	—0.28 —0.22	—0.39 —0.52	—0.50 —0.64
280	0.33 —	—0.02 3.20	—0.44 1.52	—0.61 0.61	—0.75 0.11	—0.88 —0.09	—1.00 —0.34	—1.11 —0.63	—1.21 —0.74
320	—0.34 —	—0.85 2.94	—1.06 1.32	—1.22 0.46	—1.35 0.02	—1.47 —0.21	—1.58 —0.45	—1.68 —0.73	—1.78 —0.83

*Figures are derivatives for the specified variable nutrient with the other nutrient "fixed" in quantity indicated at head of column or row. With 40 pounds of each nutrient, the amount added to yield by varying P_2O_5 is 15.96 pounds; with K_2O fixed at 40 pounds; the amount added to yield by varying K_2O is 5.57 with P_2O_5 fixed at 40 pounds.

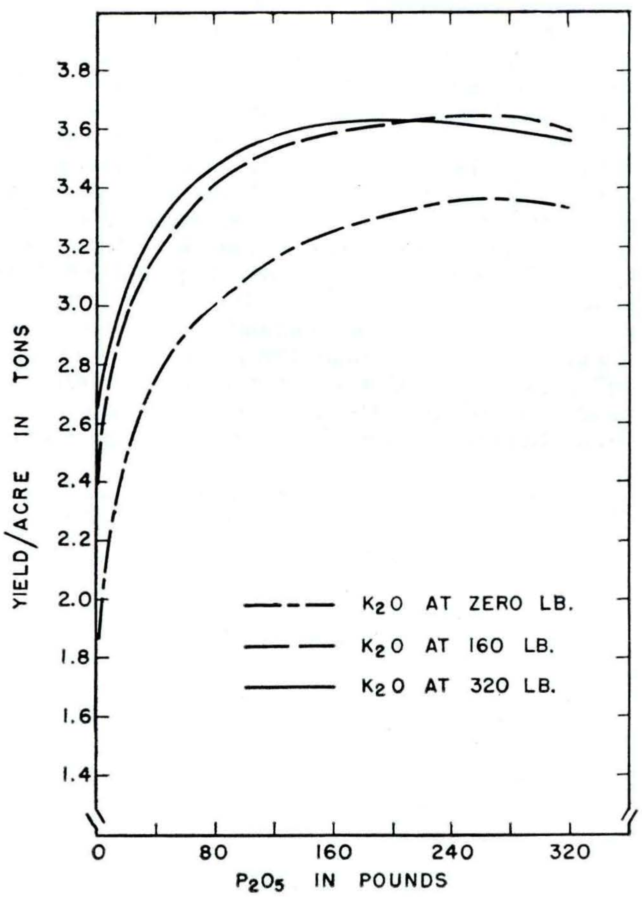

Fig. 20. Alfalfa response curves for P_2O_5 with K_2O fixed at three levels.

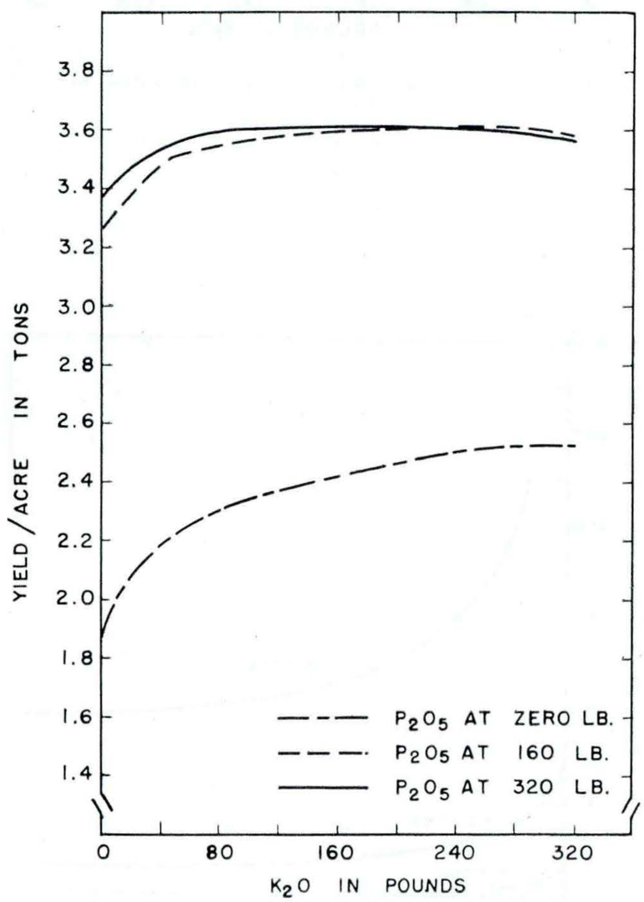

Fig. 21. Alfalfa response curves for K_2O with P_2O_5 fixed at three levels.

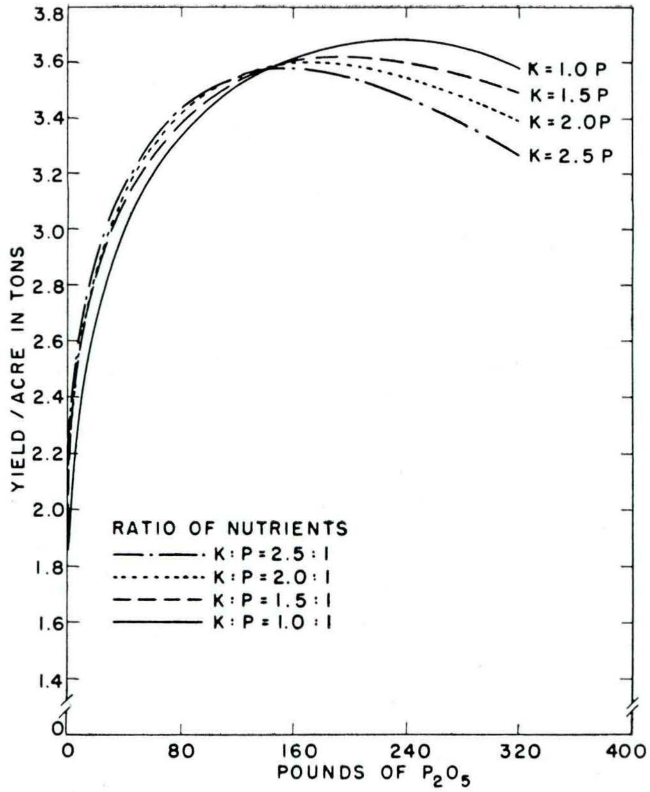

Fig. 22. Yield of alfalfa with nutrients increased in fixed proportions.

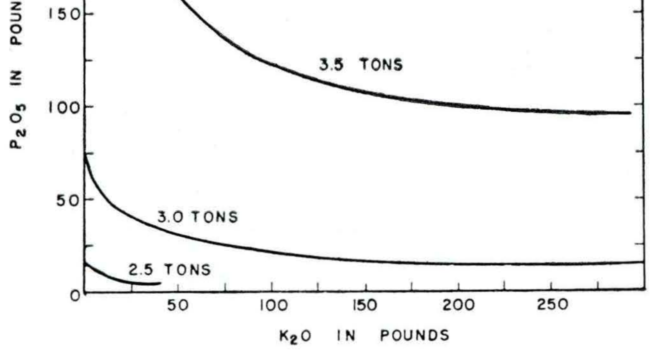

Fig. 23. Yield isoquant for alfalfa.

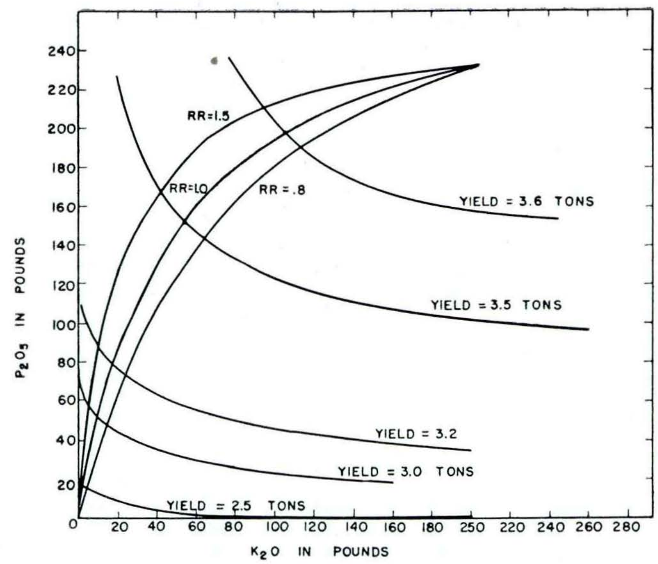

Fig. 24. Yield isoquants and isoclines for alfalfa.

ing as those presented in fig. 23, except that a few additional yields are included. The isocline curves indicate the nutrient combination which gives the same replacement rate for *each* yield level (i.e., on each successive yield contour). The isocline labeled RR of 1.0 indicates the path over which 1 pound of K_2O replaces 1 pound of P_2O_5. For a yield of 3.2 tons, the intersection point indicates that a replacement ratio of 1:1 is attained with about 78 pounds of P_2O_5 and 17 pounds of K_2O. For points on the 3.2-ton yield isoquant above this point of intersection, K_2O substitutes for P_2O_5 at a rate greater than 1:1; for points below the intersection point, 1 pound of K_2O replaces less than 1 pound of P_2O_5. Similarly, for a 3.6-ton yield, the replacement rate between nutrients is 1:1 with about 198 pounds of P_2O_5 and 107 pounds of K_2O (i.e., at the point where the yield isoquant and the isocline of 1.0 intersect). Proportionately more K_2O is required at higher

TABLE 17. NUTRIENT COMBINATIONS AND MARGINAL REPLACEMENT RATES FOR TWO YIELD LEVELS OF ALFALFA.

2.4-ton yield			3.6-ton yield		
Lbs. P_2O_5*	Lbs. K_2O*	Marginal rate of substitution ($\triangle P/\triangle K$); lbs. P_2O_5 replaced by 1 lb. K_2O†	Lbs. P_2O_5*	Lbs. K_2O*	Marginal rate of substitution ($\triangle P/\triangle K$); lbs. P_2O_5 replaced by 1 lb. K_2O†
11.28	0	—5.391	217.4	90	—1.839
4.77	10	—0.234	202.9	100	—1.176
3.09	20	—0.123	192.8	110	—0.868
2.11	30	—0.078	185.1	120	—0.680
1.46	40	—0.053	179.1	130	—0.549
1.01	50	—0.038	174.1	140	—0.452
0.69	60	—0.027	167.0	150	—0.375
0.46	70	—0.020	166.5	160	—0.312
0.30	80	—0.014	163.7	170	—0.259
0.18	90	—0.010	161.3	180	—0.214
0.09	100	—0.007	159.4	190	—0.174
0.04	110	—0.004	157.8	200	—0.138
0.01	120	—0.002	156.6	210	—0.106

* Derived from isoquant equations.

† Derivatives for isoquant equations for exactly the nutrient combinations shown.

yield levels if a 1:1 substitution ratio is maintained.

The isocline RR = 1.5 indicates the nutrient combination for each successive yield level where 1 pound of K_2O replaces 1.5 pounds of P_2O_5. Isocline RR = 0.8 indicates nutrient combinations where 1 pound of K_2O substitutes for 0.8 pound of P_2O_5. Again, these isoclines indicate the most economical combination (i.e., the least-cost combination) of nutrients for any specified yield level. With K_2O at 0.12 cents and P_2O_5 at 0.08 cents per pound, the price ratio of $\frac{12}{8}$ or 1.5. The least-cost nutrient combination includes 88 pounds of P_2O_5 and 10 pounds of K_2O for a 3.2-ton yield; it includes 213 pounds of P_2O_5 and 94 pounds of K_2O for a 3.6-ton yield (although this yield level may not itself be profitable, the nutrient combination is the one allowing the lowest fertilizer cost for the particular yield). Since the isoclines "bend" towards the K_2O axis, proportionately more K_2O must be used for higher yields if the least-cost nutrient combination is to be attained. (A single nutrient combination would provide the least-cost combination for all yield levels only if the isoclines were straight lines passing through the origin.) The isoclines converge to a single point denoting the maximum possible yield; replacement of one nutrient by the other is not possible for 3.64 tons of alfalfa.

ECONOMIC OPTIMA

Using table 16, the following results are obtained with the "successive approximation" method. With hay at 1 cent per pound and P_2O_5 and K_2O each at 7 cents, we may start in the first cell and move "across" the first row. The nutrient/hay price ratio is 7.0 $(\frac{7}{1})$ for both nutrients. Each nutrient should be added as long as its marginal yield is greater than 7.0. With P_2O_5 at zero, the 40th pound of K_2O is profitable; the 80th pound is not profitable since the marginal product of 4.21 is less than the price ratio of 7.0. Moving down this column, with K_2O at 40 pounds, the marginal product of the 80th pound of P_2O_5 is still greater than the price ratio. However, with the 80 pounds of P_2O_5 and 40 pounds of K_2O, the marginal product of K_2O drops below the price ratio of 7.0. If we start with zero pounds of K_2O, we find 80 pounds of P_2O_5 is profitable. Hence, as shown in the case of red clover, several "steps in arithmetic" must be used for the "successive approximation" method if the unique combination of nutrients is to be determined.

SIMULTANEOUS SOLUTION

The "exact" method of determining the optimum amount and combination of nutrients, gives the following results where prices or costs are $16 per ton for alfalfa, 15 cents per pound for K_2O and 12 cents per pound for P_2O_5: First, the partial derivatives are set equal to the hay/nutrient price ratios as in (46) and (47) below. Values of K and P then are expressed as in (48) and (49). By substituting for K in (49), we obtain (50) and hence the value of P in (51). Now, by substituting P into (48), we obtain (52) and hence the

$$(46)\quad \frac{dY}{dK} = -0.001394 + 0.030866K^{-0.5} - 0.000720K^{-0.5}\sqrt{P} = \frac{0.15}{16.00}$$

$$(47)\quad \frac{dY}{dP} = -0.005018 + 0.086756P^{-0.5} - 0.00072P^{-0.5}\sqrt{K} = \frac{0.12}{16.00}$$

$$(48)\quad \sqrt{K} = 2.86619 - 0.0668586\sqrt{P}$$

$$(49)\quad 0.012518\sqrt{P} = 0.086756 - 0.000720\sqrt{K}$$

$$(50)\quad 0.012518\sqrt{P} = 0.086756 - 0.002064 + 0.00004814\sqrt{P}$$

$$(51)\quad \sqrt{P} = 6.7917;\ P = 46.1272$$

$$(52)\quad \sqrt{K} = 2.412;\ K = 5.8182$$

value of K. Under the prices given, the (1) optimum rate of fertilizer application and (2) optimum combination of nutrients is 46.13 pounds of P_2O_5 and 5.82 pounds of K_2O. By substituting these values back into the original production function (equation 45) we obtain an optimum yield level of 2.938 tons of hay.

Optimum fertilizer use is specified in table 18 for particular crop and nutrient prices. Under price situation A, the nutrient input includes 87.4 percent P_2O_5 and 12.6 percent K_2O. With a fall in hay price from $16 to $10 (B) and nutrient costs remaining the same, only 41 pounds of total nutrients should be used. However, the total nutrient input now should be composed of 90.5 percent P_2O_5 and only 9.5 percent K_2O. An increase in hay price to $28 (D) requires an increase to 120.8 total pounds of nutrients for economic optimum. The 120.8 pounds is composed of 85.8 percent P_2O_5 and 14.2 pounds K_2O.

TABLE 18. OPTIMUM RATES AND COMBINATIONS OF FERTILIZER FOR SPECIFIED CROP AND NUTRIENT PRICES.

Price situation	Price per unit			Optimum quantity			
	Lbs. K_2O	Lbs. P_2O_5	Tons hay	Total lbs. nutrient	Lbs. K_2O	Lbs. P_2O_5	Yield (tons)
A	0.12	0.09	$16	71.4	8.0	63.4	3.07
B	0.12	0.09	10	41.0	3.9	37.1	2.84
C	0.12	0.09	22	98.0	12.5	85.5	3.20
D	0.12	0.09	28	120.8	17.2	103.6	3.29
E	0.09	0.12	16	58.8	13.7	45.1	2.99
F	0.09	0.12	10	31.8	6.9	24.9	2.75
G	0.09	0.12	22	84.5	21.0	63.5	3.14
H	0.09	0.12	28	107.5	28.1	79.4	3.24
I	0.08	0.08	10	50.2	7.8	42.4	2.93

A reverse of the nutrient price ratio, with crop price remaining the same, has a similar effect. The price ratio is reversed between situations A and E. Total fertilizer input should decline from 71.4 pounds under A to 58.8 pounds under E, even though hay price remains the same. The proportion of P_2O_5 of the total input should decline from 87.4 percent under situation A to 76.7 percent under E.

RESIDUAL RESPONSE FUNCTIONS FOR CORN

Residual responses also are important in determining the economic optimum use of fertilizer. Preceding discussions related to responses in the year following fertilizer application. For the corn experiment, it was possible to obtain second-year yields; the land was planted back to corn and no fertilizer was applied in the second year. Hence, the second-year response functions for 1953 reported below are due alone to the "carry-over" effect of fertilizer applied in 1952.

RESPONSE FUNCTIONS AND RELATED DATA

Two methods were used in analyzing the 1953 response data: (1) Total response surfaces for the 2 years were computed by adding the 1953 yields to the 1952 yields and fitting functions to these combined data. (2) Functions were fitted to the 1953 "carry-over" yields alone. The predicted production functions are given below.

Two-year total (1952 and 1953 data pooled and functions fitted to total yield of 2 years):

(53) Crossproduct

$$Y = -0.0965 + 0.6464N + 0.8140P - 0.00176N^2 - 0.00231P^2 + 0.00149NP$$

(54) Square root

$$Y = 12.636 - 0.2213N - 0.4614P + 4.2464\sqrt{N} + 8.7506\sqrt{P} + 0.5603\sqrt{NP}$$

Second-year residual (1953 yields only):

(55) Crossproduct

$$Y = 7.4177 + 0.0621N + 0.1502P - 0.000180N^2 - 0.000511P^2 + 0.000683PN$$

(56) Square root

$$Y = 18.317 + 0.0948N - 0.0440P - 2.1047\sqrt{N} + 0.2352\sqrt{P} + 0.2193\sqrt{NP}$$

The coefficients of determination (R^2) show the percentages of variance in yield explained by the above regressions. The R^2 values are: crossproduct 2-year total (53), 0.88 percent; square root 2-year total (54), 0.92 percent; crossproduct residual (55), 0.81 percent and square root residual (56), 0.77 percent.

TABLE 19. VALUES OF R FOR TWO-VARIABLE NUTRIENT EQUATIONS AND VALUES OF t FOR INDIVIDUAL REGRESSION COEFFICIENTS.

Equation	Value of R	Value of t for coefficients in order listed in equations				
53	0.940	8.82*	11.11*	8.65*	11.33*	8.37*
54	0.961	4.11*	8.58*	3.63*	7.49*	10.80*
55	0.900	2.00†	4.84*	2.08†	5.93*	9.03*
56	0.878	3.09*	1.43‡	3.16*	0.35§	7.41*

* $0 < P < 0.01$
† $0.01 < P < 0.05$
‡ $0.10 < P < 0.20$
§ $0.50 < P$

Although the square root function did not fit the residual data quite so well as the crossproduct, it did fit the 2-year total yields better than the crossproduct. Since the 2-year total yields combine both the first- and second-year response, the square root function was chosen for use in the following economic analysis.

If the coefficients of the second-year residual are added to the corresponding coefficients of the first-year function, the result is equal to the 2-year total functions given above. For example, the coefficients of equation (56) plus those of equation (14) equal those of equation (54). This is to be expected; the production surface for 2 years is the sum of the surfaces for the first and second years.

YIELD ISOQUANTS

Yield isoquants were derived from the basic production function in the manner outlined in previous sections. The isoquants for the 2-year production surface denote diminishing productivity over the entire surface: The segments on scale or fixed ratio lines which are intersected by yield isoquants (representing equal increments in yield) become greater for higher yield levels. In the case of the residual or second-year response surface, however, the scale lines show slightly increasing returns for small applications of fertilizer (see fig. 25 and table 19a). However, decreasing returns in the second year might have occurred if heavier fertilization had been used in the first year. Also, the second-year response may have,

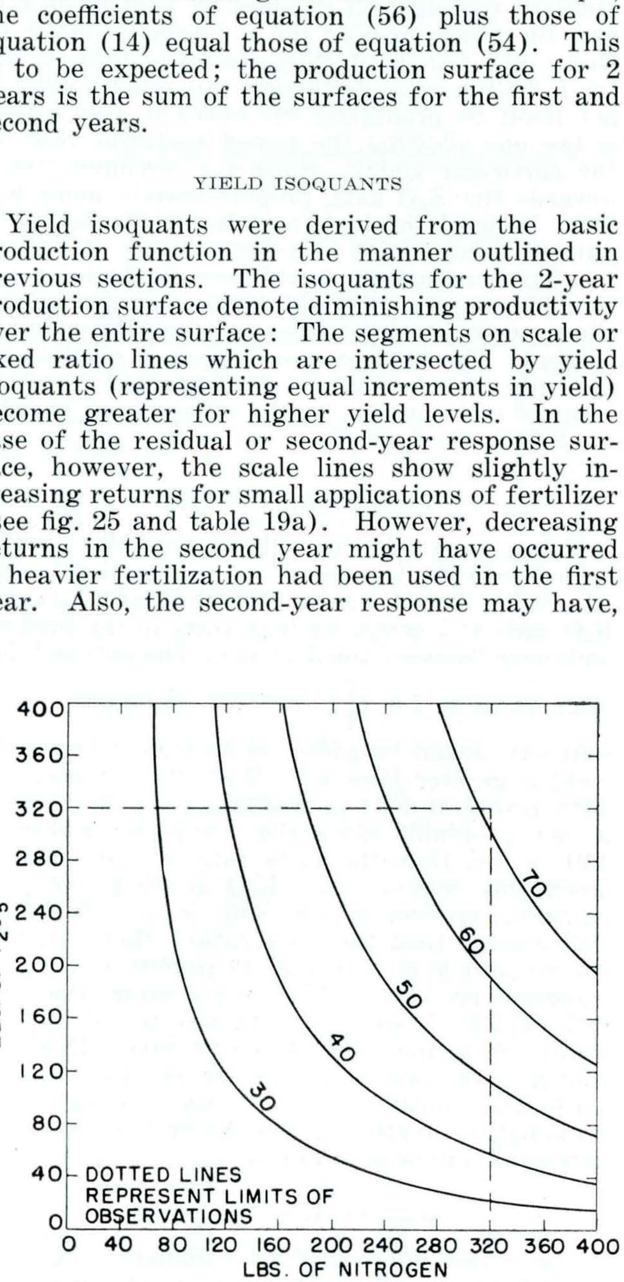

Fig. 25. Corn yield isoquants for residual function (equation 56).

TABLE 19a. ISOQUANT COMBINATIONS FOR PRODUCING SPECIFIED YIELDS; RESIDUAL FUNCTION.

40-bushel yield		50-bushel yield		60-bushel yield		70-bushel yield	
Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N
20	489.0	50	465.9	50	586.8	50	706.0
50	342.5	80	378.5	80	487.9	80	596.9
80	268.4	100	338.6	100	441.6	100	544.9
100	236.0	150	272.2	150	362.2	150	453.7
150	184.6	200	231.8	200	311.8	200	394.1
200	155.4	250	205.1	250	277.2	250	352.2
250	137.2	300	186.5	300	252.2	300	321.2
300	125.2	350	173.0	350	233.6	350	297.6
400	111.1	400	163.0	400	219.4	400	279.2

first, a stage of increasing returns and, second, a stage of decreasing returns.

For economic decisions, residual response must be considered in conjunction with the first-year response. Therefore, isoquants for the 2-year total response surface are given in fig. 26 and table 20. Five isoclines are presented along with four isoquants in fig. 26. The center isocline, RR = 1.5, is appropriate for an N/P_2O_5 price ratio of 1.5, approximately the present price relationship. If the price of N were twice that of P_2O_5, the isocline labeled RR = 2.0 would be the appropriate one. If some new process should make nitrogen one-half the price of available P_2O_5 then the isocline with RR = 0.5 would be the one to be followed to maximize profits (considering both the first- and second-year response). Of course, these isoclines will depend upon or differ with the soil type and fertility level. For other soil types and fertility conditions, a different production surface would be expected. It is obvious for the isocline presented (fig. 26) that any price ratio which would require expansion of fertilizer use along the isocline RR = 0.5 would give about the same economic results as increasing nutrients by a fixed ratio. This isocline deviates only slightly from a straight line through the origin. Also, some of the other isoclines have only slight curvature, denoting only slight profit depression if a fixed nutrient combination is used for increasing yield. Rates of fertilization were not great enough to define the point of maximum yield and convergence of isoclines for the 2-year total yield within the range of the experiment.

INPUT-OUTPUT CURVES

Input-output curves for the 2-year total response function are shown in figs. 27, 28 and 29. In fig. 27, P_2O_5 is variable and nitrogen is fixed; in fig. 28, P_2O_5 is fixed and nitrogen is variable. These curves show the same general characteristics for 2-year total yields as the curves in figs. 10 and 11 for first-year yields. However, they do have less curvature, denoting a smaller decline in marginal yields for higher fertilization levels. The residual yields in the second year cause this change. Figure 29 shows response curves when both nutrients are increased in fixed proportions (e.g., 200 pounds of P_2O_5 would also be used when input of nitrogen is 200 pounds and the ratio is P_2O_5 = N, or a 1:1 ratio). Again, because of the second-year or residual effects, maximum total yields are not denoted within the range of fertilizer applications studied.

ECONOMIC OPTIMA

With information regarding second-year response, which production surface or surfaces should be used to specify the optimum combination and application of nutrients? If the second-year total function (54) is used, the optimum inputs of nitrogen and phosphorous can be found exactly in the same way shown in the preceding sections. Optimum inputs are determined by equating marginal physical products with their corresponding factor-product price ratios as in preceding sections. This optimum solution is valid in the 2-year case only when the expected price of corn is the same for both years and the farmer does not discount the expected value of the second crop more than the first crop.

However, farmers generally discount the value of distant crops more than current crops. In this case, the problem can still be solved by adding

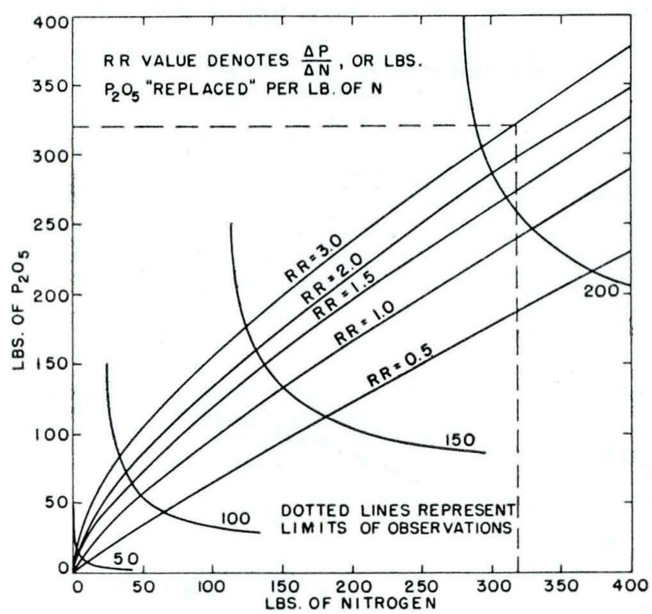

Fig. 26. Corn yield isoquants and isoclines for 2-year total function (equation 54).

TABLE 20. ISOQUANT COMBINATIONS FOR PRODUCING SPECIFIED YIELDS; 2-YEAR TOTAL FUNCTION.

50-bushel yield		100-bushel yield		150-bushel yield		200-bushel yield	
Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N
3	37.2	30	117.4	80	361.0	180	515.7
5	19.8	50	54.6	100	207.8	200	412.5
10	6.9	80	33.2	120	165.0	250	325.1
15	3.0	120	25.6	150	135.8	300	294.4
20	1.3	150	24.3	200	117.4	350	283.0
30	0.2	180	24.7	250	113.1	400	281.3

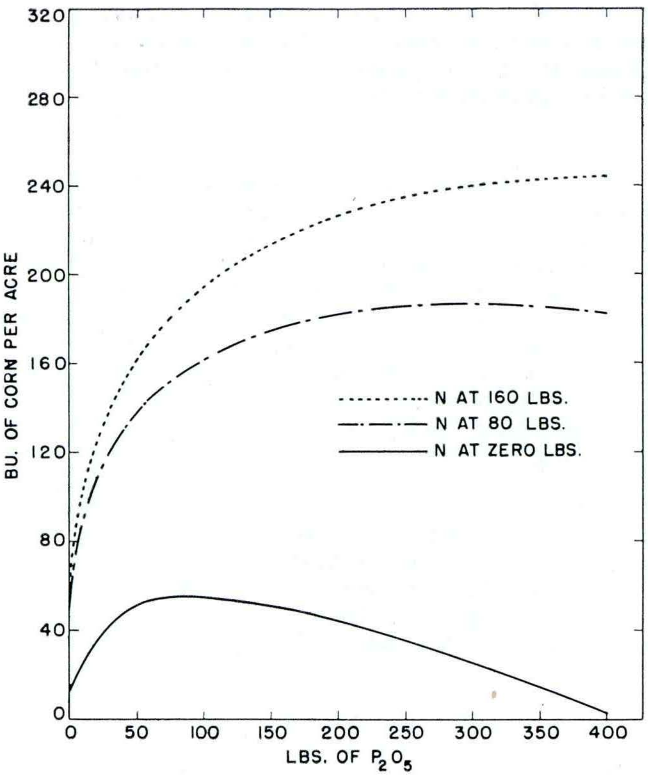

Fig. 27. Input-output curves for corn with P_2O_5 variable and nitrogen fixed at three levels (2-year total response function, equation 54).

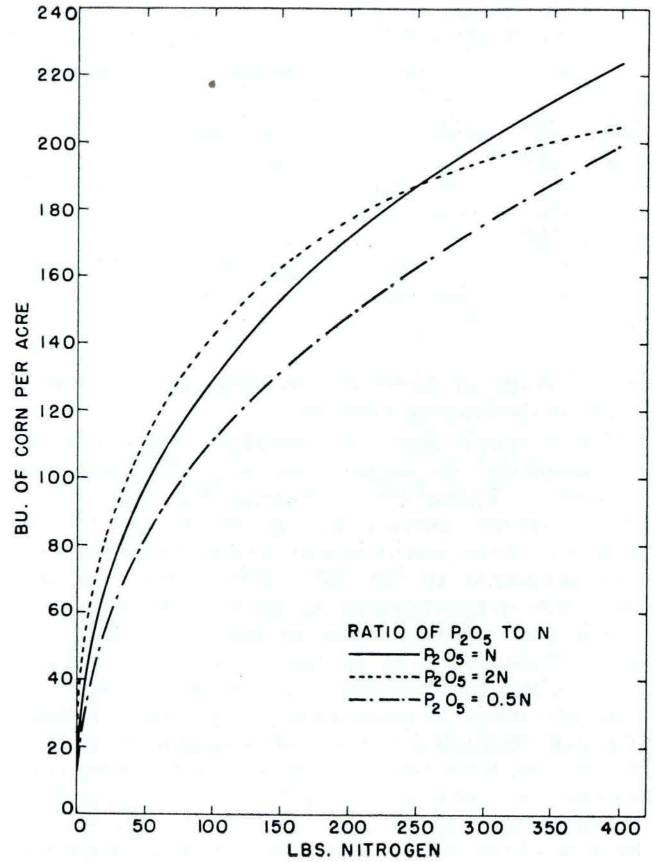

Fig. 29. Input-output curves for corn with nitrogen and P_2O_5 in fixed proportions (2-year total response function, equation 54).

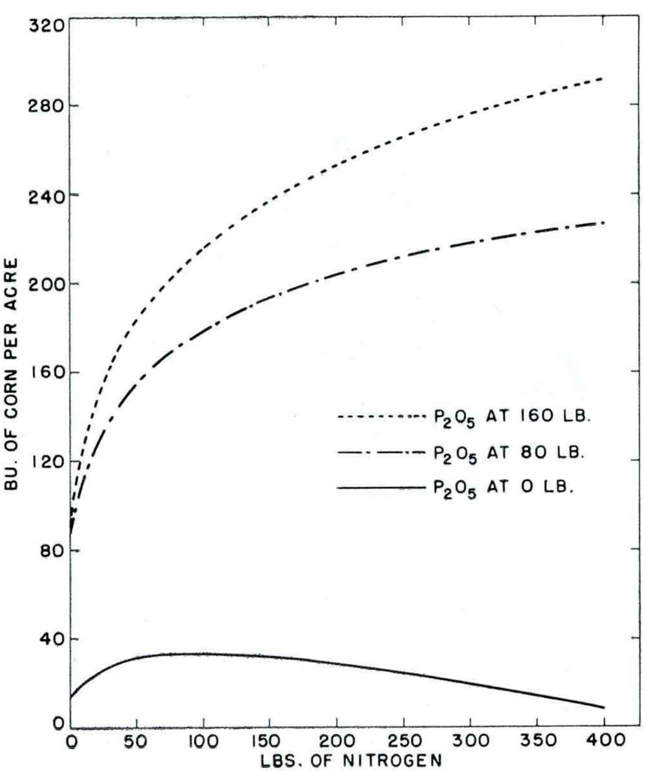

Fig. 28. Input-output curves for corn with nitrogen variable and P_2O_5 fixed at three levels (2-year total response function, equation 54).

the discounted residual production surface to the first-year surface.[27] As an example, assume the farmer expects the price of corn to be \$1.25 per bushel for both the first and second year. He discounts the value of the second crop by 20 percent due to uncertainty or other reasons. This makes the present value of the second-year corn worth \$1 per bushel. The response function is now the first-year response coefficients (equation 14) plus 0.80 times the second-year coefficients (equation 56) which gives equation (57).

$$(57)\quad Y = 8.9716 - 0.240279N - 0.452632P + 4.667464\sqrt{N} + 8.703656\sqrt{P} + 0.516464\sqrt{PN}$$

From (57) the marginal products of N and P are set equal to their respective price ratios as shown in (58) and (59).

$$(58)\quad \frac{dY}{dN} = -0.240279 + \frac{2.333732}{\sqrt{N}} + \frac{0.258232\sqrt{P}}{\sqrt{N}} = \frac{0.15}{1.25}$$

$$(59)\quad \frac{dY}{dP} = -0.452632 + \frac{4.351828}{\sqrt{P}} + \frac{0.258232\sqrt{N}}{\sqrt{P}} = \frac{0.10}{1.25}$$

[27] Whether price or yield, or both, are discounted depends on the degree of variability or uncertainty expected. In this case, we apply a relatively simple procedure and apply discounting only through the production function equation. This procedure is used for purposes of simplification and not as an indication that this is the best procedure.

Using prices per pound of \$0.15 for nitrogen and \$0.10 for P_2O_5 and solving simultaneously for N and P as shown in the preceding sections dealing with the optimum solution for two nutrients, the optimum input is 357.3 pounds for N, and 300.5 pounds for P_2O_5. These first-year inputs maximize the margin of the present value of the two corn crops over the cost of fertilizer.

Optimum inputs of N and P for various prices of corn are in table 21. The corn prices in this table are assumed to be the present *discounted* values of the crops at the time fertilizer is applied.

Although the preceding analysis determines the optimum amount of fertilizer to apply for the first year, considering both the first and second crop, it is possible that some residual response would carry over to the third or later crops. However, third- or fourth-year residual responses likely would be much weaker, just as the second-year response is much less than the first. Consequently, responses past the second year probably can be ignored without too much error in decision making.

A more important problem is to determine the optimum amount of fertilizer to be applied in the second year. This can not be answered from the present data. One hypothesis is: the optimum application for the second year will drop back to considerably less than that for a single year. More research regarding the relation of soil fertility to fertilizer response appears necessary before some of these problems can be adequately answered.

In this section and in previous ones, the appropriate economic principle has been applied in specifying optima. It is recognized, of course, that uncertainty and other factors do not allow farmers to be so "precise" in their decision making. The purpose of this study, however, has been to apply appropriate methods. Mechanical or "rule of thumb" procedures can be developed for applying these basic principles with only slight depression of profit, once additional research provides added information on basic response functions.

LIMITATIONS AND EXPERIMENTAL NEEDS

The concepts and analytical procedures employed in this study are basic for determining economic optima in the use of fertilizer. Also, they provide the basic physical or structural relationships of crop responses in relation to fertilizer application. The predictions apply to particular soils in a particular year; production surfaces obtained under other rainfall and soil conditions can be expected to differ from those obtained in the experiments reported. These limitations are not, however, unique to the type of experiment and empirical procedure reported here. Traditional experimental procedures (wherein a few rates of one or more nutrients are applied) also refer to the rainfall, climatic, insect and crop conditions of the particular year.

Further experimental work is needed, however, to provide greater knowledge of the fertilizer production surface and such related quantities as crop-yield isoquants, nutrient replacement rates and isoclines. In the case of the corn experiment reported, for example, the isoclines were only slightly curved. Is this a general situation for corn under soil-management conditions resembling those studied? Or, are the isoclines for other situations characterized by greater curvature? In our experiment, increasing nitrogen and P_2O_5 on corn in a 1:1 ratio represents only a slight depression of profits, as compared to use of the least-cost nutrient combination for each yield level.

TABLE 21. OPTIMUM FIRST-YEAR APPLICATIONS OF FERTILIZER, CONSIDERING THE RESPONSE FROM BOTH THE FIRST AND SECOND YEARS FOR SPECIFIC PRICE RELATIONSHIPS.*

Price situation				Optimum fertilizer application				
First-year corn per bu.	Second-year corn per bu.	P_2O_5 per lb.	N per lb.	Pounds N	Pounds P_2O_5	Value of first-year yield	Value of second-year yield	Margin over fertilizer
1.50	1.20	0.10	0.15	450.7	385.6	203.01	111.84	211.39
1.25	1.00	0.10	0.15	357.3	300.5	169.18	75.12	160.65
0.625	0.50	0.10	0.15	125.9	140.0	70.86	16.19	54.16
0.50	0.40	0.10	0.15	85.7	106.7	51.03	10.27	37.77
1.50	0.75	0.10	0.15	290.1	259.6	199.81	46.89	177.22
1.25	0.625	0.10	0.15	241.5	226.6	162.56	33.36	137.03
0.625	0.3125	0.10	0.15	101.5	120.8	66.96	8.84	48.50
0.50	0.25	0.10	0.15	72.6	95.1	48.59	5.90	34.08
1.50	0.00	0.10	0.15	172.4	180.1	183.60	0.00	139.73
1.25	0.00	0.10	0.15	150.6	163.0	148.45	0.00	109.56
0.625	0.00	0.10	0.15	76.8	99.2	61.80	0.00	40.35
0.50	0.00	0.10	0.15	58.5	81.1	45.36	0.00	28.47

* Prices of corn are assumed to be the present discounted value to the farmer at the time fertilizer is applied.

Additional experimental work is needed to determine whether this situation has widespread application. Perhaps depression of profit is usually small if nutrients are increased in fixed proportions. Only further experiment work can determine these relationships. However, even if isocline curvature is small (and hence fixed ratio increases of nutrients is practical), the general slope of the isoclines denoting a given substitution ratio is still important. If its slope is 45 degrees, a 1:1 ratio of nutrients is optimum; if its slope is 60.0 degrees, a 2:1 ratio is optimum, etc.

APPENDIX

The "single variable" equations, derived for the "filled out" columns, rows and diagonals, are given below. No single form of equation proved best for estimating all "single nutrient" response curves; one form of equation was best for one particular part of the data while another equation served best for another estimate. In these equations, Y refers to total yield per acre, Y′ to yield above check plot while K, N and P refer respectively to pounds of K_2O, nitrogen and P_2O_5. Yields are measured in tons for hay and bushels for corn. For the exponential and logarithmic equations, yields denoting declining total productions (negative marginal products) have been excluded in computing the functions.

I. Corn

A. Spillman functions ($Y = m - ar^x$)

1. ($N =$ zero; P_2O_5 varied)
   $Y = 37.88 - 22.53\ (0.628)^P$
2. ($N = 160$; P_2O_5 varied)
   $Y = 133.2 - 122.37\ (0.560)^P$
3. ($N = 320$; P_2O_5 varied)
   $Y = 144.96 - 122.91\ (0.621)^P$
4. ($P_2O_5 =$ zero; N varied)
   $Y = 26.84 - 11.49\ (0.562)^N$
5. ($P_2O_5 = 160$; N varied)
   $Y = 130.00 - 108.04\ (0.371)^N$
6. ($P_2O_5 = 320$; N varied)
   $Y = 136.01 - 124.07\ (0.506)^N$

B. Cobb-Douglas functions

1. Single nutrient variable ($Y = aP^b$ and $Y = aN^b$)
   a. ($N =$ zero; P_2O_5 varied) $Y' = 0.56199P^{0.6304}$
   b. ($N = 160$; P_2O_5 varied) $Y' = 8.0760P^{0.4959}$
   c. ($N = 320$; P_2O_5 varied) $Y' = 14.754P^{0.3756}$
   d. ($P_2O_5 =$ zero; N varied) $Y' = 0.41610N^{0.5066}$
   e. ($P_2O_5 = 160$; N varied) $Y' = 53.006N^{0.2283}$
   f. ($P_2O_5 = 320$; N varied) $Y' = 8.0650N^{0.4899}$

C. Quadratic functions

1. Single nutrient variable;
   squared term ($Y = a + bX + cX^2$)
   a. ($N =$ zero; P_2O_5 variable)
      $Y = 15.89165 + 0.216615P - 0.00066P^2$
   b. ($N = 160$; P_2O_5 variable)
      $Y = 35.80101 + 0.84345P - 0.00184P^2$
   c. ($N = 320$; P_2O_5 variable)
      $Y = 34.07284 + 0.90400P - 0.00202P^2$
   d. ($P_2O_5 =$ zero; N variable)
      $Y = 19.21717 - 0.06128N + 0.00019N^2$
   e. ($P_2O_5 = 160$; N variable)
      $Y = 42.60217 + 0.80231N - 0.00175N^2$
   f. ($P_2O_5 = 320$; N variable)
      $Y = 19.33370 + 1.09706N + 0.002512N^2$

D. Square root functions

1. Single nutrient variable; ($Y = a + bK + c\sqrt{X}$)
   a. ($N =$ zero; P_2O_5 variable)
      $Y = 13.6268 - 0.159596P + 3.330095\sqrt{P}$
   b. ($N = 160$; P_2O_5 variable)
      $Y = 14.3521 - 0.433523P + 13.885166\sqrt{P}$
   c. ($N = 320$; P_2O_5 variable)
      $Y = 19.7537 - 0.372048P + 12.658169\sqrt{P}$
   d. ($P_2O_5 =$ zero; N variable)
      $Y = 15.5678 - 0.004422N + 0.116429\sqrt{N}$
   e. ($P_2O_5 = 160$; N variable)
      $Y = 23.665741 - 0.392665N + 12.822205\sqrt{N}$
   f. ($P_2O_5 = 320$; N variable)
      $Y = 5.218489 - 0.433987N + 14.659665\sqrt{N}$
2. Single nutrient variable;
   ($Y = a - bX + c\sqrt{X} - dX^2$)
   a. ($N =$ zero; P_2O_5 variable)
      $Y = 16.2743 + 0.242414D - 0.269587\sqrt{P} - 0.000698P^2$
   b. ($N = 160$; P_2O_5 variable)
      $Y = 12.3406 - 0.793201P + 17.105792\sqrt{P} + 0.0006247P^2$
   c. ($N = 320$; P_2O_5 variable)
      $Y = 32.288027 + 0.081200P + 8.599693\sqrt{P} - 0.0007872P^2$
   d. ($P_2O_5 =$ zero; N variable)
      $Y = 14.665514 - 0.376290N + 3.292418\sqrt{N} + 0.000662N^2$
   e. ($P_2O_5 = 160$; N variable)
      $Y = 22.412688 - 0.614710N + 14.810331\sqrt{N} - 0.000386N^2$
   f. ($P_2O_5 = 320$; N variable)
      $Y = 10.242202 + 0.462240N + 6.634940\sqrt{N} - 0.001557N^2$

II. Red clover

A. Spillman functions

1. (K_2O = zero; P_2O_5 variable)
   $Y = 3.65 - 1.21\ (0.804)^P$

2. (K_2O = 160; P_2O_5 variable)
   $Y = 3.96 - 1.12\ (0.667)^P$

3. (K_2O = 320; P_2O_5 variable)
   $Y = 3.76 - 1.00\ (0.860)^P$

4. (P_2O_5 = zero; K_2O variable)
   $Y = 3.20 - 0.76\ (0.536)^K$

5. (P_2O_5 = 160; K_2O variable)
   $Y = 3.70 - 0.02\ (0.667)^K$

6. (P_2O_5 = 320; K_2O variable)
   $Y = 3.96 - 0.62\ (0.739)^K$

B. Cobb-Douglas functions

1. Single nutrient variable

   a. (K_2O = zero; P_2O_5 variable)
      $Y' = 0.20908P^{0.29694}$

   b. (K_2O = 160; P_2O_5 variable)
      $Y' = 0.54867P^{0.14157}$

   c. (K_2O = 320; P_2O_5 variable)
      $Y' = 0.50201P^{0.16419}$

   d. (P_2O_5 = zero; K_2O variable)
      $Y' = 0.13931K^{0.16088}$

   e. (P_2O_5 = 160; K_2O variable)
      $Y' = 1.2040K^{0.01197}$

   f. (P_2O_5 = 320; K_2O variable)
      $Y' = 1.0009K^{0.03150}$

C. Quadratic functions

1. Single nutrient variable; squared term

   a. (K_2O = zero; P_2O_5 variable)
      $Y = 2.60050 + 0.009770P - 0.000024P^2$

   b. (K_2O = 160; P_2O_5 variable)
      $Y = 3.00156 + 0.005870P - 0.000012P^2$

   c. (K_2O = 320; P_2O_5 variable)
      $Y = 3.2757 + 0.006306P - 0.000017P^2$

   d. (P_2O_5 = zero; K_2O variable)
      $Y = 2.52742 + 0.002646K - 0.000005K^2$

   e. (P_2O_5 = 160; K_2O variable)
      $Y = 3.69623 - 0.001271K + 0.000003K^2$

   f. (P_2O_5 = 320; K_2O variable)
      $Y = 3.43706 + 0.002856K - 0.000008K^2$

D. Square root functions

1. One nutrient variable ($Y = a + bX + c\sqrt{X}$

   a. (K_2O = zero; P_2O_5 variable)
      $Y = 2.426109 - 0.004936P + 0.144602\sqrt{P}$

   b. (K_2O = 160; P_2O_5 variable)
      $Y = 2.852230 - 0.002871P + 0.097104\sqrt{P}$

   c. (K_2O = 320; P_2O_5 variable)
      $Y = 2.838348 - 0.006430P + 0.149271\sqrt{P}$

   d. (P_2O_5 = zero; K_2O variable)
      $Y = 2.460013 - 0.001222K + 0.044067\sqrt{K}$

   e. (P_2O_5 = 160; K_2O variable)
      $Y = 3.699353 + 0.000800K - 0.019696\sqrt{K}$

   f. (P_2O_5 = 320; K_2O variable)
      $Y = 3.353238 - 0.002644K + 0.058793\sqrt{K}$

2. Single nutrient variable
   ($Y = a + bX + c\sqrt{X} + dX^2$)

   a. (K_2O = zero; P_2O_5 variable)
      $Y = 2.460177 + 0.001143P + 0.090169\sqrt{P}$
      $- 0.0000106P^2$

   b. (K_2O = 160; P_2O_5 variable)
      $Y = 2.830793 - 0.006705P + 0.131434\sqrt{P}$
      $+ 0.00000666P^2$

   c. (K_2O = 320; P_2O_5 variable)
      $Y = 2.777984 - 0.017218P + 0.245862\sqrt{P}$
      $+ 0.0000187P^2$

   d. (P_2O_5 = zero; K_2O variable)
      $Y = 2.446929 - 0.003553K + 0.064942\sqrt{K}$
      $+ 0.00000405K^2$

   e. (P_2O_5 = 160; K_2O variable)
      $Y = 3.692043 - 0.0005039K - 0.0080198\sqrt{K}$
      $+ 0.00000226K^2$

   f. (P_2O_5 = 320; K_2O variable)
      $Y = 3.348271 - 0.003517K + 0.066615\sqrt{K}$
      $+ 0.00000152K^2$

III. Alfalfa

A. Spillman functions

1. (K_2O = zero; P_2O_5 variable)
   $Y = 3.79 - 2.40\ (0.651)^P$

2. (K_2O = 160; P_2O_5 variable)
   $Y = 3.86 - 1.28\ (0.656)^P$

3. (K_2O = 320; P_2O_5 variable)
   $Y = 3.76 - 1.62\ (0.544)^P$

4. (P_2O_5 = zero; K_2O variable)
   $Y = 2.75 - 1.36\ (0.464)^K$

5. (P_2O_5 = 160; K_2O variable)
   $Y = 3.58 - 0.135\ (0.760)^K$

6. (P_2O_5 = 320; K_2O variable)
   $Y = 3.69 - 0.216\ (0.639)^K$

B. Cobb-Douglas functions

1. Single nutrient variable

   a. (K_2O = zero; P_2O_5 variable)
      $Y' = 0.26994P^{0.38262}$

   b. (K_2O = 160; P_2O_5 variable)
      $Y' = 1.4366P^{0.08111}$

   c. (K_2O = 320; P_2O_5 variable)
      $Y' = 1.0655P^{0.13924}$

   d. (P_2O_5 = zero; K_2O variable)
      $Y' = 0.22530K^{0.35299}$

   e. (P_2O_5 = 160; K_2O variable)
      $Y' = 2.0294K^{0.00831}$

   f. (P_2O_5 = 320; K_2O variable)
      $Y' = 2.1369K^{0.01031}$

C. Quadratic functions

1. Single nutrient variable; squared term

a. (K_2O = zero; P_2O_5 variable)
$Y = 1.83713 + 0.015611P - 0.000035P^2$

b. (K_2O = 160; P_2O_5 variable)
$Y = 2.79440 + 0.00761P - 0.000016P^2$

c. (K_2O = 320; P_2O_5 variable)
$Y = 2.51379 + 0.010707P - 0.000023P^2$

d. (P_2O_5 = zero; K_2O variable)
$Y = 1.62544 + 0.012072K - 0.000034K^2$

e. (P_2O_5 = 160; K_2O variable)
$Y = 3.38128 + 0.000676K + 0.00000026K^2$

f. (P_2O_5 = 320; K_2O variable)
$Y = 3.54024 + 0.000990K - 0.000002K^2$

D. Square root functions

1. One nutrient variable ($Y = a + bX + c\sqrt{X}$)

a. (K_2O = zero; P_2O_5 variable)
$Y = 1.446014 - 0.008430P + 0.259047\sqrt{P}$

b. (K_2O = 160; P_2O_5 variable)
$Y = 2.61441 - 0.003668P + 0.123120\sqrt{P}$

c. (K_2O = 320; P_2O_5 variable)
$Y = 2.200376 - 0.006326P + 0.192392\sqrt{P}$

d. (P_2O_5 = zero; K_2O variable)
$Y = 1.357693 - 0.009288K + 0.214010\sqrt{K}$

e. (P_2O_5 = 160; K_2O variable)
$Y = 3.429897 + 0.001497K - 0.014882\sqrt{K}$

f. (P_2O_5 = 320; K_2O variable)
$Y = 3.490534 - 0.000729K + 0.022011\sqrt{K}$

2. Single nutrient variable
($Y = a - bX + c\sqrt{X} - dX^2$)

a. (K_2O = zero; P_2O_5 variable)
$Y = 1.416920 - 0.013619P + 0.305515\sqrt{P} + 0.00000901P^2$

b. (K_2O = 160; P_2O_5 variable)
$Y = 2.596500 - 0.006886P + 0.151931\sqrt{P} + 0.00000559P^2$

c. (K_2O = 320; P_2O_5 variable)
$Y = 2.146627 - 0.015924P + 0.278341\sqrt{P} + 0.00001667P^2$

d. (P_2O_5 = zero; K_2O variable)
$Y = 1.392206 - 0.003122K + 0.58798\sqrt{K} - 0.0000107K^2$

e. (P_2O_5 = 160; K_2O variable)
$Y = 3.453417 + 0.005708K - 0.052588\sqrt{K} - 0.00000731K^2$

f. (P_2O_5 = 320; K_2O variable)
$Y = 3.475569 - 0.003426K + 0.046154\sqrt{K} + 0.00000468K^2$

Correlation Coefficients and t Values

The values of a or R, and t are presented in tables A-1, A-2 and A-3. It is evident from these statistics, that no single algebraic equation best expresses the single nutrient response curve under all situations. In terms of variance between predicted and observed yield, each of the functions appears to have both advantages and disadvantages when fitted to particular phases of the data.

TABLE A-1. VALUES OF r (OR R) AND t FOR REGRESSION EQUATIONS AND COEFFICIENTS. CORN.

Equation for which computed*	Value of r or R†	Value of t for coefficients in order of terms in equations‡		
		1st	2nd	3rd
Single variable nutrient				
Cobb-Douglas				
B1a	0.8651	6.46	....	....
B1b	0.9628	13.81	....	....
B1c	0.9670	15.17	....	....
B1d	0.8347	4.54	....	....
B1e	0.9740	16.62	....	....
B1f	0.9610	13.49	....	....
Quadratic				
C1a	0.5865	2.76	2.79	....
C1b	0.8724	5.21	3.78	....
C1c	0.9111	6.70	4.99	....
C1d	0.2093	0.81	0.83	....
C1e	0.8930	5.78	4.19	....
C1f	0.9454	9.04	6.89	....
Square-root				
D1a	0.4959	2.03	2.20	....
D1b	0.9466	4.35	7.23	....
D1c	0.9250	3.20	5.65	....
D1d	0.0273	0.06	0.08	....
D1e	0.9576	4.68	7.76	....
D1f	0.9366	3.45	6.04	....
D2a	0.5869	0.84	0.09	1.45
D2b	0.9501	2.10	4.59	0.99
D2c	0.9310	0.16	1.96	1.07
D2d	0.3750	1.44	1.26	1.51
D2e	0.9592	1.93	4.64	0.72
D2f	0.9542	1.14	1.63	2.28

* The equations for which the correlation coefficients and t values refer are those with corresponding numbers under each of types of functions indicated in text.

† The r value refers to single variable equations while R refers to the multiple correlation coefficient for equations with more than one term.

‡ The t values are in order of the coefficients presented in the corresponding text equations.

TABLE A-2. VALUES OF r (OR R) AND t FOR REGRESSION EQUATIONS AND COEFFICIENTS. RED CLOVER.

Equation for which computed*	Value of r or R†	Value of t for coefficients in order of terms in equations‡		
		1st	2nd	3rd
Single variable nutrient				
Cobb-Douglas				
B1a	0.9564	13.10	....	....
B1b	0.8359	6.09	....	....
B1c	0.8433	6.28	....	....
B1d	0.5122	2.38	....	....
B1e	0.4259	0.63	....	....
B1f	0.3929	1.71	....	....
Quadratic				
C1a	0.8580	5.57	4.47	....
C1b	0.6699	2.49	1.73	....
C1c	0.5241	2.28	2.00	....
C1d	0.5187	1.58	1.04	....
C1e	0.2113	0.81	0.73	....
C1f	0.3534	1.46	1.37	....
Square-root				
D1a	0.8690	3.13	4.76	....
D1b	0.7292	1.42	2.49	....
D1c	0.7179	3.05	3.67	....
D1d	0.5660	0.81	1.51	....
D1e	0.2054	0.55	0.70	....
D1f	0.4215	1.49	1.72	....
D2a	0.8794	0.19	1.51	1.06
D2b	0.7349	0.85	1.67	0.51
D2c	0.7606	2.23	3.18	1.45
D2d	0.5731	0.60	1.10	0.41
D2e	0.2145	0.09	0.14	0.24
D2f	0.4226	0.51	0.96	0.13

* The equations for which the correlation coefficients and t values refer are those with corresponding numbers under each of types of functions indicated in the text.

† The r value refers to single variable equations while R refers to the multiple correlation coefficient for equations with more than one term.

‡ The t values are in order of the coefficients presented in the corresponding text equations.

TABLE A-3. VALUE OF r (OR R) AND t FOR REGRESSION EQUATIONS AND COEFFICIENTS. ALFALFA.

Equation for which computed*	Value of r or R†	Value of t for coefficients in order of terms in equations‡		
		1st	2nd	3rd
Single variable nutrient				
Cobb-Douglas				
B1a	0.9799	19.64	....	....
B1b	0.9577	12.44	....	....
B1c	0.9701	15.68	....	....
B1d	0.9510	9.74	....	....
B1e	0.2344	0.96	....	....
B1f	0.3850	1.67	....	....
Quadratic				
C1a	0.8624	5.16	3.84	....
C1b	0.8553	4.68	3.31	....
C1c	0.8265	4.30	3.13	....
C1d	0.7467	4.31	3.98	....
C1e	0.4427	0.46	0.06	....
C1f	0.2857	0.77	0.51	....
Square root				
D1a	0.9371	4.33	6.90	....
D1b	0.9211	3.20	5.56	....
D1c	0.9310	4.20	6.63	....
D1d	0.7878	3.84	4.58	....
D1e	0.4616	1.11	0.57	....
D1f	0.3485	0.62	0.97	....
D2a	0.9394	1.82	4.06	0.71
D2b	0.9243	1.56	3.44	0.76
D2c	0.9454	3.02	5.27	1.89
D2d	0.7956	0.33	1.70	0.68
D2e	0.5017	1.11	1.02	0.85
D2f	0.3804	0.76	1.01	0.62

* The equations for which the correlation coefficients and t values refer are those with corresponding numbers under each of types of functions indicated in the text.

† The r value refers to single variable equations while R refers to the multiple correlation coefficient for equations with more than one term.

‡ The t values are in order of the coefficients presented in the corresponding text equations.

ANALYSIS OF VARIANCE FOR CROSSPRODUCT AND SQUARE ROOT FUNCTIONS

The analysis of variance shown below has been made to further indicate the appropriateness of the crossproduct and square root functions when applied to the three crops. These two algebraic functions have been used since they are the only ones which employ all of the observations from the experiment, including those with negative marginal products.

Corn

Statistics for analysis of variance are shown in tables A-4 and A-5 for corn.

TABLE A-4. SQUARE ROOT FUNCTION FOR CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	242,707	
Treatments	56	233,811	4,175
{Due to regression	{ 5	{ 222,828	44,566
{Deviations from regression	{ 51	{ 10,983	215
Among plots treated alike	57	8,896	156

$$F = \frac{44,566}{156} = 286$$

TABLE A-5. CROSSPRODUCT FUNCTION FOR CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	242,707	
Treatments	56	233,811	4,175
{Due to regression	{ 5	{ 201,943	40,389
{Deviations from regression	{ 51	{ 31,868	625
Among plots treated alike	57	8,896	156

$$F = \frac{40,389}{156} = 259$$

Comparing the F values of the square root function with those for the regular quadratic, it can be seen that the square root function gives a better "fit." This is emphasized further in that the treatment deviation from regression sum of squares of 31,868 for the crossproduct is almost three times the figure of 10,983 for the square root equation.

Red Clover

Comparisons between the four-term square root function and the four-term crossproduct equation are made below for red clover (sums of squares in pounds).

TABLE A-6. SQUARE ROOT FUNCTION FOR RED CLOVER.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	71,340,169	
Treatments	56	57,041,369	1,018,596
{Due to regression	{ 4	{ 45,840,232	11,460,058
{Deviations from regression	{ 52	{ 11,201,137	215,406
Among plots treated alike	57	14,298,800	250,856

$$F = \frac{11,460,058}{250,856} = 45.7$$

TABLE A-7. CROSSPRODUCT FUNCTION FOR RED CLOVER.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	71,340,169	
Treatments	56	57,041,369	1,018,596
{Due to regression	{ 4	{ 41,446,302	10,361,576
{Deviations from regression	{ 52	{ 15,595,067	299,905
Among plots treated alike	57	14,298,800	250,856

$$F = \frac{10,361,576}{250,856} = 41.3$$

The square root function again leaves a smaller residual (unexplained by regression) variance than does the crossproduct function.

Alfalfa

Analysis of variance statistics for the five-term square root and crossproduct equations follow. Again, these indicate that the square root function gives the best fit.

TABLE A-8. SQUARE ROOT FUNCTION FOR ALFALFA.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	119,276,232	
Treatments	56	107,460,032	1,918,929
{Due to regression	{ 5	{ 92,222,141	18,444,428
{Deviations from regression	{ 51	{ 15,237,891	298,782
Among plots treated alike	57	11,816,200	207,302

$$F = \frac{18,444,428}{207,302} = 89.0$$

TABLE A-9. CROSSPRODUCT FUNCTION FOR ALFALFA.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	119,276,232	
Treatments	56	107,460,032	1,918,929
{Due to regression	{ 5	{ 78,799,643	15,759,929
{Deviations from regression	{ 51	{ 28,660,389	561,968
Among plots treated alike	57	11,816,200	207,302

$$F = \frac{15,759,929}{207,302} = 76.0$$

TABLE A-10. SQUARE ROOT FUNCTION FOR RESIDUAL RESPONSE OF CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	51,216.90	
Treatments	56	45,742.37	816.83
{Due to regression	{ 5	39,485.33	7,897.07
{Deviations from regression	{ 51	6,257.04	122.69
Among plots treated alike	57	5,474.53	96.04

$$F = \frac{7,897.07}{96.04} = 82.2$$

TABLE A-11. CROSSPRODUCT FUNCTION FOR RESIDUAL RESPONSE OF CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	51,216.90	
Treatments	56	45,742.37	816.83
{Due to regression	{ 5	41,488.27	8,297.65
{Deviations from regression	{ 51	4,254.10	83.41
Among plots treated alike	57	5,474.53	96.04

$$F = \frac{8,297.65}{96.04} = 86.4$$

Analysis of variance for the total 1952 and 1953 response of corn is given in tables A-12 and A-13. The square root function's treatment deviations from regression is almost one-half that for the crossproduct function.

Residual Response of Corn

Analysis of variance for the 1953 residual response of corn is given in tables A-10 and A-11.

TABLE A-12. SQUARE ROOT FUNCTION FOR TOTAL 2-YEAR RESPONSE OF CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	466,207.68	
Treatments	56	450,794.90	8,049.91
{Due to regression	{ 5	430,186.01	86,037.20
{Deviation from regression	{ 51	20,608.89	404.10
Among plots treated alike	57	15,412.78	270.40

$$F = \frac{86,037.20}{270.40} = 318$$

TABLE A-13. CROSSPRODUCT FUNCTION FOR TOTAL 2-YEAR RESPONSE OF CORN.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	113	466,207.68	
Treatments	56	450,794.90	8,049.91
{Due to regression	{ 5	411,938.30	82,387.66
{Deviations from regression	{ 51	38,856.60	761.89
Among plots treated alike	57	15,412.78	270.40

$$F = \frac{82,387.66}{270.40} = 305$$

TABLE A-14. EXPERIMENTAL YIELDS OF CORN FOR VARYING LEVELS OF FERTILIZER APPLICATION ON CALCAREOUS IDA SILT LOAM SOIL IN WESTERN IOWA IN 1952 (YIELDS ARE IN BUSHELS PER ACRE).*

Pounds P_2O_5	Pounds nitrogen								
	0	40	80	120	160	200	240	280	320
0	24.5	23.9	28.7	25.1	17.3	7.3	16.2	26.8	25.1
	6.2	11.8	6.4	24.5	4.2	10.0	6.8	7.7	19.0
40	26.7	60.2			96.0	95.4			81.9
	29.6	82.5			107.0	95.4			76.4
80	22.1		99.5		115.9		112.4		129.0
	30.6		115.4		72.6		125.6		82.0
120	44.2			119.4	113.6			114.9	124.6
	21.9			97.3	102.1			129.2	83.0
160	12.0	96.2	102.2	133.3	129.7	105.7	130.5	123.6	135.6
	34.0	80.7	108.5	124.4	116.3	115.5	124.3	142.5	122.7
200	37.7	81.1			128.7	140.3			136.0
	34.2	51.0			109.3	142.2			118.2
240	38.0		97.2		127.6		121.1		130.9
	35.0		107.8		125.8		114.2		144.9
280	32.4			129.5	134.4			130.0	124.8
	27.4			125.2	127.6			141.9	114.1
320	5.3	79.5	116.9	135.7	122.9	138.7	127.3	131.8	127.9
	17.9	39.7	83.6	121.5	122.7	126.1	139.5	111.9	118.8

* Two figures are shown in each cell since the treatments were replicated (i.e., two plots received the same fertilizer quantities and ratios).

TABLE A-15. EXPERIMENTAL YIELDS OF RED CLOVER FOR VARYING LEVELS OF FERTILIZER APPLICATION ON WEBSTER AND NICOLLET SILT LOAM IN NORTH-CENTRAL IOWA IN 1952. (YIELDS ARE IN TONS PER ACRE).*

Pounds P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	2.58	2.68	3.06	2.76	2.91	2.83	3.03	3.21	2.67
	2.29	2.88	2.48	2.66	2.77	2.55	2.78	2.86	2.85
40	3.01	3.14			3.20	3.38			3.77
	3.49	3.05			3.57	3.28			3.73
80	3.18		3.20		3.61		3.64		4.31
	3.06		3.99		3.25		3.43		3.28
120	3.30			3.31	3.90			4.17	3.62
	3.19			3.40	3.42			4.20	3.31
160	3.59	3.56	3.74	3.04	3.81	3.45	3.68	3.70	3.45
	3.77	3.78	3.62	3.65	3.46	3.68	3.53	3.71	3.61
200	3.57	3.23			3.87	3.62			3.58
	3.55	3.51			3.53	3.78			3.82
240	3.55		3.91		3.94		3.55		3.53
	3.66		3.63		3.13		4.04		3.47
280	3.19			4.10	3.46			3.62	3.73
	3.62			3.19	3.59			4.05	3.74
320	3.35	3.55	3.61	3.64	4.09	3.65	3.49	3.85	3.57
	3.33	3.88	3.42	3.79	3.59	3.40	3.35	3.99	3.32

* See footnote for table A-14.

TABLE A-16. EXPERIMENTAL YIELDS OF ALFALFA FOR VARYING LEVELS OF FERTILIZER APPICATION ON WEBSTER AND NICOLLET SILT LOAM IN NORTH-CENTRAL IOWA IN 1952. (YIELDS ARE IN TONS PER ACRE).*

Pounds P_2O_5	Pounds K_2O								
	0	40	80	120	160	200	240	280	320
0	1.14	1.85	2.93	2.13	2.57	2.75	2.33	2.07	2.18
	1.64	2.68	2.09	2.98	2.60	2.69	2.50	2.59	2.11
40	2.86	3.40			3.46	3.52			3.05
	3.13	3.20			3.26	3.57			3.53
80	3.26		3.39		3.56		3.08		3.49
	2.86		3.51		3.22		3.28		3.56
120	2.96			3.64	3.44			4.02	3.64
	2.91			3.50	3.57			3.50	3.28
160	3.64	3.23	3.22	3.27	3.61	3.66	3.50	3.47	3.69
	3.24	3.61	3.51	3.40	3.47	3.48	3.87	3.68	3.43
200	3.65	3.12			3.43	3.62			3.34
	3.31	3.02			3.55	3.26			3.74
240	3.56		3.60		3.72		3.83		3.52
	3.66		3.29		3.81		3.64		3.65
280	3.25			3.19	3.49			3.64	3.61
	3.17			3.17	3.91			3.77	3.88
320	3.52	3.57	3.50	3.97	3.71	3.64	3.70	3.73	3.86
	3.42	3.84	3.49	3.52	3.51	3.66	3.53	3.59	3.51

* See footnote for table A-14.

TABLE A-17. RESIDUAL RESPONSE OF CORN FROM THE PRECEDING YEAR'S APPLICATION OF FERTILIZER. THESE FIGURES ARE THE 1953 YIELDS FOR THE SAME PLOTS GIVEN IN TABLE 7. NO FERTILIZER WAS APPLIED IN 1953, SO ANY INCREASE IN YIELD IS DUE TO THE RESIDUAL FERTILIZER EFFECT FROM 1952. (YIELDS ARE IN BUSHELS PER ACRE).*

Pounds P_2O_5 in 1952	Pounds nitrogen in 1952								
	0	40	80	120	160	200	240	280	320
0	5.1	4.9	7.8	25.8	4.1	14.1	9.0	9.6	7.8
	9.2	19.6	12.4	14.6	5.6	7.0	8.8	4.7	8.5
40	16.3	20.2			18.1	34.6			8.2
	20.0	10.7			63.0	23.6			37.1
80	12.3		29.6		29.2		27.8		27.9
	17.9		14.2		18.6		40.2		37.5
120	22.5			19.4	41.1			50.3	47.7
	6.1			22.1	27.0			53.2	36.8
160	5.3	44.2	21.2	23.3	47.9	36.1	56.7	60.4	51.1
	12.5	15.3	34.7	24.2	26.2	55.7	61.1	47.9	50.3
200	31.9	13.6			30.8	61.3			62.1
	11.4	15.3			59.9	62.5			67.8
240	28.4		36.5		51.9		53.1		65.8
	25.9		15.6		51.9		66.8		66.7
280	20.5			33.4	59.4			72.6	76.7
	11.5			36.2	34.5			72.1	66.1
320	8.6	10.5	14.9	41.9	60.7	52.8	67.0	60.4	69.6
	3.7	13.6	9.9	30.6	28.5	56.6	57.5	60.0	70.5

* See footnote for table A-14.

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