Physical and Mathematical Theories Of Tile and Ditch Drainage And Their Usefulness in Design

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CONTENTS

Physical and Mathematical Theories of Tile and Ditch Drainage and Their Usefulness in Design[1]

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I. INTRODUCTION

A number of theories for tile and ditch drainage have been proposed in recent years which, if valid, would enable the rational design of many drainage systems. Nevertheless, most drainage systems are still designed by rule of thumb based largely upon the observations of technicians with experience in certain restricted areas.

To develop a theoretically sound and practically valuable method of designing subsurface drainage systems, the various approaches which have been made should be critically evaluated and compared, mutually, as well as with field data. However, no such analysis has been found in the literature.

The object of this publication is to provide this type of appraisal. The assumptions underlying a number of methods of analysis will be scrutinized in detail, and various applications of these methods to field results will be tested. It is hoped that this evaluation of the *status quo* will be useful in determining to what extent present theories lend themselves to field applications and what phases of drainage design need further study.

In general, this discussion will be restricted to problems of saturated flow, while recognizing that flow in the unsaturated zone above the water table often may be important. Little progress has been made in formulating quantitative theories regarding flow in the unsaturated zone.

This bulletin also is limited to a discussion of the control of the water table. The question of what moisture conditions are required in the soil to produce the best crops is left untouched.

Finally, it should be pointed out that the inclusion of the work of any one author does not in itself imply agreement with his conclusions. In fact, several proposed analyses are discussed in detail to point out that they, or such similar type of analysis, cannot be expected to yield reliable results.

II. REVIEW OF LITERATURE

It is convenient to separate drainage flow problems, both theoretical and experimental, into two classes: steady state problems and nonsteady state problems. A steady state condition exists when a system—its flow rates and boundaries—does not change with time, i.e., when the system is in dynamic equilibrium. Otherwise, a nonsteady state condition exists.

THEORETICAL INVESTIGATIONS

STEADY STATE PROBLEMS

One of the earliest steady state solutions is based on the assumption of parallel flow. This assumption, originated by Dupuit, leads to an equation of an ellipse for the shape of the water table above parallel drains. Gustafsson (23) and Zunker (61) have credited this solution to Colding; Rothe (49) developed it independently, as did Kozeny (42), Hooghoudt (26), and Aronovici and Donnan (1).

The assumption of radial flow towards drains was used by Hooghoudt (27), Kirkham (32, 33, 34, 35, 36, 38) and Gustafsson (23), all three of whom applied the method of images to find solutions to a number of specific problems. Kirkham and Gustafsson restricted their exact solutions to problems where the soil was saturated to the surface; Hooghoudt applied the method to find an approximate solution for a curved water table over tile drains in a soil homogeneous to infinite depth. Another method was used by Kirkham (37) when he found the potential and stream functions for ditch drainage over an impervious substratum by considering the problem as a limiting case of flow into an auger hole of infinite radius surrounded by an impermeable concentric barrier located a finite distance away from the hole.

Gardner, Israelsen and McLaughlin (20), and later Farr and Gardner (14), combined the parallel flow hypothesis and the radial flow assumptions to approximate the rate of flow from an artesian vein into tile drains. Hooghoudt (27) used a similar technique for flow into drains overlying an impermeable layer.

An exact solution for the shape of the water table above tile drains in a semi-infinite, homogeneous soil was obtained by Van Deemter (52, 53) with the hodograph method. Engelund (13a) solved essentially the same problem as Van Deemter by this method, arriving at a solution in a slightly different form. Vedernikov (56) seems to have been the first to apply this approach to tile drainage, although Hamel (24) used it earlier for seepage under a dam. Gustaffson (23) found a special case of the more general Van Deemter solution,

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and also Davidson and Rosenhead (10) solved a tile drainage problem with it.

Finally, the relaxation method has been applied to tile and ditch drainage. Here, Luthin and Gaskell (43) dealt with soil saturated to the surface; Van Deemter (52, 53), with cases involving a curved water table.

NONSTEADY STATE PROBLEMS

The parallel flow assumption, leading to the so-called heat flow equation, was first used by Forchheimer (16) in connection with nonsteady groundwater flow. Recently, Ferris (15) applied the method to a ditch drainage problem and Glover, as reported by Dumm (12), to tile drains overlying an impermeable layer. Kemper (30) modified Glover's solution by introducing a correction factor obtained by comparing the equations with the results of electric analogue studies and by restricting its use to open ditch drains. Kano (29) extended the use of the ellipse equation, as derived by Kozeny, to nonsteady state conditions. Visser (57), using a different technique, also modified the ellipse equation to apply to changing water tables.

Spöttle (51), in an elaborate treatise on drainage problems, and Walker (59) modified the assumption of radial flow to find approximate solutions for the position of a falling water table.

Kirkham and Gaskell (40) treated the falling water table as a series of successive steady states and used the relaxation method to find specific solutions to four problems. Except for this last method, which in theory could be refined to any desired degree of accuracy, no exact solutions pertaining to the nonsteady state have been found in the literature.

Experimental Investigations

Three types of experiments have been conducted: field experiments, electric analogue studies and model studies.

As early as 1903, Spöttle (51, p. 106) determined the height of the water table in land with tiles spaced from 60 to 30 feet, using recording equipment at the wells midway between drains. These data were used as a qualitative check on Spöttle's theoretical considerations. Schlick (50) investigated water table behavior over many tile systems in Iowa for an 8-year period, recording also rainfall, tile discharges and soil textures. He concluded that 100-foot spacing and 4-foot depth was adequate for most Iowa soils. Weir (60) observed water table heights on bottomlands in California. He found that the water table between drains was essentially flat, in contradiction to the elliptic shape anticipated. Ferris (15) made some observations on a Michigan soil to test his theoretical findings. Similarly, Walker (59) tested his theory in Virginia.

More recently, the Iowa (28) and Minnesota (44) agricultural experiment stations have installed experiments to investigate the effect of spacing between drains on water table behavior. These are treated in more detail in Section IV, as are the data of Kirkham and De Zeeuw (39) obtained from a spacing experiment in the Netherlands. Van Schilfgaarde, Frevert and Kirkham (55) reported the installation of a field laboratory near the Missouri River in Iowa for a similar purpose.

A sand tank model, used by Gross (21), led to the development of a hyperbolic equation for the water table. Hooghoudt (26) used a sand tank to determine the hydraulic conductivity of the sand from the ellipse equation. Kirkham (31) used sands of different coarseness to check the work of Gardner *et al.* (14, 20). The effect of a hardpan was investigated by means of a model by Kirkham (35) and the case of artesian pressure by Harding and Wood (25). Gustafsson (23) experimented with varying discharges from adjacent drains, and Donnan (11) made a test of the ellipse equation; both used sand tanks. Günther (22) made a model by means of a viscous liquid flowing between parallel plates to study the flow to a drain in an impermeable layer. Besides Donnan, also Gustafsson, Günther and Kirkham (33) compared results of model studies with analytical derivations.

Childs was the first to study the flow toward drains by means of an electric analogue. He used a solid-conductor, two-dimensional analogue to study steady state (4, 5, 6, 7) as well as nonsteady state problems (8, 9). A liquid conductor was used by Dutz (13) when he adapted a three-dimensional analogue, developed by Frevert (18, 19), to study the problem of flow into the joints between tiles. Kemper (30) used a liquid-conductor, two-dimensional apparatus to check the results of his analytical work.

Several of the papers of the above literature review are critically evaluated in the next section.

III. EVALUATION OF SOME EXISTING SOLUTIONS TO DRAINAGE PROBLEMS

The Dupuit-Forchheimer Theory

THE ASSUMPTIONS

The Dupuit-Forchheimer theory of gravity-flow systems is based on assumptions which, if carried through consistently, lead to an absurdity. If the limitations of its underlying assumptions are thoroughly understood, the theory, in some cases, can lead to simpler, valuable solutions than would be obtained by a rigorous analysis based solely on Darcy's law and the Laplace equation. The theory is in widespread use.

The two basic assumptions, apparently due to Dupuit [see Muskat (45, p. 359)], are: (a) all streamlines in a system of gravity flow towards a shallow sink are horizontal; and (b) the velocity along these streamlines is proportional to the slope of the free-water surface, but independent of the depth.

Let us consider (fig. 1) a saturated soil column above an impervious layer of base $\triangle x \triangle y$ and height h(x,y) in dynamic equilibrium, and designate by v_x and v_y the velocity components in the X and Y directions. The condition of continuity may be written, if the liquid is assumed to be incompressible, as

$$v_x h \triangle y - [v_x h \triangle y + \frac{\partial(v_x h)}{\partial x} \triangle x \triangle y] + v_y h \triangle x - [v_y h \triangle x + \frac{\partial(v_y h)}{\partial y} \triangle y \triangle x] = 0. \qquad [1]$$

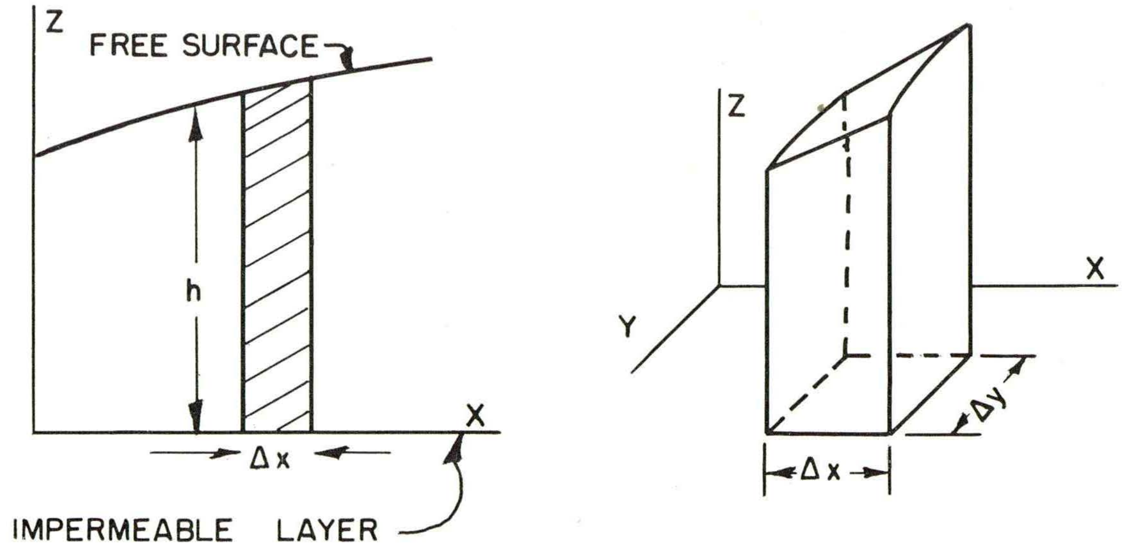

Fig. 1. Groundwater flow system in dynamic equilibrium. The shaded section at left is shown at right in three dimensions.

Dividing by $\triangle x \triangle y$, and going to the limit when $\triangle x$ and $\triangle y$ both approach zero, this relation reduces to

$$\frac{\partial}{\partial x}(hv_x) + \frac{\partial}{\partial y}(hv_y) = 0. \quad [2]$$

The second Dupuit assumption implies, if K represents the hydraulic conductivity,[3] that

$$v_x = -K\frac{\partial h}{\partial x}; \qquad v_y = -K\frac{\partial h}{\partial y}. \quad [3]$$

Combining eqs. [2] and [3], there results

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = 0. \quad [4]$$

This equation, due to Forchheimer (16, p. 83), is identical in form to the Laplace equation in two dimensions, so that the same methods of solution can be applied to both.

Strictly speaking, the Dupuit assumptions imply that there be no flow. For, by definition of potential ϕ in terms of vector velocity $\bar{v}$ (i.e.,$\bar{v} = -\nabla\phi$),

$$\frac{\partial\phi}{\partial x} = -v_x, \qquad \frac{\partial\phi}{\partial y} = -v_y, \qquad \frac{\partial\phi}{\partial z} = -v_z;$$

differentiation of these relations shows that

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_z}{\partial x}, \qquad \frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial y}.$$

However, since the velocity is assumed to be independent of depth,

$$\frac{\partial v_x}{\partial z} = \frac{\partial v_y}{\partial z} = 0,$$

so that $\partial v_z/\partial x = 0$ and $\partial v_z/\partial y = 0$ and, accordingly, the velocity in the vertical direction must be constant in every horizontal plane in the flow system. Since, however, v_z will have the constant value zero at those points where the flow is horizontal (such as, for radial flow, along the vertical inflow surface of a well), v_z will be zero everywhere. Thus the slope of the free surface must be zero everywhere, and further, according to Dupuit's second assumption, all velocities must be zero. Hence, a rigid analysis based on the Dupuit assumptions leads to the absurdity that there can be no flow at all if gravity is the only acting driving force.

Muskat (45, p. 317) has shown that, notwithstanding the above serious limitations of the Dupuit-Forchheimer theory, remarkably accurate results are obtained when it is used to determine the flux towards a well or through a dam but that the error may be large, in comparison with results obtained by more exact theoretical solutions, when the Dupuit-Forchheimer theory is used to determine the shape of the free surface and the velocity distribution. Muskat has rejected the theory entirely and credited the success of the flux determination to "fortuitous coincidence" rather than to reasonable approximations. The present writers do not agree entirely with Muskat. It seems to them that, when the flow is essentially horizontal, the loss of head in the vertical direction in a stream tube will be negligible compared with the loss in the horizontal direction in many cases, especially those cases where there is little convergence or divergence in the streamlines.

The Dupuit-Forchheimer theory has also been applied to nonsteady state problems. Letting f be the porosity, the equation of continuity for the latter case is derived by replacing the righthand side of eq. [1] by $-f\partial h/\partial t$, with the result

$$K\left[\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial h}{\partial y}\right)\right] = f\frac{\partial h}{\partial t}. \quad [5]$$

If, now, h is replaced by d + z where d represents the constant depth of the impervious layer below a reference

[3]A distinction is made between permeability, a property of the soil with the dimensions L^2 (as sq. cm.), designated as k, and hydraulic conductivity, defined as the proportionality constant K in Darcy's law $v = Ki$ and having the dimensions of a rate, LT^{-1} (as feet/day). The two are related by the equation $K = k\rho g/\mu$, where ρ, g and μ represent the fluid density, the gravitational constant and the fluid viscosity, respectively.

plane, and if $d >> |z|$ so that z may be neglected in comparison to d, this expression can be simplified to

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{f}{Kd} \frac{\partial z}{\partial t}. \qquad [6]$$

Equation [6] is often referred to as the heat flow equation since it is identical in form to the differential equation that applies to heat flow problems.

Equation [6] is, of course, no better than eq. [4] since it is based on the same assumptions. However, both equations can be expected to yield reasonably accurate results if they are applied to problems where the region of flow is of large horizontal extent relative to its depth. In such cases, the streamlines through a large part of the region will be approximately horizontal.

THE ELLIPSE EQUATION FOR THE STEADY STATE WATER TABLE

Derivation of ellipse equation. As has been cited in the Review of Literature, a number of investigators have derived the ellipse equation. Hooghoudt's reasoning (26) will be followed here.

Figure 2 represents a homogeneous soil underlain by an impermeable layer and drained by parallel, vertically walled, open ditches. Assuming that a time-constant rate of rainfall is removed equally well at all distances from the drains, the rate q_x, at which water crosses a vertical plane at any value of x, can be expressed in terms of Q_1, half the total discharge of each drain per unit length, and S, the spacing between drains, as[4]

$$q_x = \frac{S/2 - x}{S/2} Q_1 . \qquad [7]$$

From the Dupuit assumptions, it follows that

$$q_x = -yv_x = yK\, dy/dx . \qquad [8]$$

Equating the two expressions for q_x and separating the variables, one obtains the differential equation

$$ydy = (2Q_1/SK)\,(S/2 - x)\,dx . \qquad [9]$$

Integrating this from $x = 0$ and $y = h_o$ to $x = x$ and $y = y$, where h_o represents the height of the water surface in the drain above the impermeable layer, there results

$$y^2 - h^2_o = (2Q_1/SK)\,(Sx - x^2). \qquad [10]$$

This is the equation of an ellipse. Substitution of the values $x = S/2$ and $y = H_o$, the midpoint values between drains, yields

$$S = 2K(H^2_o - h^2_o)/Q_1 . \qquad [11]$$

In this form, the equation may be used to determine S if the other quantities are known, or, similarly, K or Q_1.

Applicability of ellipse equation. If the theory is restricted to ditches which are shallow compared to their spacing and which penetrate to an impermeable layer, the assumption of horizontal flow and the resulting ellipse equation appear to be reasonably correct. The fact that the flow midway between the drains is vertical is offset by the fact that the flow through the greater part of the flow region is approximately horizontal.

There is still a difficulty which has not been considered. There remains the implicit assumption that the water table would reach the ditch at the height of the ditch water level; thus, the existence of a surface of seepage has been ignored. Muskat (45, p. 289) has shown that a surface of seepage must exist, as otherwise an infinite velocity would occur at the point where the water surface in the ditch touches the ditch wall. Nevertheless, with essentially flat water tables, surfaces of seepage will be small. One is also reminded here that capillary flow, which is probably more important than a surface of seepage in problems under discussion here, has been neglected.

Considering Muskat's findings concerning the inaccuracies in the shape of the free surface as predicted by the Dupuit-Forchheimer theory for seepage in dams and also the equations as developed by Van Deemter[5] for the shape of the water table over tile lines, there is ample evidence that eq. [10] may be a poor approximation for the free surface in an actual case of drainage by ditches. Equation [11], however, which is for the flow and which involves H_o and h_o but not y, has been tested by means of various sand tank experiments (26, pp. 476-488; 11) and in the field (1) and has been

4Aronovici and Donnan (1, p. 100) arrived at this same equation without explicity stating that a time- and space-constant rate of downward seepage to the water table must be assumed. Such constancy would be present in their problem; the seepage would be excess irrigation water dripping down to the water table rather than rain water seeping through the soil as assumed by Hooghoudt.

5Van Deemter's work is discussed in Section III.

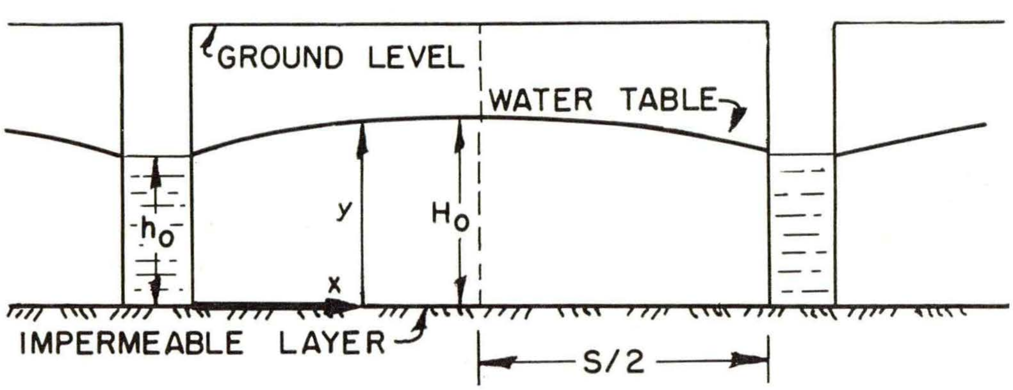

Fig. 2. Geometry and symbols used in derivation of ellipse equation.

found to agree quite accurately with the experimental data. The experiments just cited are concerned with flow into tile drains rather than into open ditches for which eq. [11] was derived. The consequences of this substitution of tile for ditches will be discussed presently.

The ellipse equation has also been applied to conditions which differ in varying degrees from the case for which it was derived. One of these is the case where ditches do not penetrate to the underlying impermeable layer. Here, the assumption of horizontal flow fails to take into account the fact that the flow lines must converge to reach the drain and that some must follow a longer path than a horizontal one. Hooghoudt (26, p. 495) has presented a refinement which allows for this longer path. He reasons as follows: If the bottom width of equally spaced ditches is 2b and the distance from each ditch bottom to the impervious stratum is L, the average streamline below each ditch will be longer than the half spacing by about $(L + b)/2$, and the proportion of streamlines below the ditch can be approximated by L/h_0. Substitution of the average length of streamline,

$$S/2 + L(L + b)/2h_0 ,$$

for $S/2$ in eq. [11] gives

$$S = 2K(H^2_0 - h^2_0)/Q_1 - L(L + b)/h_0$$

for the corrected spacing.

Another case to which the ellipse equation has been applied involves tile drains. Substitution of tile drains for open ditches requires the neglect again of the effect of convergence of flow toward the drains. Since often, at least for a time, the backfill over drains retains a higher conductivity than the undisturbed soil, the error introduced by the use of tile drains is not much greater than that caused by applying the theory to ditches which do not reach the impermeable layer. Thus, if the problem is restricted to conditions with a horizontal tight layer near the bottom of the tile, the height of the water table midway between drains should be determinable fairly closely by the last equation.

Some investigators, however, have not paid any attention to the relative position of such a tight layer. Aronovici and Donnan (1), for example, did not discuss its implications. Rothe (49) applied the ellipse analysis to a homogeneous soil of infinite depth by assuming that all flow takes place above the plane through the drain axes. This is incorrect, as is seen by inspection of Gustafsson's fig. 15 (23, p. 45). This figure shows that, when the water table is everywhere at the soil surface, nearly half the flow towards parallel drains passes below the plane through the drain axes. For the elliptically shaped water table considered by Rothe, this portion would be even greater.

If the impermeable layer, for either tile or ditch drains, is at a considerable depth below the drains, then the effect of convergence of flow can no longer be ignored. Hooghoudt substituted a radial flow method for this case, combining radial and horizontal flow for intermediate conditions. The radial method and combination method will be treated on pages 678-681.

The ellipse equation with flow in the capillary fringe. Both Hooghoudt and Donnan found that, to make the data from sand tank experiments agree with eq. [11], they had to add the height of the capillary fringe to the values found for H_0 and h_0. This procedure can be justified if it may be assumed that the upper boundary of the capillary fringe is well defined and that no flow takes place above this boundary.

The capillary fringe, as defined here, is a saturated region of less than atmospheric pressure. Its presence does not alter the total hydraulic head to be dissipated. However, the region of flow becomes larger by a constant amount equal to the height, w, of the capillary fringe. Therefore, eq. [7] still holds, but eq. [8] must be changed to read

$$q_x = (y + w)Kdy/dx.$$

Then, instead of eq. [9], the differential equation becomes

$$(y + w)dy = (2Q_1/SK)\ (S/2 - x)dx .$$

Integrating from $x = 0$ to $x = S/2$ and from $y = h_0$ to $y = H_0$, one obtains, after some manipulation,

$$Q_1S/2K = (H_0 + w)^2 - (h_0 + w)^2 .$$

Hence, addition of w to the values of H_0 and h_0 in eq. [11] does account for the capillary fringe.

THE ELLIPSE EQUATION AND THE CHANGING WATER TABLE

Visser (57) has attempted to extend the application of the ellipse equation to problems involving water table fluctuations caused over tile drains by storms of short duration and of higher intensity than a normally prevailing average (constant) rate of rainfall. The essence of this analysis is as follows.

Designating by n the rate of groundwater seepage into a unit length of tile or ditch per unit area of soil surface between tiles or ditches (so that $Q_1 = nS/2$, and n thus has the dimensions LT^{-1}), one can write eq. [11] as

$$n = 4K(H^2_0 - h^2_0)/S^2 .$$

Combining $4K/S^2$ into a factor D, and designating the head difference $H_0 - h_0$ as m, this equation takes the form

$$n_e = D(m^2_e + 2h_0m_e) , \qquad [12]$$

where the subscript e has been added to emphasize an equilibrium condition: thus, n_e is the constant discharge rate and m_e the corresponding head difference.

Now, it is assumed that during a very short interval during a period of otherwise constant normal rainfall rate, an amount of rain N inches, say, falls in excess of the amount which falls at the normal constant rate for the short interval. Further, it is assumed as a result of the excess rain N, that the water table, everywhere, rises above the equilibrium position by a height $T = N/f$ (f being the porosity of the soil), except that right over the tile the height of the water table remains h_0. Then the peak discharge, n_p, can be expressed as

$$n_p = D[\ (m_e + T)^2 + 2h_0(m_e + T)\ ] . \qquad [13]$$

Eliminating m_e from eqs. [12] and [13], it is found that

$$n_p = n_e + DT[T + 2(h^2_0 + n_e/D)^{1/2}] . \qquad [14]$$

The maximum (extra) height T to which the water table will rise on the average of once a year (or with any other desired frequency) can now be determined with the aid of weather records. Given the maximum N to be expected for a chosen storm duration, with a frequency of once per year, the corresponding rise of the water table for this storm will be simply $T = N/f$. This value of T, when substituted in eq. [14] along with the supposedly known values of n_e, h_o and D, yields n_p the peak discharge of drain per unit area of soil surface.

The above analysis enables one to design a drainage system not only on the basis of optimum average water table height, but also so that a given height ($H_o + T$ above the impermeable layer) is not exceeded more often than specified. For areas where a gentle rain of long duration, with occasionally more intense showers, is a frequent occurrence, this approach to drainage design appears valuable. Visser was primarily interested in the conditions found in the Netherlands, and his method, in application there, seems reasonable insofar as the assumptions underlying the equation and hence the equation itself are correct. Also in irrigated areas in arid regions, the method possibly could be applied profitably. In regions such as Iowa, the method does not appear feasible, as the rainfall pattern is too uneven. The method might, however, be extended to rainy periods with varying intensity by considering these as a series of consecutive periods with constant intensity and by superposing the effect of each of these periods. Even then, there remains the problem of determining an equilibrium position as a starting point.

Besides the assumptions underlying the ellipse equation, Visser also assumed that the rise of the water table would not vary with the distance from the drains—with the exception of those points immediately over the drains. This is in contradiction to the second Dupuit assumption (that the velocity is proportional to the slope of the water table) and as such invalidates the derivation of the equation. If the important factor in the Visser derivation is the increased head differential (from $H_o - h_e$ to $H_o + T - h_e$), then the equation may be a good approximation. The authors know of no experimental verification of the equation.

THE HEAT FLOW EQUATION

The heat flow equation, eq. [6], used by various investigators to solve problems concerning a falling water table, will be discussed here in connection with the work of Ferris (15) and of Glover as reported by Dumm (12).

Glover's equations. Glover, who was primarily interested in the drainage of irrigated land, has proposed a formula for the spacing required of tile drains to maintain the water table below a specified level. The major shortcoming of this formula is that it does not take into account the restricting effect of convergence of flow near the drains, even though, as will appear, Glover tacitly assumes that the drain tile is of zero radius.

Since, in essence, only the theoretical results of Glover's work have been reported (12), the derivations will be given here. We are indebted to Glover for supplying some of the missing steps.

Considering a system of equally spaced tile lines in a homogeneous soil overlying an impermeable boundary (fig. 3), the equation of continuity based on the Dupuit assumptions may be written (compare eq. [6]) as

$$\frac{\partial y}{\partial t} = \frac{KD}{f} \frac{\partial^2 y}{\partial x^2}, \qquad [15]$$

where x and y are the rectangular coordinates defined in fig. 3, t designates the variable time, K and f have the same meaning as before and D represents an average thickness of the aquifer. With the distance between the tile axes and the impermeable layer equal to d and with the initial height of the water table above the drains equal to y_o, the distance D as defined by Glover is

$$D = d + y_o/2 .$$

To treat D as a constant as is done in eq. [15], y_o must be small compared to d.

A solution of eq. [15] depends on the initial and boundary conditions, which, as used by Glover, are (fig. 3) : $y = y_o$ for $t = 0$ and for $0<x<S$ (an initially flat water table), and $y = 0$ for $t \geqslant 0$ and for $x = 0$ and $x = S$ (the water table immediately over the drain tiles always remains at drain tile level). Notice that these

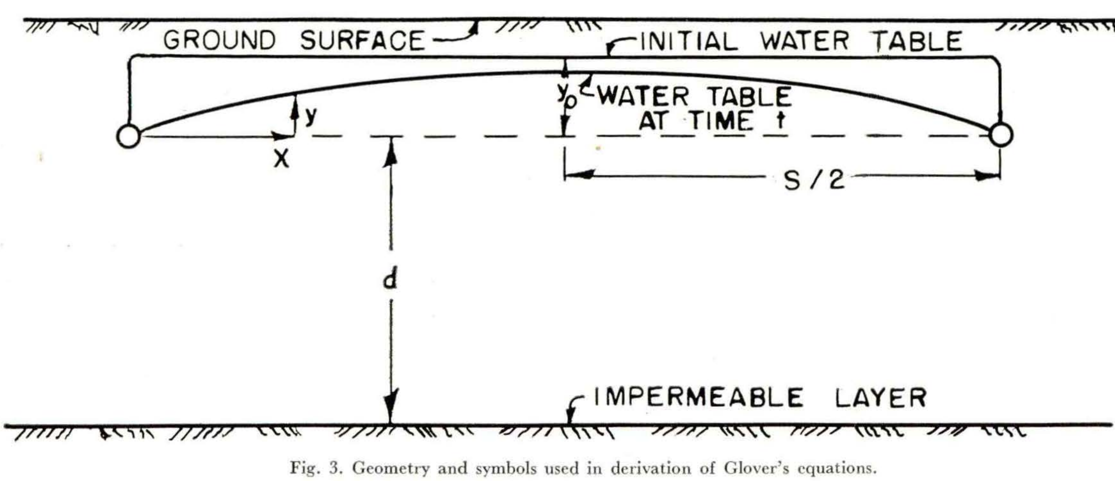

Fig. 3. Geometry and symbols used in derivation of Glover's equations.

conditions do not account for the boundaries around the tile, nor does eq. [15], unless the tile is taken to be of zero radius.

The solution of eq. [15], subject to the above conditions, can be found by conventional methods in terms of a sine series [see, for example, (47,p. 262)]. Considering the point midway between drains and neglecting all terms but the first, this series, when evaluated for the value $y_{S/2}$ of y at x = S/2, reduces to[6]

$$y_{S/2} = (4y_o/\pi) \exp(-KD\pi^2 t/fS^2) . \qquad [16]$$

The reason for retaining only the first term of the series is that, after a reasonable length of time has elapsed, the error introduced by using only the first term becomes small. In fact, when $KDt = 0.025\ S^2f$ the error is on the order of 4 percent, which, in view of the approximate nature of the continuity equation, is of no concern.

If eq. [16] is solved for S, the spacing equation is

$$S = \pi[KDt/f \ln (4y_o/\pi y_{S/2})]^{1/2}. \qquad [17]$$

To determine the amount of error introduced by assuming D to be constant when d is not large compared to y_o, Glover also developed an equation for the case where d = 0. The appropriate continuity equation, again based on the Dupuit assumptions, is

$$\frac{\partial}{\partial x}(Ky\frac{\partial y}{\partial x}) = f\frac{\partial y}{\partial t}, \qquad [18]$$

which is equivalent to eq. [5] for one-dimensional flow.

This time Glover, without specifying an initial condition for y except at x = S/2, assumed

$$y = y_o \text{ at } x = S/2 \text{ for } t = 0, \qquad [19a]$$

$$y = 0 \text{ at } x = 0 \text{ and } x = S \text{ for } t \geqslant 0. \qquad [19b]$$

With these assumptions, the solution of eq. [18] gives a bowed water table at t = 0 rather than the flat water table incorporated in the solution of eq. [15].

By means of the transformation

$$U = y/y_o,\ \alpha = x/S, \beta = Ky_ot/fS^2,$$

the continuity equation, eq. [18], may be written as

$$\frac{\partial}{\partial \alpha}(U\frac{\partial U}{\partial \alpha}) = \frac{\partial U}{\partial \beta}.$$

Assuming a solution of the form $U = V(\beta)W(\alpha)$, one obtains by substitution

$$(W')^2/W + W'' = V'/V^2, \qquad [20]$$

where the primes indicate differentation with respect to the appropriate independent variable.

The left hand member of eq. [20], since this member is not a function of β, can be held constant when β in the right hand member varies; also, the right hand member of eq. [20], since this member is not a function of α, can be held constant when α varies. Therefore, the two sides must be equal to the same constant, say, –k.

Writing, then,

$$V'/V^2 = -k$$

and integrating, the solution for V is

$$V^{-1} = kKy_ot/fS^2 + c_1, \qquad [21]$$

where c_1 is a constant of integration.

The equation in W is

$$(W')^2 + WW'' = -kW,$$

which may be written as

$$\frac{d}{d\alpha}(W\frac{dW}{d\alpha}) = -kW.$$

Dividing by W and replacing $W^2/2$ by γ, one obtains

$$\pm (2\gamma)^{-1/2}\frac{d^2\gamma}{d\alpha^2} = -k.$$

Defining p by $p = \frac{d\gamma}{d\alpha}$, so that $p\frac{dp}{d\gamma} = \frac{d^2\gamma}{d\alpha^2}$, the above equation simplifies to

$$p\,dp = \mp k(2\gamma)^{1/2}\,d\gamma,$$

which can be integrated to

$$p^2 = c_2 \mp 2^{5/2} k \gamma^{3/2}/3.$$

Since $p = d\gamma/d\alpha$, one finds the differential equation

$$\frac{d\gamma}{\pm (c_2 \mp 2^{5/2}k\gamma^{3/2}/3)^{1/2}} = d\alpha,$$

which, in terms of W, becomes

$$\frac{WdW}{(c_2-2kW^3/3)^{1/2}} = d\alpha. \qquad [22]$$

This relationship is equivalent to the one found by Glover (12, eq. [8]) except that the values of the constants c_2 and k in eq. [22] and c_1 in eq. [21] are given there without derivation. The values of these constants will now be derived.

To satisfy eq. [19a], one must have $U = V(0)W(1/2) = 1$, which may be accomplished by the arbitrary selection of $W(1/2) = 1$, $V(0) = 1$. It then follows immediately from eq. [21] that $c_1 = 1$. Furthermore, the slope of the water table at x = S/2 must be zero at all times; that is,

$$dU/d\alpha = (S/y_o)dy/dx = 0 \text{ at } \alpha = 1/2.$$

This relationship must also hold for t = 0, where $U(\alpha,0) = V(0)W(\alpha) = W(\alpha)$, so that then $dW/d\alpha = dU/d\alpha = 0$ at $\alpha = 1/2$. Rewriting eq. [22] as

$$\int_0^W \frac{WdW}{(3c_2/2k - W^3)^{1/2}} = \alpha(2k/3)^{1/2} \qquad [23]$$

and differentiating each side with respect to α, there results, for the midpoint where $\alpha = 1/2$ (47, p. 247, eq. [7]),

$$\int_0^W \left[\frac{d}{d\alpha}\frac{W}{(c_3{}^3-W^3)^{1/2}}\right]_{\alpha=1/2} dW + \left[\frac{W}{(c_3{}^3-W^3)^{1/2}}\frac{dW}{d\alpha}\right]_{\alpha=1/2} = (2k/3)^{1/2}/2,$$

[6]The notation A exp B represents Ae^B where e is the base of natural logarithms.

where $c_3^3 = 3c_2/2k$. Carrying out the differentiation in the first term, it is found that the integrand contains the factor $dW/d\alpha$ and thus vanishes at $\alpha = 1/2$, which is also true for the second term just as it stands. Thus, the denominators must also vanish for k in the right hand side to be different from zero. Hence, $(c_3^3 - W^3) = 0$; that is, $c_3 - W = 0$ or $c_3 = W(1/2) = 1$, from which it follows that $3c_2 = 2k$. Finally, eq. [23] may be evaluated for $\alpha = 1/2$. For then the upper limit of the integral becomes unity, and, by the substitution $\mu = W^3$, one obtains for the left hand side of eq. [23]

$$\int_0^W \frac{W dW}{(1-W^3)^{1/2}} = 1/3$$

$$\int_0^1 \mu^{2/3-1} (1-\mu)^{1/2-1} d\mu = \tfrac{1}{3} B(\tfrac{2}{3},\tfrac{1}{2}) ,$$

where B is the beta function. It may be evaluated, by utilizing gamma functions, as

$$\frac{1}{3} B\left(\frac{2}{3}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{1}{2}\right)}{3\,\Gamma\left(\frac{7}{6}\right)} = 0.862 \ldots .$$

Considering the approximate nature of the assumptions underlying the derivation, Glover took the number 0.862 . . . to be equal to $(1/2)(3)^{1/2}$. The latter value, when substituted in the left hand side of eq. [23] (with $\alpha = 1/2$ in the right hand side), results in $k = 9/2$ and, since $3c_2 = 2k$, one has $c_2 = 3$.

The equation for the height of the water table at the midpoint, where $W = W(1/2) = 1$, may now be found from eq. [21]. One finds, since now $V = U = y_{S/2}/y_o$, the result

$$y_{S/2}/y_o = 2S^2f/(9Ky_ot + 2S^2f) .$$

Solving for S, the spacing is

$$S = [9Ky_ot/2f(y_o/y_{S/2}-1)]^{1/2} , \qquad [24]$$

which is Glover's result.

Comparing eqs. [17] and [24], Dumm reported that the difference in spacings calculated from these, for equal values of the parameters (d being zero for both cases), was always less than 10 percent. From this he concluded that the use of eq. [17] was justified, independently of the relative depth to the impermeable layer. This conclusion presently will be shown to be erroneous.

The major difficulty with Glover's analysis is that it is based on the assumption of horizontal flow. Aside from the general objections to this assumption, which were discussed earlier, there is the problem that eq. [17] was originally developed for tile drains with the restriction that d be large compared to y_o. Hooghoudt has shown (see p. 680), however, that the effect of convergence of flow toward tile drains becomes marked at relatively low values of d/S. Inspection of his table 5 (27, pp. 656-694) shows that this effect must be taken into account when d/S reaches a value around 0.05 or 0.10, depending on the permeability and other factors. Hence the assumption of horizontal flow cannot be expected to yield reliable results unless $d << S$. But with $d << S$, the continuity equation (eq. [15]) still requires that $d >> y_o$, a condition which normally will not be met simultaneously with the condition $d << S$ in the field. Therefore, eq. [17] should not be expected to apply to field conditions. However, to offset this observation

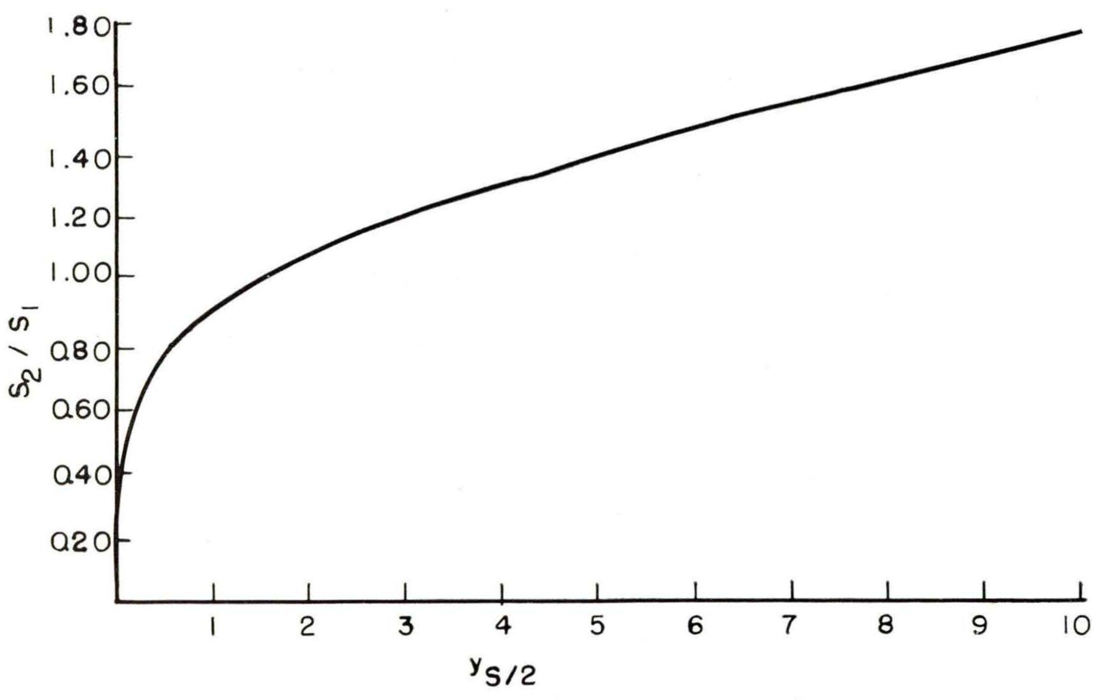

Fig. 4. Change in ratio of spacings as calculated from first and second of Glover's equations for different depths of water table.

there is Dumm's claim that eqs. [17] and [24] are in agreement to within 10 percent for practical conditions. But our findings are not in agreement with this claim as the following calculation shows.

Let S_1 be the spacing given by eq. [17] and S_2 that given by eq. [24]. Then, solving eqs. [17] and [24] for tK/f, eliminating tK/f and solving for S_2/S_1, one obtains eq. [25] if it is assumed that $y_o = 1 + y_{S/2}$ and $D = y_o/2$.

$$S_2/S_1 = (9y_{S/2}/\pi^2) \ln [(4+4y_{S/2})/\pi y_{S/2}]^{1/2}. \qquad [25]$$

In other words, eq. [25] represents the relative magnitude of S_2 compared to S_1 for a drop in the water table of 1 foot when the drains are placed on an impermeable layer. A plot of this equation (fig. 4) shows that the ratio S_2/S_1 varies from zero to infinity, with a percentage difference on the order of 20 to 70 percent—not 10 percent—for those conditions of greatest practical importance.

In view of the above considerations, eq. [17] cannot be justified when d is large, because of the convergence effect, or when d is small, because then the heat flow equation, eq. [15], does not hold.

Some further comments are in order regarding Glover's first and second solutions, eqs. [17] and [24]. The flat water table, assumed as the initial condition applicable to eq. [15], is unrealistic. Unless the drains were plugged during irrigation, such a condition seldom would develop. This assumption was used only to evaluate the constants in the sine series. These may be evaluated as well, however, (see eq. [19a]) if the initial condition is restricted to a consideration of the height of the water table over and midway between drains, leaving the initial water table shape unspecified. If the equation is applied to land initially flooded to the surface, the flat water table assumption could be used.

The assumption that no water stands over the drains at any time is not quite correct either, unless a highly permeable backfill is assumed. Then a surface of seepage would have to be taken into account.

A comparison (fig. 5) of eqs. [17] and [24] can be made with some of the solutions worked out by Kirkham and Gaskell (40). Considering open ditches 5 feet deep which penetrate to an impermeable layer, the values of tK/f have been calculated from eq. [17] as well as from eq. [24] for a water table drop of 1 foot and spacings varying from zero to 80 feet, and the results have been plotted together with the curve determined by Kirkham and Gaskell for the same conditions. The first Glover equation gives higher values of tK/f, and the second lower values than found by Kirkham and Gaskell.

The comparison in fig. 5 tends to put the Glover equations in favorable light in that open ditch conditions have been substituted for tile drainage conditions, for which Glover's formulas were developed. Thus, the convergence effect neglected by Glover has largely been eliminated. Since this convergence would cause more resistance to flow in the tile case than in the ditch case, the drop of the water table over tile drains would be slower and the time factor larger; that is, the middle curve in fig. 5 would be higher. Hence, the first Glover equation, eq. [17], would be expected to yield better results for tile drains located on an impervious layer than is indicated in fig. 5 for open ditches, if the Kirkham and Gaskell solutions are assumed to be correct. On the other hand, the error in eq. [24] (the second Glover equation) would become greater than is indicated in fig. 5.

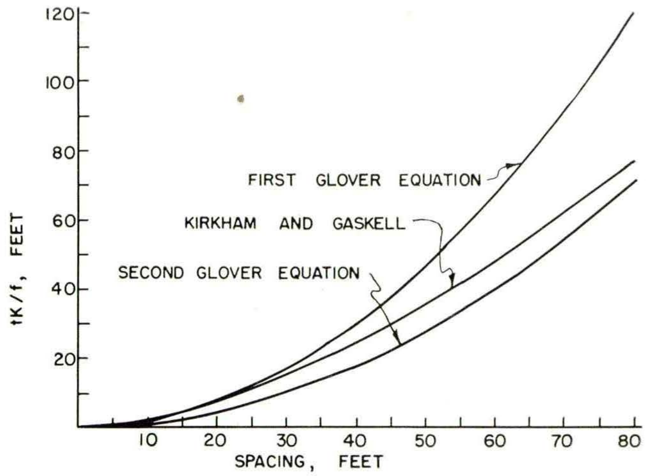

Fig. 5. Comparison of Glover's equations with the results of the relaxation solution of Kirkham and Gaskell.

Especially as long as no better analytical solution is available, it appears from all the above that Glover's equation may be used advantageously to approximate the actual problem. Caution must be used, however, not to rely upon the results too heavily.[7] One should note, however, that it is eq. [24] and not eq. [17] which was developed for the case d = 0 here under consideration.

Ferris' analysis. Another solution based on the heat flow equation was presented by Ferris (15). He considered a homogeneous, isotropic aquifer of infinite areal extent, of constant thickness b, bounded above and below by impermeable strata and with a ditch drain of infinitesimal width penetrating to or below the lower aquiclude. He assumed the Dupuit assumptions to be valid. He also assumed the drainable pore space f and the coefficient of transmissibility, which he took equal to the product Kb, to be constants.

Aside from the weakness of the Dupuit assumptions and the fact that the coefficient of transmissibility (see fig. 6) is K(b − y) and hence not constant, a point to be discussed later, Ferris equated one of the variables,

[7]Since the completion of this study, Kemper (30) has developed an empirical correction factor for the Glover equations based on electric analogue studies. According to Kemper, this correction has resulted in better agreement between field data and theory.

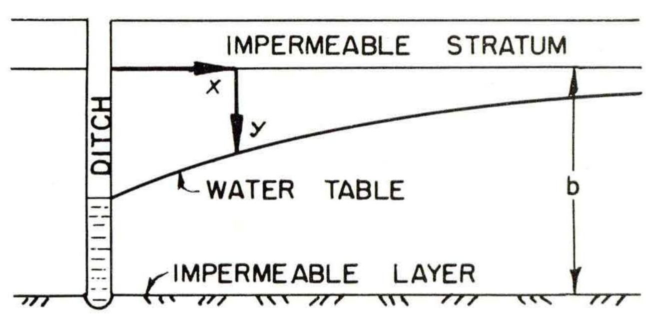

Fig. 6. Geometry and symbols used in discussion of Ferris's equation.

V, to a constant first and then considered it as a variable, thus invalidating his final result.

Although Ferris' analysis is faulty, he is to be credited with an ingenious effort to find a solution to a complicated problem. The slips which invalidated his final result are of such a nature that they can be easily overlooked. In fact, it is the subtlety of argument required to show the invalidity of his solution which justifies the inclusion of its discussion in this bulletin.

With coordinates as shown in fig. 6, Ferris gave the continuity equation as

$$\frac{\partial^2 y}{\partial x^2} = \frac{f}{Kb}\frac{\partial y}{\partial t}.$$

He applied the initial condition $y = 0$ for all x and the boundary condition $y = 0$ at $x = \infty$ for any t. These resulted in a solution of the form

$$y = ct^{-1/2} \exp(-fx^2/4Kbt) . \qquad [26]$$

To evaluate c, he reasoned that the total volume of water yielded by the aquifer must equal the quantity V to be removed by the drain; that is, in terms of an equation,

$$V/2 = \int_0^{\infty} fydx . \qquad [27]$$

Substituting eq. [26] into this expression and integrating, holding t constant, yields

$$c = V/2(\pi fKb)^{1/2},$$

showing that V is independent of t (and of x) and, hence, a constant. Thus, Ferris wrote

$$y = [V/2(\pi fKbt)^{1/2}] \exp(-fx^2/4Kbt) . \qquad [28]$$

Assuming a constant discharge rate $Q = V/t$, Ferris reasoned that eq. [28] should also apply for an infinitesimal drop $\triangle y$ corresponding to a discharge $Q\triangle t$. This enabled Ferris to write eq. [28] as

$$y = [Q/2(\pi fKb)^{1/2}] \int_0^t t^{-1/2} [\exp(-x^2 f/4Kbt)]dt.$$

Using the substitution

$$u = x(f/4Kbt)^{1/2}, \qquad [29]$$

this equation can be partially integrated by parts to

$$y = (Qx/2Kb)[\pi^{-1/2}u^{-1} \exp(-u^2) - 1 + 2\pi^{-1/2} \int_0^u \exp(-u^2) \cdot du] .$$

Designating the quantity in brackets by the "drain function u," D(u), this equation may be written

$$y = (Qx/2Kb)D(u) . \qquad [30]$$

Since D(u) can be plotted against u with the help of tables of the normal distribution, eqs. [29] and [30] give a relationship between the variables of the problem.

For a known Q, b and x, if y has been measured for several values of t, these equations permit the determination of the constants f and Kb, according to Ferris.

Even if the Dupuit assumptions are accepted as correct, the above analysis is fallacious. To evaluate c in eq. [26], Ferris integrated eq. [27] at constant t, thus finding c and V independent of t. However, physically V cannot be independent of t, because the volume of water to be removed by the ditch will depend on the time t that the ditch has already been removing water.

If one assumes that a quasi-steady state has been reached so that the water table drawdown y will be given by eq. [26] with c the constant given below eq. [27] and hence V a constant, then the analysis still will be in error. It will be in error because it will then be incorrect to take $Q = V/t =$ a constant (as was done below eq. [28], for now Q could only be a constant if V varied directly as t. Thus c below eq. [27] would have to be directly proportional to t rather than being constant, and eq. [26] accordingly would not be a solution of the equation of continuity. This can be verified by differentiating [26] with c being replaced by At, where A is a constant. That Q is not nearly constant was shown by Ferris when he reported that, in comparing field data with the equations, a period of 2 hours elapsed before the flux "approached the constancy assumed for the derivation of the drain function" (15, p. 289).

Furthermore, eqs. [29] and [30] give two relationships between three unknowns—that is, between u, Kb and f—which would allow an infinite number of solutions. It appears that Ferris used the additional relationship that y as a function of x^2/t be identical in form to D(u) as a function of u^2. This, however, is not necessarily true.

Finally, there remains the question of constancy of the product Kb. An assumption to that effect is justified only if the total depth of the aquifer is large compared to the amount of drawdown as was supposed below eq. [5] in deriving the heat flow equation. Ferris' problem, however, is theoretically restricted to ditches penetrating to the underlying aquiclude. When he applied his analysis to a field ditch of shallow depth in a 50-foot-deep aquifer, he warned that ". . . the greatest departure of the field conditions from the initial assumptions is the limited penetration of the drain into the aquifer" (15, p. 289). It was this condition that caused the rate of change of Q to become so small that he called it constant—even though his example showed a drop from 22 to 12 gal./min. in about 18 hours.

The Assumption of Radial Flow

Since a tile line can be thought of as a horizontal well, there is an analogy between the radial flow into a well and the flow into a tile drain. This analogy has led to some solutions of drainage flow problems which are poor approximations, some solutions which are good approximations and some which are exact. Examples of each of these types will be discussed.

THE ANALYSES OF SPÖTTLE AND WALKER

As early as 1911, Spöttle (51, p. 104) proposed an equation for the shape of the water table over drains based on the following reasoning. A tile line placed in a homogeneous soil initially with a flat water table at a depth d above the center of the tile (fig. 7) will cause a particle of water at A to drop with a constant velocity v; it will arrive at A′ after a length of time t. The distance traveled will equal

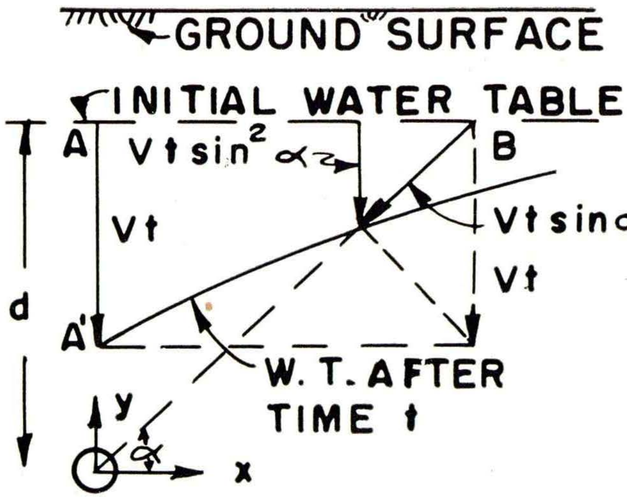

Fig. 7. Symbols used in discussion of Spöttle's analysis.

$$\overline{AA'} = vt\,.$$

A particle initially at B will move towards the drain along a radius from B towards the drain, but with a velocity equal to v sin α where α is the angle the path of flow makes with the horizontal. The drop in water table in time t would be vt $\sin^2\alpha$, so that the height of the water table above the drain at time t could be expressed as

$$y = d - vt\ \sin^2\alpha\,,$$

or, writing sin α in terms of x and y, i.e., $\sin\alpha = y/(x^2+y^2)^{1/2}$, one has

$$x^2(d-y) = y^2(y-d+vt)\,,$$

which is Spöttle's result.

The first objection that must be raised is, of course, that the velocity is not constant along the path from water surface to drain. On the contrary, it will increase rapidly when the drain is approached. The assumption that the streamlines will be radial also must be rejected, for the medium is not of infinite extent.

Walker (59) recently proposed a method of analysis similar to that of Spöttle. He considered parallel drains (fig. 8a), either open ditch or tile, in an isotropic soil, a distance S apart. The analysis was not restricted to homogeneous media but also was applied to stratified soil with each layer homogeneous and isotropic within itself. In the case of stratified soil, it was assumed that the hydraulic conductivity of the tightest layer would govern the flow. It was also assumed that the paths of flow would be along radii originating from the drains.

The most important objection to Walker's analysis is his assumption that the velocity, everywhere in the soil, equals the hydraulic conductivity. This implies a unit hydraulic gradient everywhere. Actually this condition can occur only if the draining water is falling vertically under the action of gravity alone, as for ponded water of essentially zero thickness, draining vertically into a horizontal bed of gravel kept at atmospheric pressure. (In this example, ponded water is specified; otherwise capillarity would result in a hydraulic gradient of less than unity. A zero thickness of the ponded water is further specified; otherwise, the hydraulic gradient would be greater than unity.)

As illustrated in fig. 8a, the velocity of the water moving from an arbitrary point (x_1,y_1) on the water table

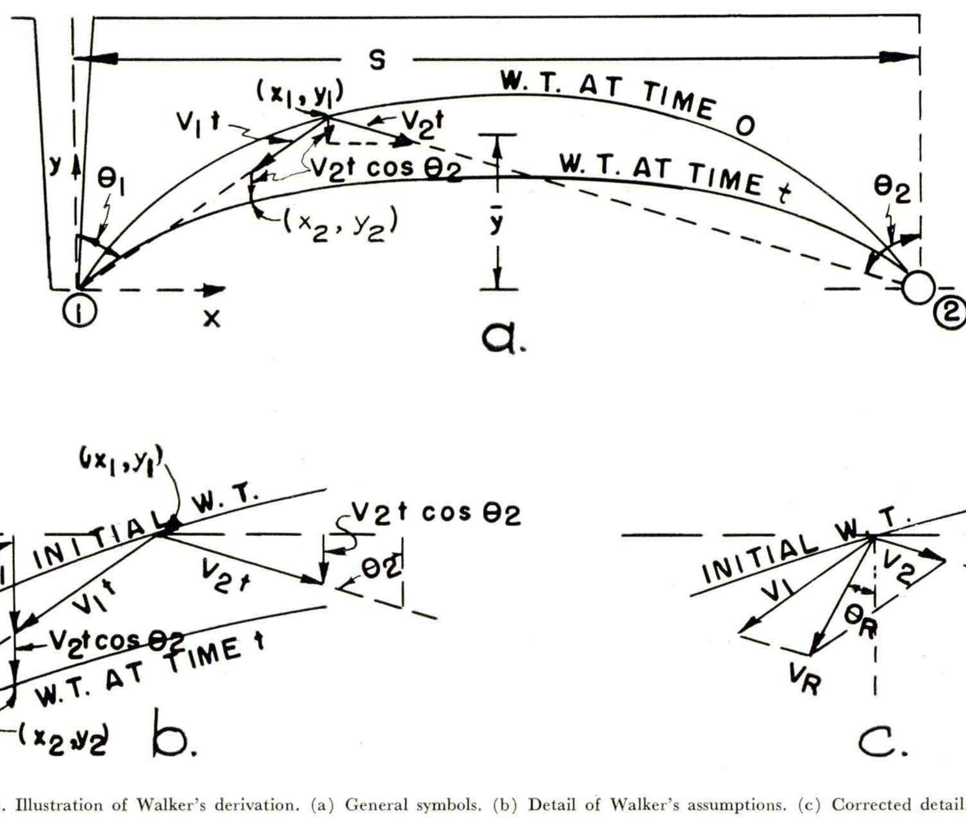

Fig. 8. Illustration of Walker's derivation. (a) General symbols. (b) Detail of Walker's assumptions. (c) Corrected detail.

towards drain 1 in Walker's analysis was taken as approximately constant over a short period such as 1 day, and the velocity was taken as directed towards drain 1. Thus, the distance traveled by a particle of water originally at (x_1,y_1) after a time t would be the product of the velocity v_1 and the time t. If Θ_1 is the angle between the vertical and the assumed line of flow, the water table recession $\triangle y_1$ due to flow towards drain 1 in time t would equal $v_1 t \cos \Theta_1$. Likewise, the recession due to flow towards drain 2 would be written

$$\triangle y_2 = v_2 t \cos \Theta_2 ,$$

and the total water table recession

$$\triangle y = \triangle y_1 + \triangle y_2 = v_1 t \cos \Theta_1 + v_2 t \cos \Theta_2 . \quad [31]$$

Figure 8b shows the vectorial treatment as presented by Walker.

Next Walker expressed the velocities in terms of the lowest hydraulic conductivity, K, of the hydraulic conductivities K_a, K_b, K_c, . . . of the various layers through which the water must pass, and the aeration porosity, f, of the layer in which the phreatic surface occurs. His expression—in which he tacitly assumes that the hydraulic gradient is unity—was

$$v_1 = v_2 = K/f .$$

Substitution of this expression in eq. [31] yields

$$\triangle y = (Kt/f)\ (\cos \Theta_1 + \cos \Theta_2) . \quad [32]$$

If the point midway between the drains is considered, $\Theta_1 = \Theta_2 = \Theta$ and eq. [32] reduces to

$$\triangle y = (2Kt/f) \cos \Theta . \quad [33]$$

Suppose that the optimum rate of drawdown for crop growth were known and substituted into eq. [33] for $\triangle y/t$, then the angle Θ would be known and the required spacing could be calculated from Walker's spacing equation,

$$S = 2\,\overline{y} \tan \Theta . \quad [34]$$

where $\overline{y}$ is the average height of the water table above the drains at the midpoint during the time period t considered.

As pointed out in connection with Spöttle's problem, the streamlines for the problem under consideration cannot be radial. Even if this approximation were accepted, however, a number of objections to Walker's analysis remain. First of all, fig. 8b indicates that the distance traveled from point (x_1,y_1) to point (x_2,y_2) in time t is made up of the sum of the two vector distances $v_1 t$ and $v_2 t \cos \Theta$ rather than the distance made up of the resultant of $v_1 t$ and $v_2 t$. Furthermore, not the distances traveled toward each drain, but the potential differences or velocity components caused by each drain should be considered as additive vectors.

Reasoning correctly from the assumption of radial flow, one might consider a potential Φ_1 due to drain 1 and a Φ_2 due to 2, with corresponding velocity components v_1 and v_2. The resulting velocity vector v_R (see fig. 8c), multiplied by a time factor t, would properly represent the distance traveled; the corresponding water table recession would be

$$\triangle y = v_R t \cos \Theta_R ,$$

or, in component notation,

$$\triangle y = v_1 t \cos \Theta_1 + v_2 t \cos \Theta_2 , \quad [34a]$$

which is the same as eq. [31]; and the result thus far is the same as Walker's.

Using Darcy's law, the velocities may be expressed as

$$v_1 = -(K/f)\ \partial\Phi_1/\partial s \ , \ v_2 = -(K/f)\ \partial\Phi_2/\partial s ,$$

where s is the direction of flow. Considering the point midway between drains and assuming each drain to be at the same potential, eq. [34a] can be rewritten as

$$\triangle y = -2(K/f)\ t \cos \Theta\ \partial\Phi/\partial s ,$$

since then $\Theta_1 = \Theta_2 = \Theta$ and $\partial\Phi_1/\partial s = \partial\Phi_2/\partial s = \partial\Phi/\partial s$.

The last expression for $\triangle y$ differs from Walker's (see eq. [33]) by the factor $-\partial\Phi/\partial s$. The discrepancy is due to Walker's assumption that

$$v_1 = v_2 = K/f$$

which implies a unit hydraulic gradient. Since no simple expression for the hydraulic gradient at the water table midway between drains can be given, one can do no more than replace eq. [33] with the inequality

$$\triangle y < (2Kt/f) \cos \Theta .$$

This expression is of no use in design because the inequality is too wide.

Finally, there remains Walker's statement that the permeability of the tightest layer governs the rate of drainage. Except in the extreme and practically useless case where there is a layer of zero permeability above the tile, this would not be the case. Kirkham, for example, (38, fig. 11) has shown that the flow depends not only on the permeability of a layer but also on its thickness. For 6-inch tile drains placed 4 feet deep and 100 feet apart in stratified soil with the upper layer 100 times as permeable as the lower, he found that the discharge is 1.6 times as large when the upper layer is 3 feet deep as when it is 1 foot deep. This figure of 1.6 applies to the flooded condition only.

Whereas it has been shown that Walker's derivation is far from an exact solution, it is possible that the effects of some of the assumptions and approximations tend to cancel each other. This might explain the relatively good agreement he found between field data and analysis. The values of K used by Walker, for example, were determined by the core sample method and thus may tend to be considerably lower than the actual hydraulic conductivity of undisturbed soil *in situ*. This would at least partly offset the error introduced by assuming a unit hydraulic gradient.

HOOGHOUDT'S RADIAL FLOW SOLUTIONS

To supplement the ellipse equation, Hooghoudt proposed the use of a radial flow pattern for soils homogeneous to great depths and the use of a combination of the radial and horizontal flow assumptions for intermediate cases (27).

For a vertical well in a homogeneous medium, the equipotentials have the form of concentric circles. If Φ designates the potential, then the hydraulic gradient at any point along a radius r towards the well is $d\Phi/dr$.

Thus, the flow q per unit of arc length of such an equipotential circle and per unit length of well is, by Darcy's law,

$$q = -K\ d\Phi/dr\ ,$$

and the total flow per unit length of well (taken positive when flowing towards the well) is

$$Q = +\ 2\pi Kr\ d\Phi/dr$$

which yields, after integration,

$$\Phi = (Q/2\pi K)\ \ln r$$

as a general expression for the potential, a constant of integration having been omitted.

If, considering flow in a plane, two radii emanating from the well center and separated by an angle π were made impervious, then the flow through both halves of the circular region would be unchanged and hence would be equal. If the flow region on one side of the two radii (which form a straight line) were removed, the total flow into the well would be just half of the original flow and the potential distribution would be given by

$$\Phi = (Q/\pi K)\ \ln r\ . \qquad [35]$$

If a tile drain were installed with its upper half in an impermeable layer and with its lower half in a soil of constant permeability and infinite depth, the potential distribution due to artesian pressure generated at great depth would be described by eq. [35]. For two such tile lines installed parallel to each other, the potential at a point could be expressed as

$$\Phi = (Q/\pi K)\quad (\ln r_1 + \ln r_2)$$

where r_1 and r_2 are the distances of this point from the two drains. Q designates here, as all through this discussion, the flux per unit length of drain. For an infinite number of equally spaced drains, eq. [35] would take the form

$$\Phi = (Q/\pi K)\quad \sum_{n=1}^{\infty}\ \ln r_n\ .$$

If the artesian pressure is not too great, the shape of the water table, when the upper impermeable medium is replaced by homogeneous soil, will approach a plane through the drain axes. The foregoing analysis then still would apply, even though only approximately. If one point, designated as A, is taken on a drain circumference and another, B, midway between adjacent drains and on the plane through their axes, then the approximate potential difference between B and A may be written as

$$\triangle\Phi = \Phi_B - \Phi_A = (Q/\pi K)\ (\sum_n \ln\ r_{Bn} - \sum_n\ \ln r_{An})\ ,$$

$$n = 0,\ 1,\ -1,\ 2,\ -2,\ .\ .\ .\ . \qquad [36]$$

Here r_{An} represents the distance from the center of the nth drain to point A and r_{Bn} the distance to point B. Except for the term r_{Ao}, where O refers to the drain on which point A is located, the drains may be considered as line sinks with negligible radius. Carrying out the indicated summing in eq. [36], one finds, upon defining S as the drain spacing, r_o as the drain radius and A by

$$A = (2.30/\pi)\ \ (-0.197\ +\ \log_{10}\ S/2r_o)\ ,$$

the result

$$\triangle\Phi = (Q/K)\,A\ . \qquad [36a]$$

Tables of A for values of r_o from 0.03 to 0.45 meters are available (27, p. 652—where P is Φ, sl is Q, l is S and s is rainfall rate).

Whereas eq. [36] has been derived here for the case of artesian pressure, Hooghoudt has shown—a very surprising result—that it also applies to the case of steady rainfall. If a series of M point sources are imagined on the plane through the drain axes between each two drains, spaced so that the distance between adjacent sources is S/M, and if each of these sources has a strength Q/M, then the potential difference equation must be written as

$$\triangle\Phi = (Q/\pi K)\ \ (\sum_n\ \ln\ r_{Bn}\ -\ \sum_n\ \ \ln r_{An})$$
$$+\ \sum_{m=1}^{M}\ (Q/M\pi K)\ \ (\sum_n\ \ln r_{Amn} -\ \sum_n\ \ \ln r_{Bmn})\ . \qquad [37]$$

The distance r_{Amn} represents the distance from point A to the mth source to the right of the nth drain.

The second term may be written as a series of 2M sums which cancel in pairs[8], leaving as final result again eq. [36]. If M approaches infinity, the effect of the sources is equivalent to that of a uniform rate of rainfall. Thus Hooghoudt has shown that, if the drop in potential from the water table to the plane through the drain axes can be neglected, the potential difference $\triangle\Phi$ in the case of constant rainfall on a homogeneous soil is given by eq. [36]. When the rainfall rate is small compared to K, so that the rise of the water table midway between tiles is small compared to the spacing, the percentage of head dissipated above the plane through the drains is small. In such a case one would expect the equation to give a good approximation for the true physical condition.

The above analysis is restricted to a soil homogeneous to infinite depth. If an impermeable layer occurs at a relatively great finite depth d below the drain axes, the analysis can be modified by considering a series of image drains at a distance d below the impermeable layer (fig. 9). The potential difference between A and B as given by eq. [36] must then be corrected for the effects of the image drains 0′, 1′, -1′, 2′, -2′, . . . , resulting in

$$\triangle\Phi = (Q/\pi K)\ \ (\sum_n \ln r_{Bn} - \sum_n \ln r_{An}$$
$$+ \sum_n\ \ln r_{Bn'} -\ \sum_n\ \ln\ r_{An'})\ . \qquad [38]$$

Since here the first two sums are the same as those in eq. [36], one may, upon writing the second two sums as an infinite product, etc., obtain the result, equivalent to Hooghoudt's eq. [67] (27, p. 574),

[8]The proof of this statement is given in Appendix A.

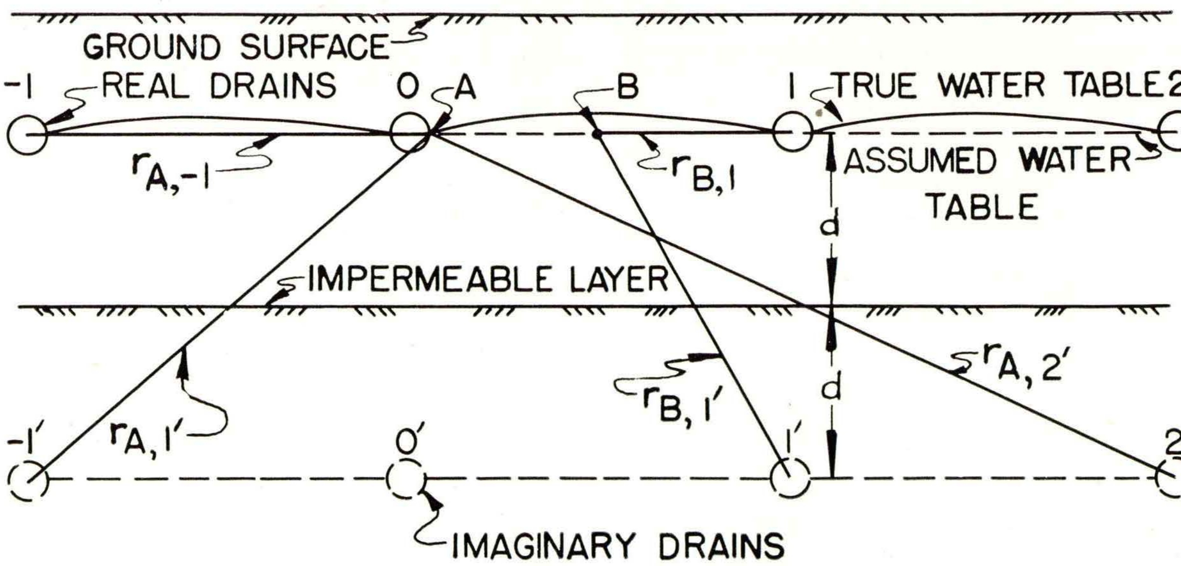

Fig. 9. Hooghoudt's arrangement of images for tile drains above an impermeable layer with examples of notation.

$$\triangle\Phi = 2.30\,(Q/\pi K)\,\{-0.197 + \log_{10} S/2r_o + \frac{1}{2}\sum_{n=1}^{\infty} \log_{10} \frac{[(2n-1)^2S^2/4 + 4d^2]^2}{(n^2S^2 + 4d^2)[(n-1)^2S^2 + 4d^2]}\}. \quad [38a]$$

Both eqs. [36] and [38] (and hence [36a] and [38a]) are only approximate in that they are based on the assumption of a flat water table. Aside from the additional head loss that occurs, eq. [35] implies that the flux across each equipotential is the same. However, with the curved water table (which is not an equipotential) this is not the case: The flux through the equipotentials farther from the drains is less than that through those nearer the drains. In the case of a considerable difference m in height of the water table between points over the drains and midway between them (the distance m is not related to the subscript m in eq. [37]), Hooghoudt (27, p. 562) has suggested that eqs. [37] and [38] may be improved by adding the term Qm/KS to the right sides of the equations. This term may be derived by assuming strictly vertical flow through the region above the drain axes. If the height of the water table above the drains at the midpoint is m_1 and at the drains m_2, then the flow Q/S per unit area, which moves downward from the water table to the level of the drains may be expressed for the midpoint between drains as

$$Q/S = K\triangle\Phi_1/m_1$$

and for a point over the drains as

$$Q/S = K\triangle\Phi_2/m_2\,.$$

The head difference due to flow in the region above a plane through the drains is (assuming strictly vertical flow in this region above the drains) consequently

$$\triangle\Phi_1 - \triangle\Phi_2 = Qm/KS\,.$$

Moreover, eq. [38] must be restricted to cases where d is equal to or greater than about S/4 [compare (45, p. 526)]. When the impermeable layer approaches the drain, the assumption of radial flow gives a poor approximation. For those cases where the impermeable layer is very near the drain, the assumption of horizontal flow may be used (see eq. [11]). In intermediate cases, the two types of flow can be combined by assuming that (fig. 10) the flow in the region $x < x_1$ is radial and in the region $x > x_1$ horizontal in nature. The plane $x = x_1$ must be chosen so that the potential difference between points a and b of fig. 10, as calculated from eq. [38], is a minimum. That will cause the (vertical) plane $x = x_1$ to be nearly an equipotential plane, as it should be for the horizontal flow analysis to apply. Hooghoudt (27, p. 576) has shown that this requires that $x_1 = 0.707d$.

The potential difference $\triangle\Phi$ between A and B in fig. 10 can be obtained by applying eq. [38] to the difference $\triangle\Phi_1'$ from A to b, and eq. [9] to the difference $\triangle\Phi_2'$ from b to B. There results (27, pp. 576-578, especially eq. [78])

$$\triangle\Phi = \triangle\Phi_1' + \triangle\Phi_2', \quad [39]$$

where (27, eq. [69])

$$\triangle\Phi_1' = \frac{2.30Q}{\pi K}\Big[\log_{10}\frac{0.707d}{r_o} + \sum_{n=1}^{\infty}\log_{10}\frac{(nS)^2-d^2/2}{(nS)^2} + \frac{1}{2}\sum_{n=0}^{\infty}\log_{10}\frac{(nS+0.707d)^2+4d^2}{(nS)^2+4d^2} + \frac{1}{2}\sum_{n=1}\log_{10}\frac{(nS-0.707d)^2+4d^2}{(nS)^2+4d^2}\Big] \quad [40]$$

and (27, eq. [77])

$$\triangle\Phi'_2 = (Q/K)\;(S - 1.414d)^2/8dS\,. \quad [41]$$

Defining the right hand sides of eqs. [40] and [41],

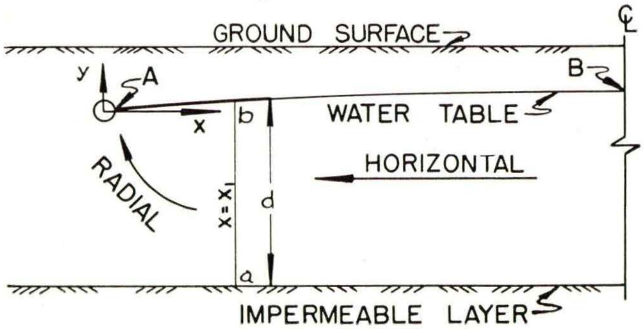

Fig. 10. Combination of radial and horizontal flow.

except for the coefficient Q/K, as B and C, respectively, eq. [39] becomes

$$\triangle\Phi = (Q/K)\ (B + C)\,. \qquad [41a]$$

Tables of B, C and of B + C for a drain radius r_o of 0.03 meters are available (27, pp. 653-655).

In addition to tables for B, C and B + C, Hooghoudt has prepared an extensive table of values of d_e, where d_e refers to the thickness of an "equivalent layer." It is defined as a permeable layer overlying a fictitious impermeable layer of such thickness that, if the spacing is computed from eq. [11] with h_o replaced by d_e, the same answer will be obtained as when the appropriate formula, whether eq. [11], [36a], [38a] or [41a], is applied.[9] The tables for d_e may also be used as an aid in computing the hydraulic conductivity K.

Use of this table of values of d_e introduces an error of less than 10 percent in spacing calculations and of less than 20 percent in hydraulic conductivity calculations, according to Hooghoudt, except in some extreme cases that are unlikely to occur in practice.

Hooghoudt's analysis of the drainage problem constituted one of the first comprehensive treatments of the subject to be found in the literature. Although none of his solutions is exact, his approximations are clever, and most of them cannot be criticized fairly. Comparison with Van Deemter's exact solution for a homogeneous soil shows that Hooghoudt's equations for that case result in very nearly the same answers. The assumption of horizontal flow has already been discussed in detail. Hooghoudt restricted its use, however, to those cases where the assumption is most reasonable. Since the height of the water table at the midpoint between drains is considered rather than the shape of the water table, good results can be expected. His results apply only insofar as steady state rainfall or its equivalent may be assumed.

EXACT SOLUTIONS WITH THE METHOD OF IMAGES

The single series of image drains used by Hooghoudt in deriving eq. [38] was insufficient to make the solution exact. The exact flow pattern for an actually flat water table over an impermeable layer could have been obtained by an infinite number of reflections so that image drains, for fig. 10 (but not shown there), would be placed along the planes y = 2nd, n = ± 1, 2, 3, ... Considering the neglect of the effect of the curvature of the water table, the accuracy obtained by the single row of images certainly was sufficient. It is this added approximation, however, which causes the failure of Hooghoudt's method when the drain comes near the impermeable layer.

Avoiding the uncertainties of the curved water table, Kirkham and Gustafsson, as noted in the Review of Literature, solved a number of specific problems involving land flooded to the surface with the method of images. The potential in their problems, as in Hooghoudt's, can be found by adding the potentials of each of the real and image drains, using the basic expression

$$\Phi = (Q/2\pi K)\ \ln r\,. \qquad [42]$$

Kirkham found in each case the potential difference between an arbitrary point and a point on the drain circumference. He worked with the complex potential $\Omega = \Phi + i\,\Psi$ with the corresponding basic equation, in terms of the complex variable $z = x + iy$,

$$\Omega = (Q/2\pi K)\ln z\,. \qquad [43]$$

That eq. [42] is the real part of this last expression can be shown by writing z as $re^{i\Theta}$ and separating the real and imaginary parts:

$$\Omega = \Phi + i\Psi = (Q/2\pi K)\ \ln re^{i\Theta} =$$
$$(Q/2\pi K)\ (\ln r + i\Theta)\,.$$

Gustafsson also used complex potentials and found general solutions for several problems, but he started with the potential distribution for a line source and sink as may be found elsewhere (e.g., 47, p. 406). Using this expression as a basis, he developed the required relationships by summation procedures similar to Kirkham's.

Whereas Kirkham's solutions are given in relatively simple form, Gustafsson's are expressed in terms of elliptic functions, which restricts their practical value.

The Hodograph Analysis

In the following discussion, the potential Φ, defined as

$$\Phi = p/\rho g + y\,, \qquad [44]$$

where p designates the pressure of the fluid, ρ its density and y the height above an arbitrary reference level, will be replaced by

$$\phi = K\Phi\,. \qquad [44a]$$

Thus, physically, Φ is the height, referred to the level y = 0, to which water would stand in a piezometer, the lower end of which is at the point (x, y) in the flow medium. Furthermore, the discussion will be restricted to steady state flow in a two-dimensional region designated as the z-plane where $z = x + iy$, y being vertically upward and x horizontally to the right. It is assumed that the reader has a knowledge of functions of a complex variable and, in particular, of the so-called analytic functions.

Taking the fluid to be incompressible, the continuity equation may be written

$$\partial v_x/\partial x + \partial v_y/\partial y = 0$$

[9]The use of this table is illustrated in Appendix B. Visser (57) has presented a nomographic solution to replace the table, based on a series of relaxation solutions.

where v_x and v_y are the velocity components in the x- and y-directions. Then, since by Darcy's law

$$v_x = -\partial\phi/\partial x, \; v_y = -\partial\phi/\partial y, \qquad [45]$$

one must have

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0.$$

Defining a stream function, ψ, as the conjugate of ϕ by the Cauchy-Riemann relations

$$\partial\phi/\partial x = \partial\psi/\partial y, \; \partial\phi/\partial y = -\partial\psi/\partial x,$$

it is apparent that ψ also will satisfy Laplace's equation and that the complex function ω defined by

$$\omega = \phi + i\psi$$

is analytic. This function ω will be designated as the complex potential.

It can easily be shown that the family of curves

$$\phi = \text{constant}$$

forms the orthogonal trajectories of the family of curves

$$\psi = \text{constant}.$$

The first of these will be designated as equipotentials, the second as streamlines.

THE THEORY

The solution of steady state potential flow problems is accomplished by finding a solution of differential equations which satisfies certain boundary conditions. From the theory of analytic functions, it is known that any one set of boundary conditions will yield a unique solution.

The differential equations that must be solved are

$$\nabla^2\phi = 0 \text{ and } \nabla^2\psi = 0.$$

The boundary conditions vary with the problem, but, in general, four types may be considered:

(a) Along a streamline and therefore also along an impermeable boundary,

$$\psi = \text{constant and } \partial\phi/\partial n = 0, \qquad [46]$$

where n is the direction perpendicular to that of the streamline.

(b) Along an equipotential, such as the wetted perimeter of a ditch,

$$\phi = Kh \text{ and } \partial\psi/\partial n = 0,$$

where h represents the height of the water above the reference plane in a piezometer terminating at the point in question and where n is now the direction orthogonal to that of the equipotential.

(c) Along a surface of seepage, such as along that portion of a ditch wall between the water level in the ditch and the water table, one has $p = 0$ and, consequently, by eqs. [44] and [44a],

$$\phi = Ky. \qquad [47]$$

Notice that, in view of eq. [45], the vertical component of the velocity of seepage along a vertical surface of seepage is $-K$.

(d) Along the water table, defined as the locus of points within the saturated region at atmospheric pressure, one also has

$$\phi = Ky. \qquad [48]$$

But here, unlike the situation for eq. [47], $\dfrac{\partial\phi}{\partial y}$ does not equal K because p of eq. [44] does not stay constant as y varies at constant x. Aside from the condition expressed by eq. [48], there is, at the water table, one of two conditions imposed on the stream function. If there is no infiltration to the water table, the condition is

$$\psi = \text{constant};$$

that is, the water table is a streamline. If water is added or removed along the water table at a constant rate N (cubic inches per square inch per hour, say), the condition is

$$\psi = Nx + \text{constant}. \qquad [49]$$

In eq. [49] the water table is not now a streamline but a surface along which streamlines begin or terminate.

The problem now may be restated as follows: The complex potential ω must be found as a function of z. If that has been accomplished, the equipotentials and streamlines in the z-plane are known. Put differently: If a conformal transformation or a series of such transformations can be found that will change the original flow region, with arbitrarily shaped boundaries, into a simple region for which the flow pattern is known or can be determined, the problem is, in principle, solved.

It has been found convenient not to deal directly with the flow region in the z-plane but to consider the corresponding region in a so-called w-plane, where w is the complex velocity. It may be defined, analogously to the definition of the velocity components, by the relationship

$$w = u + iv = -d\omega/dz. \qquad [50]$$

Now

$$d\omega/dz = (d\phi + id\psi)/(dx + idy) =$$

$$\frac{(\partial\phi/\partial x)\,dx + (\partial\phi/\partial y)\,dy + i[(\partial\psi/\partial x)\,dx + (\partial\psi/\partial y)\,dy]}{dx + idy}$$

so that, using the Cauchy-Riemann relations, one obtains

$$d\omega/dz = [(\partial\phi/\partial x)\;(dx + idy) -$$

$$i(\partial\phi/\partial y)\;(dx + idy)]/(dx + idy).$$

That is, cancelling out the terms $dx + idy$ and using eq. [45], one has

$$d\omega/dz = -v_x + iv_y;$$

whence, comparing with eq. [50]

$$u = +v_x, \quad v = -v_y. \qquad [51]$$

The use of the w-plane has given rise to the name "hodograph" analysis. A hodograph is the plot of the velocities at each point of a flow system on axes which represent two mutually perpendicular velocity components. Whereas generally the positive axes are taken as v_x

and v_y, it will be more convenient here to consider the hodograph as a plot of u versus v. Thus, the w-plane is the hodograph plane.

If w can be expressed as a function of ω, then by eq. [50] z is known as a function of ω, or conversely, ω as a function of z. Thus, the problem would be solved.

The boundary conditions to be imposed on w can be derived from those set up for z. Considering the same four types of boundaries, one finds the following relationships:

(a) Along a streamline, if s is the direction of flow and α the angle between streamline and x-axis,

$$v_x = -\partial\phi/\partial x = -(\partial\phi/\partial s)\cos\alpha$$

and

$$v_y = -\partial\phi/\partial y = -(\partial\phi/\partial s)\sin\alpha$$

which may be written, using eq. [51],

$$v/u = -\tan\alpha\,. \qquad [52]$$

(b) Along an equipotential, there results in similar manner, with α now the angle between the equipotential and the x-axis,

$$v/u = \cot\alpha\,.$$

(c) Along a surface of seepage, by differentiation of eq. [47], α now being the angle between the surface of seepage and the x-axis,

$$(\partial\phi/\partial x)\,(dx/ds) + (\partial\phi/\partial y)\,(dy/ds) = K(dy/ds)$$

or

$$-u\cos\alpha + v\sin\alpha = K\sin\alpha.$$

(d) Along the water table one obtains similarly from eq. [48], α now being the angle between the water table at the point in question and the x-axis,

$$-u\cos\alpha + v\sin\alpha = K\sin\alpha\,;$$

and from eq. [49], the Cauchy-Reimann relations and eq. [51]

$$-v\cos\alpha - u\sin\alpha = N\cos\alpha\,.$$

Rewriting these last two equations, respectively, as

$$(K-v)\sin\alpha = -u\cos\alpha \qquad [53]$$

and

$$(N+v)\cos\alpha = -u\sin\alpha\,, \qquad [54]$$

α may be eliminated. There results

$$-\tan\alpha = u/(K-v) = (N+v)/u\,, \qquad [55]$$

which may be written as

$$u^2 + [v-(K-N)/2]^2 = (K+N)^2/4\,. \qquad [56]$$

The above equations show that, if the bounding streamlines, equipotentials and surfaces of seepage are straight lines (α = constant), then the corresponding velocity plots (of u versus v) also will be straight lines. The shape of the water table is unknown in the z-plane, but in the w-plane it is, by eq. [56], a circular segment with its center at $[0, (K-N)/2]$ and having radius $(K+N)/2$. Thus the boundaries are known in the hodograph plane.

The foregoing discussion, in a somewhat different form, may be found in a number of publications. Publications of Hamel (24), Muskat (45), Breitenöder (3), Gustafsson (23, pp. 101-113) and Van Deemter (52, 53) may be mentioned.

THE ANALYSIS OF VAN DEEMTER

Whereas Gustafsson gave the first complete solution of a tile drainage problem by means of the hodograph method, his problem is a special case of a more general one solved by Engelund (13a) and of an even more general problem solved by Van Deemter (52). Only Van Deemter's work will be discussed here. His treatment of tile drainage problems will be analyzed in detail, and those steps not adequately explained in his publications will be filled in. It also will be shown that Van Deemter's solution does not always lend itself to direct application to field cases. This limitation results from the implicit demand of his solution that there be a relation between the drain size and the pressure in the drains. Only for the case where the drains run full and the water table just reaches the drain top will it be possible to present a direct procedure for solving the problem completely. Even so, Van Deemter's solution is of great value and represents the nearest solution yet given analytically to the actual field problem.

Statement of the problem. Van Deemter considered a homogeneous, isotropic, semi-infinite soil drained by parallel, equally spaced tile lines. He restricted the problem to steady state conditions by assuming a steady rate of rainfall (Dutch "neerslag") or evaporation, N, and a steady rate of deep seepage or artesian flow, L. With the origin of coordinates at the center of a drain and the positive y-direction upwards, positive values of L and N would designate artesian pressure and evaporation, whereas negative values would indicate deep seepage and rainfall. If the difference L–N is positive, the drains will remove water. If it is negative, infiltration from the tile line into the soil, that is, subsurface irrigation, will result.

Only the condition $K + N > 0$ will be considered; that is, in the case of rainfall, the magnitude of the rainfall rate must be less than K. The condition $K + N < 0$ is not amenable to solution as it implies a steadily rising water table, which thus is not an equilibrium condition, as required here.

The drains will be considered initially as line sinks or, in two dimensions, as points. First, those cases where $L - N > 0$ will be considered. For definiteness, it will be assumed that $L \geqslant 0$ and $N < 0$. With reference to fig. 11, and remembering eqs. [46], [48] and [49], the boundary conditions may be stated as

$$\begin{aligned} &\text{(a) Along PQ:} && \psi = 0\,,\\ &\text{(b) Along QR:} && \phi = Ky\,,\\ & && \psi = Nx\,, \qquad [57]\\ &\text{(c) Along RS:} && \psi = Na\,,\\ &\text{(d) Along SP:} && \psi = (N-L)a\,, \end{aligned}$$

where condition (c) also applies along TP.

As a check on the boundary conditions, the following is observed: Through the region PQRT will flow $N/(N-L)$ of the total flow, through PTS, $-L/(N-L)$ of the total. Thus, using obvious subscript notation,

$$(\psi_{RS} - \psi_{PQ})/(\psi_{SP} - \psi_{RS}) = N/(-L)\,,$$

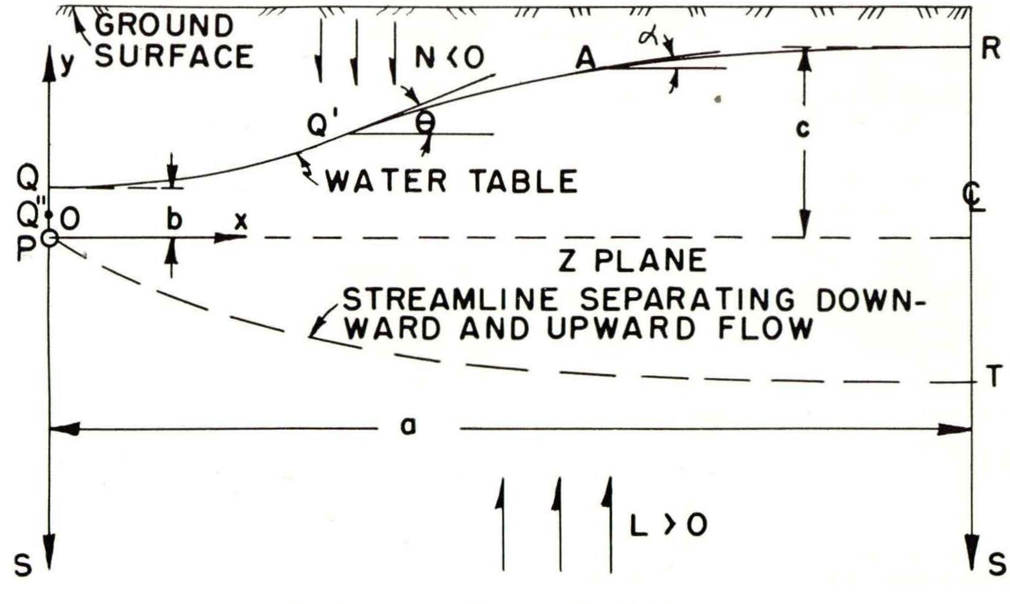

Fig. 11. Drainage problem as posed by Van Deemter.

which is a relation in keeping with conditions (a), (c) and (d).

The boundary conditions at $y = -\infty$ and at the point P have not yet been specified. Since the velocity $-\partial\phi/\partial y$ must equal L at $y = -\infty$, ϕ approaches the boundary value $-Ly$ at great depths. That is

$$\phi = -Ly, \quad y = -\infty .$$

At P the velocity will be infinite, and hence $\phi = -\infty$ at P.

Development of the hodograph. To develop the w-plane, that is, the hodograph (fig. 12), one must consider the velocity distribution along the boundaries. Along the streamlines, eq. [52] gives

$$v/u = -\tan\alpha .$$

Since PQ, RS and SP are all parallel to the y-axis, u must be zero; hence the points P, Q, R, T and S must all lie on the v-axis of the w-plane. At T, which is a stagnation point, the velocity must be zero; along TS and SP one has $v_y > 0$ and along PQ and RT, $v_y < 0$. Hence, in view of eq. [51], S and P must fall on the negative v-axis, and Q and P on the positive v-axis. Since the velocity at P is infinite, this is no contradiction. As has been mentioned, $v_y = L$ at S, or $v = -L$. At Q and R, $\alpha = 0$, so that there, from eqs. [53] and [54],

$$u = 0, \quad v = -N ;$$

this is in agreement with physical conditions. Along the water table, the velocity components must satisfy eq. [56]. Thus, the curve segment QR must lie along the circle with center at $w = i(K - N)/2$ and radius $(K + N)/2$. From eq. [55] it is seen that the angle α at an arbitrary point A along QR in the z-plane corresponds to the angle between the line from $w = iK$ to A and the vertical in the w-plane. Thus, the point of inflection Q′, where $\alpha = \Theta$, will be the point farthest from Q and R along the circle Q, Q′, Q″. The negative sign in eq. [55] indicates that the circular arc must be in the region $u < 0$.

Notice in figs. 11 and 12 that if one proceeds along the path PSSTRQ′QP in the z-plane and along the path PSTRQ′QQ″P in the hodograph plane, the region to the left in the z-plane corresponds to the region to the left in the hodograph plane. A similar statement holds for subsequent transformations. The point Q″ which lies between P and Q in fig. 11 has the special significance that there the seepage velocity has the magnitude of the hydraulic conductivity. Above Q″ the velocity is less, below greater.

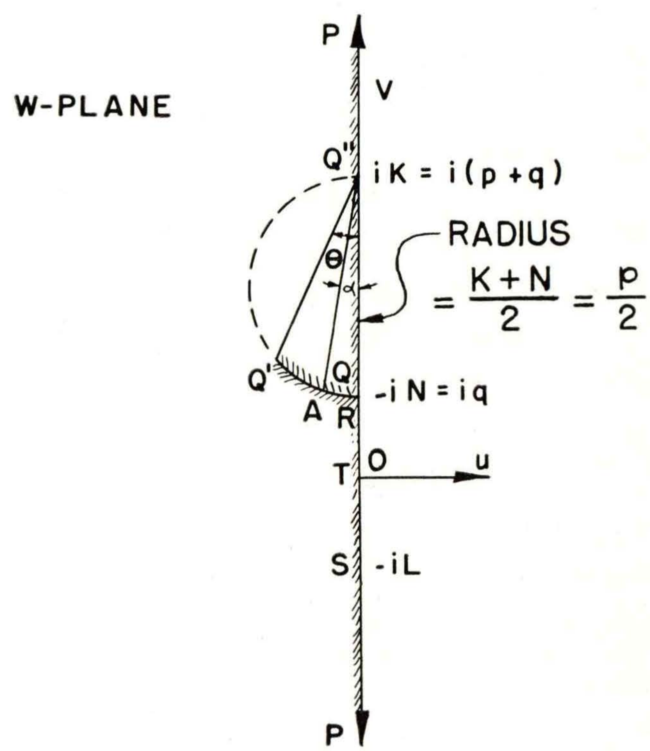

Fig. 12. Hodograph corresponding to flow region of fig. 11.

Development of $\omega = \omega(w)$. Along the boundary of the hodograph, the stream function ψ must satisfy the conditions listed under eqs. [57]. Along QR, this condition is given in terms of x rather than in terms of u and v. To alleviate this difficulty, a new function Ω is introduced, defined as

$$\Omega = \omega - iNz = \phi + Ny + i(\psi - Nx) . \qquad [58]$$

This function has the boundary condition, obtained from eqs. [57]:

(a) Along PQ: $\Omega = \phi + Ny$,

(b) Along QR: $\Omega = (K + N)y$,

(c) Along RS: $\Omega = \phi + Ny$,

(d) Along SP: $\Omega = \phi + Ny + i(N-L)a$.

The flow region in the Ω-plane (fig. 13) can be plotted from the above relations. Since it is only along SP that Ω has an imaginary part, the points P, Q, R and S will fall along the horizontal axis. These points will now be located more precisely. Referring to fig. 11, y equals b at Q and c at R. Hence, in fig. 13 at Q, from eq. [57b] and eq. [58], $\Omega_Q = (K + N)b$; and similarly at R, $\Omega_R = (K + N)c$. At S, as has been seen below eqs. [57], $\phi = -Ly$. Hence from condition (c), $\Omega_S = (N - L)y$; or since at S, $y = -\infty$ and $N - L < 0$, there results at S: $\Omega_S = \infty$. If condition (d) is used it is also clear that $\Omega_S = \infty + i(N-L)a$. At P, $y = 0$ and $\phi = -\infty$. Therefore $\Omega_P = -\infty$ or $\Omega_P = -\infty + i(N - L)a$.

The problem, in view of eqs. [58] and [50], has now been reduced, within an integration, to that of transforming the strip PSSRQOP of the Ω-plane to the flow region of the w-plane. This may be done by transforming this strip to the upper half of an η-plane (fig. 14), transforming the flow region of the w-plane, by means of an intermediate ξ-plane (fig. 15), to the upper half of a σ-plane (fig. 16) and matching the σ- and η-planes.

Considering first the strip on the Ω-plane and recalling that the transformation $w = \exp z$ (w here is not to be confused with w of fig. 12) maps the strip $0 \leqslant y < \pi$ onto the upper half of a w-plane [see (41, p. 85) or (47, p. 388)], it appears that the transformation desired, since $L - N > 0$, is

$$\eta = \exp[-\pi\Omega/(L - N)a] . \qquad [59]$$

From fig. 13, $\Omega_P = -\infty, -\infty + i(N - L)a$; and $\Omega_S = +\infty, \infty + i(N - L)a$. Hence $\eta_P = \pm\infty$, and $\eta_S = 0$. Defining γ as

$$\gamma = (K + N)/(L - N) > 0 , \qquad [60]$$

it also is readily seen that $\eta_R = \exp(-\pi c\gamma/a)$ and $\eta_Q = \exp(-\pi b\gamma/a)$.

Next is the mapping of the w-plane onto the σ-plane by means of the intermediate ξ-plane. Let

$$\xi = \delta + i\epsilon = (K + N)/(-N + iw) . \qquad [61]$$

For simplicity, let $(K + N) = p$ and $N = -q$, with p and q both positive. Then

$$\xi = p/(q + iw) .$$

Separating the real and imaginary parts, one finds

$$\delta = -p(v - q)/[u^2 + (v - q)^2] , \qquad [62a]$$

$$\epsilon = -pu/[u^2 + (v - q)^2] , \qquad [62b]$$

with the inverse relationships

$$u = -p\epsilon/(\delta^2 + \epsilon^2) , \qquad [63a]$$

$$v = q - p\delta/(\delta^2 + \epsilon^2) . \qquad [63b]$$

The line $u = 0$ in the w-plane corresponds to the δ-axis in the ξ-plane, because from eqs. [62]

$$\xi = -p/(v - q) \qquad [64]$$

when $u = 0$.

The circle of eq. [56], that is, the circle

$$u^2 + (v - q - p/2)^2 = p^2/4$$

in the w-plane is mapped onto the ξ-plane as the line $\delta = -1$, as may be seen by substituting the values of u and v given by eqs. [63a] and [63b] into the equation of the circle.

The points w_R and w_Q lie on the v-axis with $v = q$. Hence, from eq. [64], $\xi_R = \infty$ and $\xi_Q = -\infty$, where both points are on the δ-axis, and the signs have been obtained by considering the direction of approach of v to q. These points also lie on the circle in the w-plane, or the line $\delta = -1$ in the ξ-plane. Hence, from eq. [63b], with $v = q$ and $\delta = -1$,

$$q - q = -p(-1)/(\epsilon^2 + 1) ,$$

so that

$$\epsilon = \infty ,$$

and the points ξ_R and ξ_Q are also at $\epsilon = \infty$ and $\delta = -1$. Furthermore, since from fig. 12 $w_T = 0$, it follows, with the use of eq. [64], that $\xi_T = p/q$. Similarly, $w_P = \pm\infty$ and $\xi_P = 0$. At S, $w_S = -iL$ which, again using eq. [64], corresponds to

$$\xi_S = -p/(-L - q) = (K + N)/(L - N) = \gamma .$$

Inspection of fig. 12 shows that

$$w_{Q'} = -(p/2)\sin 2\Theta - i(p/2)\cos 2\Theta + i(p/2 + q) .$$

Substitution of the u-part and v-part of $w_{Q'}$ into eq. [62a] and then into eq. [62b] results in

$$\xi_{Q'} = -1 + i\cot\Theta .$$

Similarly, substitution of $w_{Q''} = i(p + q)$ results in $\xi_{Q''} = -1$.

The configuration in the ξ-plane is a rectilinear polygon, so that a Schwarz-Christoffel transformation may be used to map it onto the upper half of the σ-plane. Such transformations are of the form (47, p. 398)

$$\frac{d\xi}{d\sigma} = A(\sigma - \sigma_1)^{-\alpha_1/\pi}(\sigma - \sigma_2)^{-\alpha_2/\pi}\ldots(\sigma - \sigma_n)^{-\alpha_n/\pi} ,$$

where A is a complex constant, the α's are the exterior angles of the polygon and $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the fixed points on the real σ-axis corresponding to the vertices of the polygon. Three of these points $\sigma_1, \sigma_2, \ldots, \sigma_n$ may be chosen arbitrarily (2, p. 74) as long as their order of magnitude is the same as the order in which the corresponding vertices occur when the sides of the polygon are traced. It is convenient here to choose $\sigma_R = \pm\infty$, $\sigma_{Q'} = -1$ and $\sigma_Q = 0$. Since the polygon has only three sides, all of the σ_i values may be assigned independently.

To visualize the transformation in fig. 15, consider the vertical line QQ′ or RQ′ as split down the middle

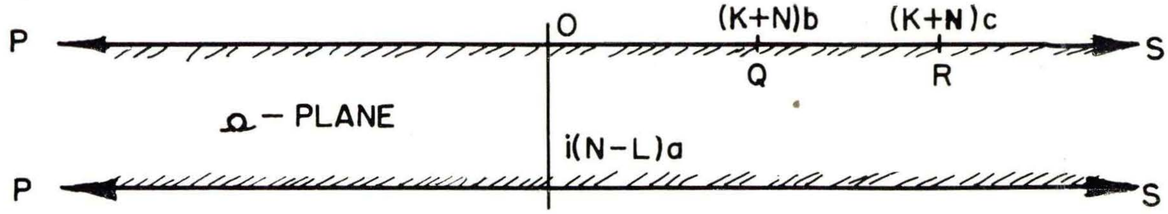

Fig. 13. Plot of flow region in the Ω -plane.

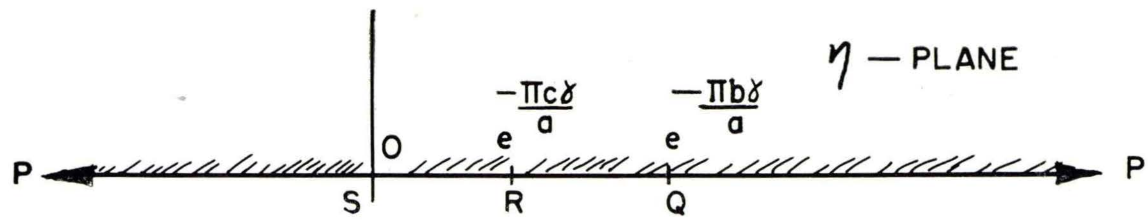

Fig. 14. Plot of flow region in the η -plane.

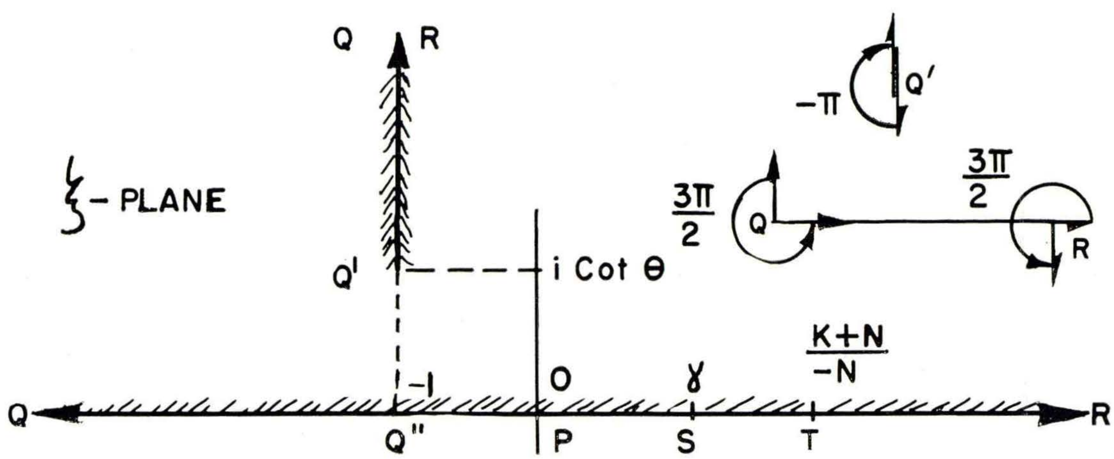

Fig. 15. Plot of flow region in the ξ -plane with insert to show angles of polygon.

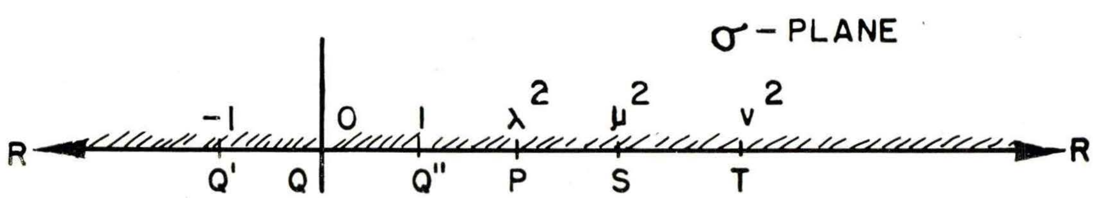

Fig. 16. Plot of hodograph in the σ -plane.

from QR to Q′. In the transformation to the σ-plane the point Q′ may be thought of as being pushed up while Q and R spread, respectively, to the left and right until QQ′R is a horizontal line. This horizontal line then may be considered as translated and rotated counterclockwise until the two points Q in fig. 15 merge into the single point Q in fig. 16, the points Q′ and R falling simultaneously into the positions Q′ and R shown at the left hand portion of the real axis in the σ-plane.

To obtain the exterior angles, the above depiction need not be considered. The angles are obtained (see insert of fig. 15) by traversing the boundaries counterclockwise. The angle turned through at R is $\alpha_1 = 3\pi/2$; at Q′, $\alpha_2 = -\pi$; at Q, $\alpha_3 = 3\pi/2$.

After substituting the angles and the corresponding σ_R, $\sigma_{Q'}$ and σ_Q into the differential equation, the choice of $\sigma_R = \pm\infty$ allows the cancellation of the corresponding factor (here having an exponent −3/2) from the differential equation (17, p. 542), so that there results

$$d\xi/d\sigma = A\sigma^{-3/2}(\sigma + 1)\ .$$

Integrating this equation, one obtains

$$\xi = A(2\sigma^{1/2} - 2\sigma^{-1/2}) + B = 2A(\sigma - 1)/\sigma^{1/2} + B, \quad [65]$$

B being a constant of integration.

In eq. [65], A and B are considered as complex. However, their imaginary parts are zero as will be shown: By the choice of the constants σ_Q and σ_R, the real axis of the ξ-plane has been mapped onto the positive real axis of the σ-plane. Thus, for $\sigma = 1$, ξ must be real and, by eq. [65], ξ must equal B. Hence, B must be real. Also, since $(\sigma - 1)/\sigma^{1/2}$ is real for σ real and positive, and since ξ and B are also real when σ is positive, A must be a real constant.

To evaluate A and B substitute $\sigma_{Q'} = -1$ and $\xi_{Q'} = -1 + i \cot \Theta$ into eq. [65] to find

$$-1 + i \cot \Theta = 2A(-2/i) + B = B + 4Ai.$$

Hence $B = -1$ and $A = (1/4) \cot \Theta$, and eq. [65] becomes

$$\xi = (\sigma - 1)(\cot \Theta)/2\sigma^{1/2} - 1. \quad [66]$$

Since $\xi_{Q''} = -1$, one obtains from eq. [66] $\sigma_{Q''} = 1$.

Let $\sigma_P = \lambda^2$, $\sigma_S = \mu^2$ and $\sigma_T = v^2$; then it follows further from eq. [66] that

$$\tan \Theta = (\lambda^2 - 1)/2\lambda, \quad [67]$$

$$(1 + \gamma) \tan \Theta = (\mu^2 - 1)/2\mu, \quad [68]$$

$$(-K/N) \tan \Theta = (v^2 - 1)/2v. \quad [69]$$

Since $\gamma > 0$ and $(-K/N) > 1 + \gamma$, as is seen with the aid of fig. 15, one has, from eqs. [67], [68] and [69], the result $\lambda^2 < \mu^2 < v^2$. One has further because $\sigma_{Q''} = 1$, the relation $1 < \lambda^2 < \mu^2 < v^2$ (as is shown in fig. 16); also $1 < \lambda < \mu < v$. Equations [67], [68] and [69] may be considered definitions of λ, μ and v, with Θ, however, still unknown. In view of eq. [60], one observes that eq. [68] is the same as

$$\frac{L + K}{L - N} \tan \Theta = (\mu^2 - 1)/2\mu. \quad [68a]$$

There remains the matching of the upper halves of the η- and the σ-planes. This must be done with a broken linear (bilinear) transformation. The general form of this transformation (2, p. 175; 17, p. 512; 20a, p. 84) may be given as

$$\eta = (a_1\sigma + a_2)/(\sigma + a_3),$$

where a_1, a_2 and a_3 are generally complex constants. This transformation causes a shift of the σ-origin to $-a_3$, reflexion about a line through the new origin with amplitude 1/2 arg $(a_2 - a_1a_3)$ and inversion about this origin with the inversion constant $(a_2 - a_1a_3)$. The result is the σ-plane mapped onto the η-plane, but with the η-origin shifted to a_1.

In the present problem, there are four points which could be matched, namely P, S, R and Q. Three of these may be matched at will. Choosing to match P, S and R, the transformation takes the form

$$\eta = [(\sigma - \mu^2)/(\sigma - \lambda^2)] \exp(-\pi c\gamma/a). \quad [70]$$

This corresponds to a choice of

$$a_1 = \exp(-\pi c\gamma/a),$$
$$a_2 = -\mu^2 \exp(-\pi c\gamma/a),$$
$$a_3 = -\lambda^2.$$

Subtracting $\exp(-\pi c\gamma/a)$ from each side of eq. [70] and simplifying the expression, there results

$$\eta - \exp(-\pi c\gamma/a) = -[(\mu^2 - \lambda^2)/(\sigma - \lambda^2)] \exp(-\pi c\gamma/a).$$

In this form it is apparent that the above choice of the constants may be interpreted as a shift of the η-origin of $\exp(-\pi c\gamma/a)$, a shift of the σ-origin of λ^2 and inversion of the distances from the new origins $[\sigma = \lambda^2 = \sigma_P$ and $\eta = \exp(-\pi c\gamma/a) = \eta_R]$ by the relation

$$\overline{(P\sigma)}\ \overline{(R\eta)} = -(\mu^2 - \lambda^2) \exp(-\pi c\gamma/a),$$

where $\overline{(P\sigma)}$ and $\overline{(R\eta)}$ denote the magnitudes of the distances P to σ and R to η. The negative sign in this last expression causes the upper half of the η-plane to be mapped onto the upper half of the σ-plane.

Substituting $\sigma_Q = 0$ and $\eta_Q = \exp(-\pi b\gamma/a)$ into eq. [70] yields the additional relationship

$$\exp[\pi(c - b)\gamma/2a] = \mu/\lambda. \quad [71]$$

For later use the constant β is introduced by the relation

$$1 + \beta = \mu/\lambda. \quad [71a]$$

Hence,

$$\exp[\pi(c - b)\gamma/2a] = 1 + \beta. \quad [71b]$$

Since $\mu^2 > \lambda^2$, and since also $\mu > \lambda$ (because in eqs. [67] and [68], $\Theta > 0$ by fig. 11 and $\gamma > 0$ by eq. [60]), one has $\beta > 0$, and incidentally $c > b$, as was to be expected.

In summary, a functional relationship has been developed between ω and w by having followed two paths which schematically may be depicted as

$$\begin{array}{ccccc} \omega & \longrightarrow & \Omega & \longrightarrow & \eta \\ & & & & \downarrow\uparrow \\ w & \longrightarrow & \xi & \longrightarrow & \sigma . \end{array}$$

Since $w = d\omega/dz$, the differential equation for ω as a function of z is now known. The integration of this equation is all that remains.

Integration of $\omega = \omega(z)$. Integration of the differential equation will be simpler if the parameter t is introduced, where t is defined by

$$\lambda t = \sigma^{1/2}. \quad [72]$$

This relationship maps the upper half of the σ-plane onto the first quadrant of the t-plane (47, p. 399). It yields (fig. 17): $t_R = (\infty, i\infty)$; $t_{Q'} = i/\lambda$; $t_Q = 0$;

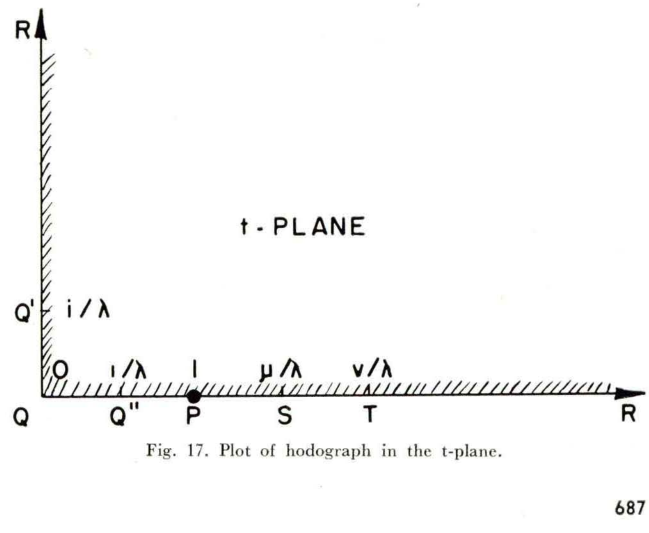

Fig. 17. Plot of hodograph in the t-plane.

$t_{Q''} = 1/\lambda$; $t_P = 1$; $t_S = \mu/\lambda$; $t_T = v/\lambda$. It is interesting to note that the phreatic surface is mapped onto the imaginary t-axis and the rest of the boundary of the original flow region onto the real t-axis.

It follows from eqs. [59] and [70] that

$$\Omega = \gamma c(L-N) - [a(L-N)/\pi] \ln [(\sigma-\mu^2)/(\sigma-\lambda^2)],$$

which, with eqs. [60] and [72], may be written as

$$\Omega = (K+N)c - [a(L-N)/\pi] \ln [(t^2-\mu^2/\lambda^2)/(t^2-1)]. \quad [73]$$

Differentiating,

$$d\Omega/dt = -2a(L-N)(\mu^2/\lambda^2-1)\, t/[\pi(t^2-\mu^2/\lambda^2)(t^2-1)]. \quad [74]$$

To express Ω in terms of z, a result obtained from eqs. [50] and [58] is used:

$$d\Omega = d\omega - iNdz = -(w+iN)dz.$$

From eq. [61] it is seen that

$$-(w+iN) = i(K+N)/\xi$$

so that

$$d\Omega = [i(K+N)/\xi]dz.$$

Using eqs. [66] and [72], this relationship may be changed into

$$i(K+N)dz = \{[(t^2\lambda^2-1)/2t\lambda] \cot\Theta - 1\}d\Omega,$$

which, by substitution into eq. [74], gives the differential equation for z(t) as

$$i(K+N)dz = -\frac{a(L-N)(\mu^2/\lambda^2-1)(\lambda^2t^2-1)\cot\Theta}{\pi\lambda(t^2-\mu^2/\lambda^2)(t^2-1)}dt - d\Omega. \quad [75]$$

Breaking up the coefficient of dt into partial fractions and integrating, one finds

$$i(K+N)z = -\frac{a(L-N)\cot\Theta}{\pi\lambda} \int [\lambda^2 (\frac{\mu^2/\lambda^2}{t^2-\mu^2/\lambda^2} - \frac{1}{t^2-1}) - (\frac{1}{t^2-\mu^2/\lambda^2} - \frac{1}{t^2-1})]dt - \int d\Omega$$

$$= -\frac{a(L-N)\cot\Theta}{\pi\lambda} (\frac{\mu^2-1}{2\mu/\lambda} \ln\frac{t-\mu/\lambda}{t+\mu/\lambda} - \frac{\lambda^2-1}{2} \ln\frac{t-1}{t+1}) - \Omega + C,$$

where C is a constant of integration. If Θ is eliminated by means of eq. [68a] for the first of the terms within parentheses and by means of eq. [67] for the second, and if eq. [73] is used to eliminate Ω, there results

$$i(K+N)z = -(K+N)c + \frac{a}{\pi}(L-N)(\ln\frac{t-1}{t+1} + \ln\frac{t^2-\mu^2/\lambda^2}{t^2-1}) + \frac{a}{\pi}(K+L)\ln\frac{t+\mu/\lambda}{t-\mu/\lambda} + C. \quad [76]$$

To evalute C, one substitutes $t_R = \infty$ and $z_R = a + ic$ into the above relationship, which then yields $C = ia(K+N)$.

If both sides of eq. [76], after using C, are divided by (K + N) and multiplied by -i, there results, after rearrangement and with use of eq. [71a] and eq. [60],

$$z = a + ic + \frac{ia}{\pi\gamma}[2\ln(t+1) - \ln(t+1+\beta) - \ln(t-1-\beta)] + \frac{ia}{\pi}(1+\frac{1}{\gamma})[\ln(t-1-\beta) - \ln(t+1+\beta)].$$

Simplification yields

$$z = a + ic + i\frac{a}{\pi}(\ln\frac{t-1-\beta}{t+1+\beta} + \frac{2}{\gamma}\ln\frac{t+1}{t+1+\beta}). \quad [77]$$

From eq. [58], $\omega = \Omega + iNz$. Hence ω (t) is found by substituting Ω (t) of eq. [73] and z(t) of eq. [77] into eq. [58]. The result is, after use of eq. [71] and eq. [60],

$$\omega = Kc + iNa + \frac{a}{\pi}(L\ln\frac{t-1}{t-1-\beta} + N\ln\frac{t+1+\beta}{t-1} + \frac{K-N}{\gamma}\ln\frac{t+1}{t+1+\beta}). \quad [78]$$

Thus z and ω are both given as functions of t, so that a relationship between $\omega = \phi + i\psi$ and $z = x + iy$ has been established. Equations [77] and [78] constitute an incomplete solution of the problem since the quantity β has not yet been specified. Before specifying β, the equations for the water table and in particular for the heights b and c will be obtained.

Equations for the water table. Since the imaginary axis of the t-plane represents the water table, the equation of the water table may be obtained from eq. [77] by introducing a real quantity s and equating t to is, where the quantity is represents the values of t along the axis of imaginaries. One finds:

$$z = a + ic + i\frac{a}{\pi}(\ln\frac{is-1-\beta}{is+1+\beta} + \frac{2}{\gamma}\ln\frac{is+1}{is+1+\beta}). \quad [79]$$

To simplify eq. [79] one has

$$\ln\frac{is-1-\beta}{is+1+\beta} = \ln(-1) + \ln\frac{1-is/(1+\beta)}{1+is/(1+\beta)},$$

or, using the relation $\ln(-1) = \ln\exp\pi i = \pi i$ and formula 645 of Peirce (46),

$$\ln\frac{is-1-\beta}{is+1+\beta} = \pi i + (2/i)\tan^{-1}s/(1+\beta).$$

Also, since for any (real) A and B

$$\ln(A+iB) = (\tfrac{1}{2})\ln(A^2+B^2) + i\tan^{-1}B/A,$$

one obtains

$$\ln\frac{is+1}{is+1+\beta} = \frac{1}{2}\ln\frac{1+s^2}{(1+\beta)^2+s^2} + i[\tan^{-1}s - \tan^{-1}s/(1+\beta)].$$

Substitution into eq. [79] gives, after some simplification,

$$z = \frac{2a}{\pi}\left(\frac{1+\gamma}{\gamma}\tan^{-1}\frac{s}{1+\beta} - \frac{1}{\gamma}\tan^{-1}s\right) + ic + \frac{ia}{\pi\gamma}\ln\frac{s^2+1}{s^2+(1+\beta)^2}.$$

Separating the real and imaginary parts, the two parametric equations for the coordinates of the water table result: $(0 \leqslant s \leqslant \infty)$

$$\frac{x}{a} = \frac{2}{\pi}\left(\frac{1+\gamma}{\gamma}\tan^{-1}\frac{s}{1+\beta} - \frac{1}{\gamma}\tan^{-1}s\right)$$

[80]

$$\frac{y}{a} = \frac{c}{a} + \frac{1}{\pi\gamma}\ln\frac{s^2+1}{s^2+(1+\beta)^2}.$$

These equations still involve the unknown height of the water table c midway between drains. An equation for c may be obtained by substituting the corresponding values $z_p = 0$ and $t_p = 1$ into eq. [77]:

$$0 = a + ic + i\frac{a}{\pi}\left(\ln\frac{-\beta}{2+\beta} + \frac{2}{\gamma}\ln\frac{2}{2+\beta}\right)$$

$$= a + ic + \frac{ia}{\pi}\ln(-1) + \frac{ia}{\pi}\ln\frac{\beta}{2+\beta} + \frac{2ia}{\gamma\pi}\ln\frac{2}{2+\beta}.$$

That is,

$$\frac{\pi c}{a} = \ln\frac{2+\beta}{\beta} + \frac{2}{\gamma}\ln\frac{2+\beta}{2}. \qquad [81]$$

One can now obtain $\pi b/a$ by use of eq. [81] in eq. [71], with the result

$$\frac{\pi b}{a} = \ln\frac{2+\beta}{\beta} + \frac{2}{\gamma}\ln\frac{2+\beta}{2+2\beta}. \qquad [82]$$

The quantity β. It is recalled (see eq. [71a] and below eq. [71b]) that $1 + \beta = \mu/\lambda$, with $\beta > 0$ and $\mu > \lambda > 1$. Also, eqs. [67] and [68] require, for finite $\tan\Theta$, that

$$(1+\gamma)(\lambda^2-1)/\lambda = (\mu^2-1)/\mu.$$

If one defines k for the moment by $\mu/\lambda = k(1+\gamma)$, where it is remembered that by eq. [60] $\gamma > 0$, then one obtains

$$k = (\lambda^2-1)\mu^2/(\mu^2-1)\lambda^2;$$

and here, since $\mu > \lambda > 1$, it follows (for finite $\tan\Theta$) that $k < 1$, and therefore, since $(1+\beta) = k(1+\gamma)$, that $\beta < \gamma$. When $\tan\Theta = \infty$, $\beta = \gamma$, as will be shown presently. When $\tan\Theta = 0$, $b = c$ and hence (eq. [71b]) $\beta = 0$; but this last case is ruled out, since it has been agreed to take $\mu > \lambda$. Therefore, one has this important result for β:

$$0 < \beta \leqslant \gamma.$$

Equations [81] and [82] show, for a given value of γ, that b and c both increase as β decreases; eq. [71b] shows that simultaneously there occurs a decrease in the difference $(c-b)$ for a fixed a and γ.

It may be seen, by solving eqs. [67], [68] and [71a] for $\tan^2\Theta$ in terms of β and γ, that

$$\tan^2\Theta = \frac{[(1+\beta)^2-1]^2}{4[(1+\beta)(1+\gamma)-(1+\beta)^2][(1+\beta)(1+\gamma)-1]};$$

that is, if one puts for brevity $X = 1 + \beta$ and $A = 1 + \gamma$, one has

$$4\tan^2\Theta = \frac{X-1}{X}\,\frac{X+1}{A-X}\,\frac{X^2-1}{AX-1}.$$

Here each of the three quotients increases steadily with X (without inflexion points) in the permissible range $1 < X \leqslant A$. Therefore $\tan\Theta$ increases steadily with β as β ranges from 0 to γ; that is, Θ is a single-valued function of β.

In the last result a special case is important. When $X = A$, that is, when $\beta = \gamma$, one has $\Theta = \pi/2$; this occurs when the surface of the water table above P in fig. 11 enters parallel to PQ at the height b above P. This height b may be obtained by putting $\beta = \gamma$ in eq. [82]; it is the lowest height the water table can have for a fixed γ and spacing 2a.

The equation for $\tan^2\Theta$ also shows that small values of β correspond to small values of Θ, that is, by physical reasoning, to small values of $(c-b)$, as is further clear from fig. 11 and eq. [71b].

To relate z and ω in eqs. [77] and [78], an independent relation for β still is needed. This relation may be obtained by recognizing that in actuality a drain will not be a line sink but a surface of finite cross-sectional area. This surface thus will constitute an additional boundary to the potential problem and, by a theorem of potential theory, it accordingly will be required that, for all points on this surface, either (a) the potential, (b) the normal derivative of the potential or (c) a combination of these two be known. Boundary conditions of this type already have been taken into account for the other boundaries of the flow system. Condition (a) seems easiest and most natural to use. Therefore, one of the equipotentials about P will be taken as having a known value ϕ_0. Accordingly, the constant β must be chosen so as to make $\phi = \phi_0$ for some specified value of z on the equipotential ϕ_0.

Let the position of the equipotential in question be specified by having it pass through the point $(x, y) = (0, -r_0)$; that is, it must pass through the point $z_0 = -ir_0$. If r_0 is small it is clear that the equipotential will be of nearly circular cross-section so that a drain of radius r_0 could be identified with such an equipotential. If, however, r_0 is large, the equipotential will not be of circular cross-section but will have the form of a horseshoe, and the upper ends of the horseshoe will touch points on the phreatic surface where, by definition, the gauge pressure is zero and where, accordingly, the potential ϕ_0 equals (see eq. [44]) Ky_0, y_0 being the height of the water table at the points in question above the plane $y = 0$.

Corresponding to z_0, there is a point $t_0 = 1 + \delta$.

Since this point lies on the line PS, one sees that $t_0 < t_S$, and thus that $\delta < \beta$. The relationship between r_0 and δ may be found from eqs. [77] and [81] in the form

$$\pi r_0/a = \ln \frac{\beta}{\beta - \delta} + \frac{2+\gamma}{\gamma} \ln \frac{2+\beta+\delta}{2+\beta} - \frac{2}{\gamma} \ln (1 + \delta/2) . \quad [83]$$

The potential ϕ_0 is the real part of eq. [78] evaluated at t_0:

$$\phi_0 = Kc + (a/\pi) \{L \ln[\delta/(\beta - \delta)] + N \ln[(2 + \beta + \delta)/\delta] + [(K - N)/\gamma] \ln[(2 + \delta)/(2 + \beta + \delta)]\} . \quad [84]$$

The corresponding value ψ_0 of the stream function, obtained simultaneously and only mentioned incidentally, is

$$\psi_0 = a(N - L).$$

If the pressure in the drain at $y = -r_0$ is denoted by p_0 and the corresponding piezometric height (referred to the level $y = -r_0$) is h_0, then from eq. [44],

$$\phi_0 = K(p_0/\rho g - r_0) = K(h_0 - r_0)$$

where $h_0 - r_0$ is the hydraulic head now referred to the level $y = 0$. Define H_0 by

$$H_0 = h_0 - r_0 .$$

Then,

$$\phi_0 = KH_0 , \quad [85]$$

and H_0 is the hydraulic head (referred to the level $y = 0$) at $y = -r_0$ and at all other points on the equipotential surface passing through $y = -r_0$.

One now has eqs. [81], [83], [84] and [85] to solve for the four unknowns ϕ_0, β, δ and c, it being assumed that values of γ, H_0, r_0 and a are given. Notice that physically H_0 must always be less than b.

Let us return to eqs. [81] and [82]. If one equates β to its highest value γ in these equations, the lowest possible values of b and c are obtained. The equations become

$$\pi c_{min}/a = \ln[(2+\gamma)/\gamma] + (2/\gamma) \ln[(2+\gamma)/2], \quad \gamma/\beta = 1 \quad [86]$$

and

$$\pi b_{min}/a = \ln[(2+\gamma)/\gamma] + (2/\gamma) \ln[(2+\gamma)/(2+2\gamma)], \gamma/\beta = 1 . \quad [87]$$

A plot of these equations is shown as curve 1, $\gamma/\beta = 1$, in fig. 18. Figure 18 also shows plots of eqs. [81] and [82] for $\gamma/\beta = 2$, 10 and 50, all taken from Van Deemter (52, p. 19). One sees for $\gamma/\beta = 1$ and for $\gamma/\beta = 2$ that the curves for c/a nearly coincide. In fact, if $\gamma/\beta < 2$ the difference between c and c_{min} is less than 10 percent. The ratio b/b_{min}, however, is on the order of 4 when $\gamma/\beta = 2$. Now in practice, b is generally small compared to c and often near zero. Thus fairly large values of β, and hence fairly small

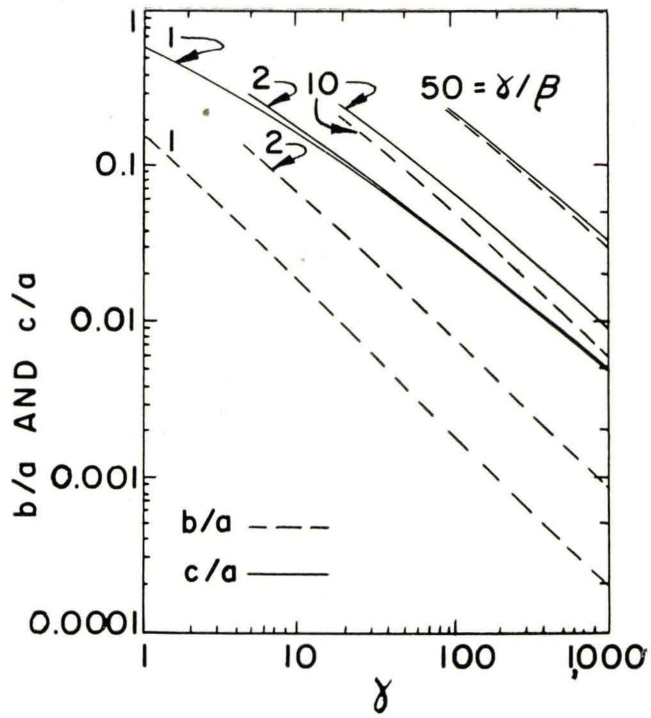

Fig. 18. Effect of change in γ/β on c and b for a range of γ. Traced from Van Deemter (52, p. 19).

values of γ/β as, say, $\gamma/\beta = 2$ or less, would be expected; and accordingly one may conclude from fig. 18, as did Van Deemter, that the right hand sides of eqs. [86] and [87] are useful and simple approximations for $\pi c/a$ and $\pi b/a$.

However, eqs. [81] and [82] (and the discussion below them) are only strictly true if the drains in question are of the non-circular shape of a certain equipotential surface ϕ_{om} passing through a certain point $z = -ir_{om}$. An infinite number of pairs of (r_{om}, ϕ_{om}) may be obtained. For practical reasons, only those pairs of (ϕ_{om}, r_{om}) should be chosen for which $\phi_{om} > 0$; otherwise suction, which seldom exists in practice, would have to operate inside the drain. For a specified pair of values of β and ϕ_{om}, there will be only one value of r_{om}; also, for a specified pair of values of β and r_{om}, there will be only one value of ϕ_{om}.

A more detailed examination, for the important special case $L = 0$, of the relationships governing the influence of β on the flow has brought out some interesting facts about the Van Deemter solution. In practice, as has been mentioned, a non-zero drain radius must be taken into account. To treat the drain surface as an equipotential, one of two conditions must hold. Either the drain must run at least full, to avoid a surface of seepage to which the theory does not apply, or the water table must intersect the drain, with the water in the drain standing to the height at which the water table intersects it. Thus, since in either case the equipotential ϕ_0 must pass through points higher than $y = 0$, one must have, ruling out the artificial cases of drains operating under suction, $H_0 > 0$. If $b >> r_0$, the equipotential ϕ_0 will very nearly pass through both $y = -r_0$ and $y = +r_0$, and then the drain will run just full for

$H_0 = r_0$, approximately. If $b \leqslant r_0$, the equipotential ϕ_0 will intersect the water table and the curve ϕ_0 would become a "ditch" drain of the horseshoe shape previously mentioned. If b is not much greater than r_0, then the equipotential ϕ_0 may pass through the y-axis at a level lower than $y = b$, or it may intersect the water table.

Reducing eqs. [81], [84] and [85] to one by eliminating ϕ_0 and c, and setting $L = 0$, there results, with the aid of eq. [60],

$$\pi H_0/a = \pi(h_0 - r_0)/a = \ln \frac{2+\beta}{\beta} + \frac{2}{\gamma} \ln \frac{2+\beta}{2} - \frac{1}{1+\gamma} \ln \frac{2+\beta+\delta}{\delta} + \frac{2+\gamma}{\gamma(1+\gamma)} \ln \frac{2+\delta}{2+\beta+\delta}$$

which, together with eq. [83], fixes β and δ for a given set of values of H_0, r_0, a and γ $(= -K/N - 1 > 0)$. Calculations from these formulas bring out that H_0, h_0 and r_0 all increase with increasing δ for constant β and γ [and hence for constant $(c - b)/a$]. Figure 19 is an example of the type of relationships obtained. The curves for r_0 and h_0 intersect at a relatively low value of δ and meet again asymptotically when δ approaches β. From such curves, the values for $\pi r_0/a$ can be obtained for which $r_0 = h_0$ (or $H_0 = 0$) and for which $2r_0 = h_0$ (or $H_0 = r_0$). These are plotted against the corresponding values of β in fig. 20. Curves I and II of this figure represent the condition $r_0 = h_0$ and curves III, IV, V and VI refer to the case $2r_0 = h_0$. Also shown in fig. 20 are curves VII and VIII which represent the case, similarly obtained, where $r_0 + b$

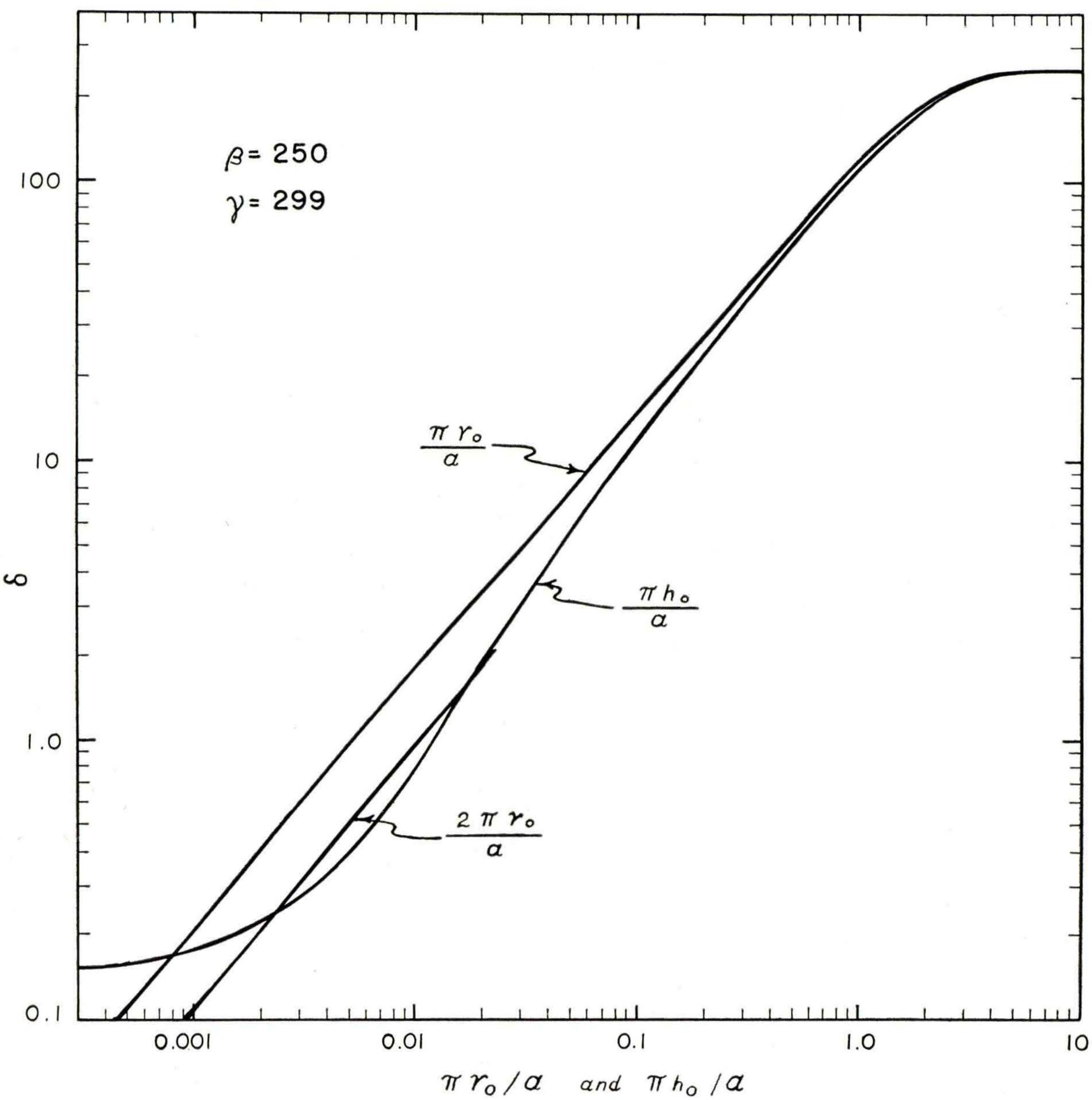

Fig. 19. Effect of changing δ on r_0 and h_0 when $L = 0$, that is when $\gamma = (K/-N)-1$.

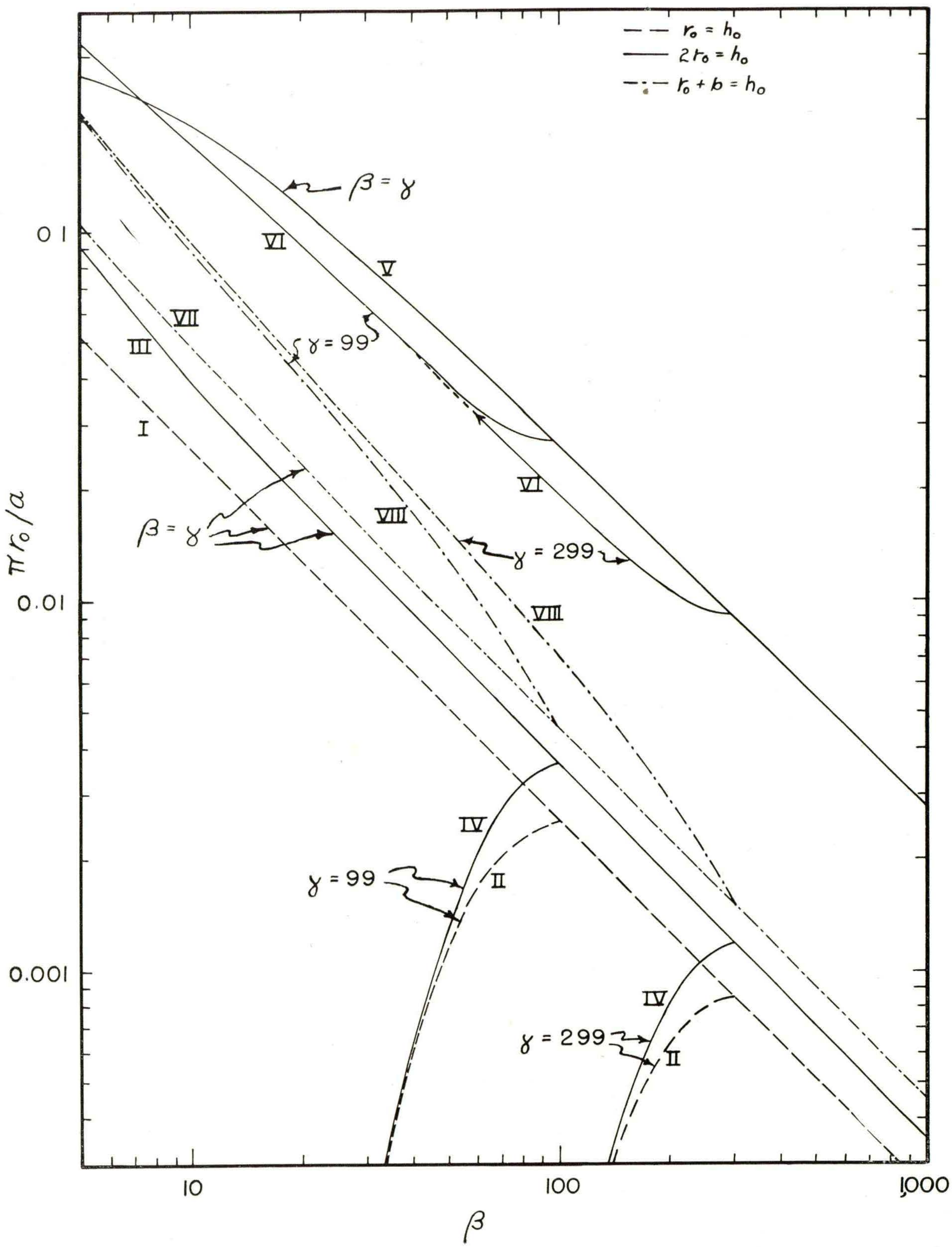

Fig. 20. Relationship between β and tile "radius" for various pressure conditions in the drain.

$= h_0$ (or where $H_0 = b$). The physical significance of these curves will now be investigated.

Curve I, for which the values on the axis of abcissas are $\beta = \gamma$, shows that, for $h_0 = r_0$, the Van Deemter solution for the lowest water table will require a lower r_0 for a higher γ. Thus, since $\gamma = -K/N - 1$, a decrease in rate of rainfall will increase γ and hence decrease, as one would expect, the "drain size" required to make the drain flow under the condition $h_o = r_o$ — that is, under zero gauge pressure at the level $y = 0$. Meanwhile, in accordance with eq. [71b], one sees that $(c - b)/a$ varies as

$$(c - b)/a = (2/\pi)\ (1/\gamma) \ln (1 + \gamma) ,$$

or that $(c - b)$ decreases as γ increases.

Curves I, II, III and IV all correspond to cases where the equipotential ϕ_0 crosses the positive y-axis (axis of ordinates) at a value less than b. Thus they represent conditions where the water table passes above the equipotential drain circumference.

Considering curves II and IV, one sees, for a fixed permeability and a given rate of rainfall (i.e., $\gamma =$ constant), that a decrease in β is associated with a decrease in r_0 if r_0/h_0 remains constant. One also sees, by passing horizontally to the left from a point on curve II to a point on the corresponding curve IV, that for a fixed γ and r_o the drain pressure h_o and the hydraulic head H_0 increase as β decreases. Both of these observations are in accord with the earlier conclusion that decreasing β involves raising the water table and decreasing the difference $c - b$. For, the decrease in r_0 for decreasing β with constant r_o/h_o reflects the fact that a smaller drain size will offer more resistance to flow, just as the increase in drain pressure resulting when β is decreased at constant γ and r_o implies a smaller total head difference causing flow.

From an applied point of view, curves I and II are of little value. They represent a condition where the pressure in the drain ($h_o = r_o$) is insufficient to prevent a surface of seepage. Curves III and IV, on the other hand, represent the condition where r_0 is considerably less than b and where, consequently, it is reasonable to assume that the equipotential ϕ_o crosses the y-axis at both $y = -r_0$ and $y = +r_0$. Since for curves III and IV $h_0 = 2r_0$ or $H_o = r_0$, they very nearly represent the condition where a circular drain of diameter $2r_o$ runs just full and where the water table crosses the y-axis at a height such that $b >> r_0$.

Inspection of fig. 20 shows that Van Deemter's solution requires that the drains must run under back pressure (i.e., more than full) if the drain radius is greater than indicated by curve III and if the drain circumference passes below the point $y = b$. Taking as an example the values $N = -1.20$ inches/day, $L = 0$, $K = 10$ feet/day and $a = 50$ feet, then $\gamma = 99$ and curve III yields $\pi r_o/a = 0.00365$. Thus a drain larger than $(2 \times 50 \times 12 \times 0.00365)/\pi = 1.39$ inches in diameter must run under pressure to cause a water table as given by eqs. [86] and [87]. Similarly, a 10-inch diameter drain at 100-foot spacing ($\pi r_o/a = 0.0262$) will require $\gamma = 14 = (120/-N) + 1$ (that is, $-N = 9.2$ inches of rain per day) if the drain is not to run under pressure and still have water standing above it. In other words, since drainage installations are not designed for 9.2 inches of rain per day but ordinarily for ½ inch per day or less, Van Deemter's equations show that full-running subsurface drains with water over the drains are most unlikely to occur under practical conditions.

Curves V and VI, just as III and IV, represent the case where $H_o = r_o$. They correspond, however, to an equipotential ϕ_0 which intersects the water table. No special significance can be attached to this condition; it simply represents the case where the water table intersects the drain circumference at a height above the origin equal to the maximum depth r_o of the curve $\phi = \phi_o$ below the origin and where the water level in the drain stands to the level at which the water table intersects the drain circumference. For example, when $\gamma = 299$ and $\beta = 250$, curve VI yields $\pi r_o/a = 0.00940$. Thus, for $a = 50$ feet, the height at which the water table intersects the drain circumference and the height above the origin to which the water stands in the drain in this case both are $y = 0.150$ feet, as is the depth r_o of the drain below the origin.

The case where the equipotential ϕ_o just touches the water table for any given β and γ is characterized by the relationship $h_0 = r_0 + b$ or its equivalent $H_o = b$. Curves VII and VIII represent this condition. It is significant in that it represents the second possible drain size for which a given water table position, as fixed by β and γ, can be obtained with the drain running just full. The first possible solution to this condition was given by curves III and IV, and these resulted in unrealistically small drain sizes or high rainfall rates. In the case of curves VII and VIII, one cannot assume the drain diameter to equal $2r_o$ as was done for curves III and IV. However, one may assume the drain diameter to be $b + r_o$. Figure 21 shows a plot of $(b + r_o)/a$ versus β for $\beta = \gamma$ and for some arbitrarily selected values of γ. Taking the same example as before, namely $\beta = \gamma = 99$ ($-N = 1.20$ inches/day, $L = 0$ and $K = 10$ feet/day) and $a = 50$ feet, one obtains from fig. 21 $(b + r_o)/a = 0.00344$ or $b + r_o = 0.172$ feet $= 2.06$ inches. Thus an only slightly larger drain diameter is found from curve VII than was found from curve III, both of which are restricted to the extreme condition $\beta = \gamma$. Since it was previously found that r_o decreases as β decreases at constant γ for curve IV, it was then concluded that curves III and IV did not present a realistic solution to the problem. Inspection of curves VIII shows that here r_o increases as β decreases at constant γ, so that now a realistic solution can be found. For example, if one takes 6-inch diameter drains at 100-foot spacing, one has $(b + r_o)/a = 0.5/50 = 0.01$ and, from fig. 21 for $\gamma = 99$, one finds $\beta = 58$.

The curves of fig. 21 then can be used to determine the proper value of β, given the values of γ, of a and of the drain diameter ($= b + r_o$). This value of β, when used to calculate the shape of the water table from eqs. [80] and [81] or the shape of the equipotentials and streamlines from eqs. [77] and [78], will result in the proper solution for the condition where the drain runs just full and where the water table just touches the top of the drain.

In conclusion, it is again emphasized that, according

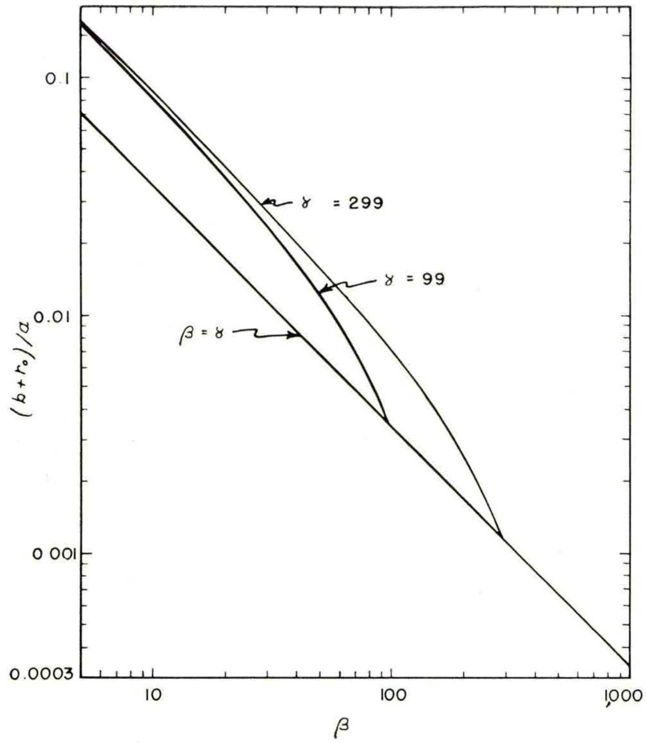

Fig. 21. Relationship between β and tile diameter for a full-flowing drain in contact with the water table.

to Van Deemter's theory, one would seldom expect to find the water table in the field standing at any distance above the drains. This is in accordance with field experience. The case where the water table just contacts a full-running drain is a limiting case which could occur in practice, although the "horseshoe" shape of drain is more likely to occur. Since no relationship between drain pressure and drain size can be given which applies generally to drains running only partly full, no generalized solution has been offered to cover such cases.

Equivalence of Childs' analogue and Van Deemter's hodograph solution. The similarity between Childs' analogue studies (4) and the foregoing analytic solution suggests that results from the two methods should agree and that the conclusions drawn from the Van Deemter solution should carry over to Childs' results. The assumptions underlying both analyses are identical except that Childs used a finite size drain with circumference at zero potential, compared to Van Deemter's point drain with $\phi = -\infty$. Childs' solution was restricted, however, to cases where water stands above the drains. Hence one would expect that his results are limited in practical application.

Using the values for permeability obtained from rainfall and electric analogue currents reported by Childs for his figs. 3, 4 and 5 (4, p. 326) to determine γ, then scaling the values of b/a from the figures and finally calculating β from eq. [82], one can find from eq. [81] the values of c/a in the last column of table 1. Comparison of the seventh column with the last one shows good agreement between the two solutions.[10]

[10]In table 1 certain of the values do not agree with those quoted from Childs by Van Deemter. The latter's column labeled γ probably represents K/—N. The discrepancies in β and c/a have not been explained. The values of —N in table 1 are, quoting Childs, "not quite constant" but are in proportion 1.67:1.73:1.66, the value —N = 0.100 cm./hr. having been chosen arbitrarily. Multiply Childs' permeabilities, namely 1.15 x 10^{-7}, 1.18 x 10^{-7} and 1.43 x 10^{-7} by 980 x 3,600 to obtain the values of K in table 1.

Another check can be made by determining β from the values of γ and r_0 given by Childs, using eqs. [81], [83], [84] and [85]. For Childs' fig. 3, this check also yields fair agreement as shown in table 2.

The small discrepancy between the two methods can be attributed to the difficulty of ascertaining the proper value of r_0 in the analogue; also the equipotential $\phi_0 = r_0$ in the hodograph case is not circular as in the analogue.

In view of the findings from the hodograph analysis showing that water seldom stands above tile drains, it is of interest to investigate the practicality of Childs' results. If $-N = 0.5$ inches/day, the corresponding value of K for his fig. 3 would be (see table 1) (0.4057) (0.5/0.1)/12 = 0.169 feet/day and, similarly, for his figs. 4 and 5, K = 0.173 and 0.210 feet/day, respectively. These K–values are far lower than one would expect to find for soils which could be economically tile drained, at least in the United States. The Clarion and Webster soils of Iowa, for example, generally have a hydraulic conductivity of 10 to 20 feet/day (54, p. 161); also, Kemper (30, p. 61) reported values from about 5 to 20 feet/day for the Bladen soil in eastern North Carolina. The work of Reeve and Kirkham (48) indicates that soils of $K < 1$ foot/day are not likely to be economically tiled. It is possible, however, that the electric analogue results realistically portray the conditions found in certain heavy clay soils which are mole drained.

Applicability of the solution for drainage problems. The derivation of the solution presented was originally restricted to the cases where $K + N > 0$, $L - N > 0$, $L \geqslant 0$ and $N < 0$. Besides $L \geqslant 0$ and $N < 0$, there are two other possibilities. One could have rainfall and deep seepage, or evaporation and upward seepage. The three cases may be summarized as follows:

Case I: $N < 0 < L$, rainfall and upward seepage;

Case II: $N < L < 0$, rainfall and deep seepage;

Case III: $L > N > 0$, evaporation and upward seepage.

FOOTNOTE 10 (cont'd)

TABLE 1. COMPARISON OF CHILDS' RESULTS WITH EQS. [81] AND [82] ASSUMING EQUIVALENCE OF b.

Childs' fig. no.	r_0/a	K cm. per hr.	—N cm. per hr.	$\gamma = \frac{K+N}{-N}$	Childs' values b/a	Childs' values c/a	From eqs. [81] and [82] β	From eqs. [81] and [82] c/a
3	0.0052	0.4057	0.1000	3.057	0.42	0.50	0.61	0.518
4	0.022	0.4163	0.1036	3.018	0.25	0.38	1.17	0.414
5	0.041	0.5045	0.0994	4.075	0.10	0.29	2.80	0.308

TABLE 2. COMPARISON OF CHILDS' RESULTS WITH EQS. [81] AND [82] ASSUMING EQUIVALENCE OF r_0.

Source	r_0/a	γ	b/a	c/a	β	δ
Childs	0.0052	3.057	0.42	0.50	....	....
Van Deemter	0.0052	3.057	0.365	0.480	0.745	0.0101

In all three cases the drains must remove water, since by hypothesis, in each case, L − N > 0. In each case, however, the solutions as originally derived remain valid. The relative magnitude of L and N affects the position of the point T (fig. 11), as will be discussed analytically in the next section. For Case I, point T will lie on the line $\overline{RS}$; for cases II and III it will lie on the line $\overline{PQ}$. For Case II, it must be between P and S; for Case III, between P and Q. Thus, for cases II and III, figs. 11, 12, 15, 16 and 17 no longer apply, insofar as they show the position of point T.

Finally, it was implicity assumed that there was only one point of inflection on the water table. That there can be only one point of inflection may be proved by assuming initially that there is more than one. Then the hodograph would, along its circular boundary, reverse direction once for each point of inflection. Since the hodograph is traced once for one traverse around the flow region in the z-plane, there must be a one to one correspondence between the points of the flow region in the z-plane and the points inside the hodograph in the w-plane. Since the boundaries form part of the conformal regions, this one to one correspondence contradicts the "backtracking" of the hodograph. Hence, only one point of inflection can occur.

The location of the point T. The point T (fig. 11) is the stagnation point where the upward and downward streamlines midway between drains meet. The position of the point T may be obtained from eq. [77] by the substitution $t = t_T = v/\lambda$.
Thus

$$z_T = a + ic + \frac{ia}{\pi}\left(\ln\frac{v/\lambda - 1 - \beta}{v/\lambda + 1 + \beta} + \frac{2}{\gamma}\ln\frac{v/\lambda + 1}{v/\lambda + 1 + \beta}\right). \quad [88]$$

If β is known, then v and λ (and μ and tan Θ) may be calculated from eqs. [67], [68], [69] and [71a]. Accordingly, z_T would be known from eq. [88]. The three cases of the previous section must be recognized. In all three, the results are simplified when $\beta = \gamma$.

In Case I, where $N < 0 < L$, eqs. [67] and [69] may be solved for v and λ to yield

$$\lambda = \tan\Theta + (\tan^2\Theta + 1)^{1/2}$$

$$v = (K/-N)\tan\Theta + [(K/-N)^2\tan^2\Theta + 1]^{1/2},$$

where positive signs have been chosen for the square roots, since in Case I (see fig. 17) $v > \lambda > 0$, and $-K/N > 0$. Thus

$$t_T = \frac{v}{\lambda} = \frac{(K/-N) + [(K/-N)^2 + \cot^2\Theta]^{1/2}}{1 + (\cot^2\Theta + 1)^{1/2}}.$$

If further $\beta = \gamma$, then $\Theta = \pi/2$ and

$$t_T = K/-N.$$

With the aid of fig. 17 and eq. [71a], one has $v/\lambda > \mu/\lambda = 1 + \beta = 1 + \gamma$ so that $t_T = K/-N = v/\lambda > 1 + \gamma$. If now the ordinate at T is designated by h, one has $z_T = a + ih$, and eq. [88] yields

$$\pi(c_{min} - h)/a = \ln\frac{(K/-N) + 1 + \gamma}{(K/-N) - (1 + \gamma)} + \frac{2}{\gamma}\ln\frac{(K/-N) + 1 + \gamma}{(K/-N) + 1}. \quad [89]$$

Equation [89] is important in the problem of salinification, especially in the Netherlands where farm lands lie below sea level and salt water can seep upward into soils. The equation is of more general interest in that calculations with it show (52, p. 22) negligible influence of upward seepage on the height c_{min} when $-h > a$. If one assumes that a horizontal impermeable layer halfway between P and T in fig. 11 has the same effect on the height c_{min} as does the actual curved streamline (surface) PTS, then these calculations show that, whenever an impermeable layer is more than one-fourth the drain spacing below drain center, sensibly the same value of c_{min}/a is obtained.

In Case II, $t_T = K/-N$ as before and, by hypothesis one has, as in all drainage cases, $K + N > 0$ and therefore $1 < K/-N$. Hence, using eq. [60] one finds $1 < K/-N < 1 + \gamma$. Thus, since here T lies between P and S and remembering that $\ln(-1) = \pi i$, one finds for Case II

$$\pi(c_{min} - h)/a = \ln\frac{1 + \gamma - K/N}{1 + \gamma + K/N} + \frac{2}{\gamma}\ln\frac{1 + \gamma - K/N}{1 - K/N}.$$

The equation is of interest since in most drainage systems there is deep downward seepage which never reaches the drains.

In Case III, one has $-K/N < 0$, and it has been seen that T lies between P and Q. Hence (see fig. 17) $0 < v < 1$. If $\beta = \gamma$, then $\Theta = \pi/2$ and Q, Q′ and Q″ coincides in the w-plane (fig 11) and also in the t-plane (fig. 17). Thus, $\lambda = \infty$ and $v/\lambda = 0$, so that with $z_T = ih$ as in Case II, one obtains from eq. [88]

$$\pi(c_{min} - h)/a = (2/\gamma)\ln(1 + \gamma).$$

Now from eqs. [86] and [87] (for which $\beta = \gamma$) one obtains by subtraction

$$\pi(c_{min} - b_{min})/a = (2/\gamma)\ln(1 + \gamma).$$

Therefore, it is concluded for Case III, where there is evaporation at the surface but the upward seepage rate L is greater than this evaporation rate, that

$$h = b_{min}.$$

Sub-irrigation problems. When $L - N < 0$ the drains will not act as sinks but as sources. Accordingly, the case applies to sub-irrigation. This problem is of limited practical importance but will be mentioned nevertheless. It can be shown by the foregoing procedures that the same differential equation will be obtained when $L - N < 0$ as when $L - N > 0$. Hence eqs. [80], [81] and [82] also apply to the sub-irrigation problem. The only difference is that now $\gamma < 0$ and $\beta > 0$. Therefore, the approximate solution for b_{min} and c_{min} is no longer valid.

IV. COMPARISON OF THEORY WITH FIELD DATA

Steady State Data

Because of the uneven distribution of precipitation, a condition approaching equilibrium in rainfall-seepage conditions is seldom, if ever, encountered in the Midwest. Elsewhere, however, the rainfall pattern during the winter season often can be approximated by a constant low rate maintained over a relatively long period. Consequently, Kirkham and De Zeeuw (39) succeeded in obtaining data from a drainage experiment in the Netherlands which lend themselves to steady state interpretation.

Their data consist of water table heights, tile and ditch discharge, rainfall and soil permeability, and were obtained for installations of tile drains and open ditches at 4 spacings—8, 10, 12 and 16 meters—over a 3-week period in 1950. Each of the spacings was replicated three times in each of two plots for both types of drains. The data of Kirkham and De Zeeuw now will be compared with the Dupuit-Forchheimer theory as given in the ellipse equation of Aronovici and Donnan, with the theory of Hooghoudt and finally with the theory of Van Deemter.

THE ELLIPSE EQUATION OF ARONOVICI AND DONNAN

As pointed out before, eq. [11] also was developed by Aronovici and Donnan. They wrote it in the form

$$S = 4K(b^2 - a^2)/Q, \qquad [90]$$

where b and a correspond to H_0 and h_0 in eq. [11] except that the impermeable layer, from the upper level of which b and a are measured, is not necessarily at the same depth as the bottom of the drain. Also, Q is the discharge per unit length of drain per unit time, from both sides of the drain, rather than from one side as Q_1 in eq. [11].

Certain assumptions must be made before eq. [90] can be tested against the field data. In view of the permeability data reported (39, fig. 5) and the geological description of the area, it appears sound to assume an impermeable layer 180 cm. below the surface. This is the upper boundary of a reportedly slowly permeable peat layer with K = 5 mm./day. Above that layer, the permeability measurements varied greatly, but an average of $75 < K < 100$ mm./day probably is a reasonable estimate. The tile depth was reported as 97 ± 5 cm.; the value of 97 cm. will be used here. The value of a for use in eq. [90] for tile drains is thus $1.800 - 0.970 = 0.830$ meters.

Using the average water table height for the period Nov. 27 to Dec. 9, 1950, inclusive (39, fig. 6), with the corresponding average rainfall N = –2.82 mm./day (evapo-transpiration was negligible), one obtains the comparisons listed in table 3. Also included in this table are comparisons with the data for open ditches 50 cm. deep, for which the value for a is $1.800 - 0.500 = 1.300$ meters. In the calculations, the value K = 75 mm./day was used, and Q was determined as the product of the actual spacing and the rainfall rate. A sample calculation with eq. [90] for tile at 8-meter spacing is $S = 4 \times 0.075\ (1.196^2 - 0.830^2)/0.0226 = 9.85$ meters, as recorded in table 3.

The discrepancy between the calculated and actual spacings, especially in the case of the open ditches, points out the danger of using the ellipse equation for design purposes. Probably more important than the above observation is the fact that for the tile drains the calculated spacings vary more slowly than the actual

TABLE 3. COMPARISON OF ACTUAL SPACINGS OF TILES AND DITCHES WITH SPACINGS COMPUTED FROM THE ELLIPSE EQUATION (EQ. [90]); FIRST FOUR COLUMNS ARE FROM OBSERVED VALUES (39).

Actual spacing S (meters)	Q* ($meter^3$ per day per meter of tile)	Water table ht. b above impermeable layer (meters)†		Spacing S‡ (meters) calculated from eq. [90]		Percent deviations from actual spacings	
		Tile drains	Open ditches	Tile drains	Open ditches	Tile drains	Open ditches
8	0.0226	1.196	1.401	9.85	3.64	+23	—54
10	0.0282	1.327	1.450	11.40	4.38	+14	—56
12	0.0338	1.428	1.538	11.96	5.98	— 0	—50
16	0.0451	1.645	1.723	13.42	8.44	—16	—47

*Q = 0.00282 S.
†1.800 less 10^{-3} x numbers recorded in fig. 6 of ref. (39); a = 0.83 meters for tile, a = 1.30 meters for ditches.
‡With K = 75 mm./day.

spacings. This trend will be discussed in more detail later on.

A second test can be made of the ellipse equation. Discharge measurements (39, table 2) and corresponding water table heights [not tabulated in (39) but incorporated in the text] were recorded at three times on Dec. 13 and 14. Using these discharge figures, the spacings again can be calculated with eq. [90]. The results of the calculations are given in table 4.

That greater tile spacings were calculated by the second test than by the first can partly be explained on the basis of non-equilibrium conditions for the second test. Measurements showed that the water table rose Dec. 13 between 11 a. m. and 5 p. m., but dropped considerably between 5 p. m. Dec. 13 and 3 p. m. Dec. 14. Such a dropping water table would result in the prediction of larger spacings by the ellipse equation than would have been predicted on the basis of the equilibrium conditions of table 3, since a greater rainfall and therefore a greater discharge than reported in table 4 would be required to maintain a steady water table. Again, however, the calculated spacings vary less than the actual, as evidenced by the last column of the table.

HOOGHOUDT'S ANALYSIS

Similarly to the calculations leading to table 3, the tables prepared by Hooghoudt (27) can be checked against the average conditions observed between Nov. 27 and Dec. 9. This requires, in addition to the data used previously, the radius of the tile. Since the tile used had an inside diameter of 5 cm. and outside diameter of 7 cm., it was assumed for the present purpose that $r_0 = 3$ cm.

The calculations yield the comparison presented in

TABLE 4. COMPARISON OF ELLIPSE EQUATION WITH FIELD MEASUREMENTS (39) OF TILE DISCHARGE AND WATER TABLE HEIGHT.

Actual spacing (meters)	Spacing in m calculated for:* Dec. 13 5 p.m.	Dec. 14 3 p.m.	Dec. 14 5 p.m.	Average calculated spacing, m	Difference (percent)
8	10.90	10.83	11.96	11.23	+40.3
10	13.60	12.85	13.23	13.23	+32.3
12	15.90	14.50	14.38	14.93	+24.4
16	14.90	22.28	22.20	19.79	+23.7

*With K = 75 mm./day.

table 5. Whereas these data are insufficient in scope to warrant any far-reaching conclusions, they do show better agreement between theory and field results than was found in the case of the ellipse equation. Considering Hooghoudt's assertion that the tables yield a solution of his equations with an accuracy of ± 10 percent, essential agreement may be claimed in this case. Also, it was assumed that K = 75 mm./day, but the data would warrant a selection of K = 100 mm./day as well. Of particular importance is the nearly constant percentage of difference between actual and calculated spacings. The value of r_0 also probably could have been taken larger than 3 cm. as the tile were covered with a matty highly permeable peat. But changing r_0 doesn't change Q very much. Hooghoudt's results are for half-filled tile.

The available evidence corroborates Hooghoudt's statement that his tables are sufficiently accurate for design work where steady state conditions can be assumed.

Comparison of the calculations based on the ellipse equation with those based on Hooghoudt's tables suggests that the neglect of convergence is the most serious shortcoming of the ellipse equation because the large decrease in percentage difference with increasing spacing observed in tables 3 (23 to –16 percent) and 4 (40 to 24 percent) was not evident in table 5 (–12 to –17 percent). Hooghoudt's tables combine radial flow near the drains with horizontal flow in the section midway between drains, but otherwise both analyses are essentially identical.

VAN DEEMTER'S SOLUTION

Strictly speaking, a comparison of the field data of Kirkham and De Zeeuw with Van Deemter's solution is not possible, because Van Deemter's solution applies only to an infinitely deep homogeneous soil. Nevertheless, a comparison will be made, and the effect of the less permeable peat layer, noted by Kirkham and De Zeeuw, will be investigated afterwards.

It will be assumed for the present that the hydraulic conductivity was uniformly 100 mm./day and the rainfall 2.82 mm./day. Kirkham and De Zeeuw have shown that there was some upward seepage but that it was less than 2 mm./day. Assuming then that L =2, 1 and 0 mm./day, three values for γ are found: γ = 20.2, 25.4 and 34.5.

It also is necessary to choose a reasonable value for β. It has been shown before that a large β corresponds to low pressures in the drain, and that varying β over the range $\gamma/2 \leqslant \beta \leqslant \gamma$ has little effect on the solution if b is small. Inspection of the data shows that b = 0 or nearly so. A check of the tile capacity with Manning's formula showed that the tile would run at only about one-third capacity if it had a 0.05-percent slope. Although the exact slope was not given, the data lead one to believe that the actual slope was around 0.2 percent. Thus it seems safe to state that the tile drains never ran full. One may conclude that the assumption $\beta = \gamma$ is probably very nearly correct.

TABLE 5. TILE SPACINGS CALCULATED FROM HOOGHOUDT'S TABLES (27) FOR FIELD DATA (39) AVERAGED OVER 13 DAYS.*

Actual spacing, m	Calculated† spacing, m	Difference, percent
8	7.0	-12.5
10	9.0	-10.0
12	10.4	-13.3
16	13.2	-17.5

*Nov. 27—Dec. 9, 1950.
†For K = 75 mm./day.

Using eqs. [80] and [81], and remembering that any arbitrary value of s, $0 \lesseqgtr s \lesseqgtr \infty$, represents a point on the water table, the water table was plotted for the three values of γ corresponding to the three rates of upward seepage listed above. Curves 1, 2 and 3 of fig. 22 depict the resulting curves. As was to be expected, decreasing L, that is, increasing γ, resulted in a lower water table. Since the right hand sides of eqs. [80] do not involve the spacing, the shapes of the curves in fig. 22 are not affected by the spacing. It is known that $0 < L < 2$ mm./day, so that curves 1 and 3 represent the upper and lower boundaries between which the water table must lie if the assumed value K = 100 mm./day is correct. Curve 1 may also be interpreted as the case where K = 80, L = 1 and N = –2.82. Hence, curves 1 and 2 may be considered the limits between which the water table must fall for $80 \leqslant K \leqslant 100$ and L = 1.

Superimposed on the three curves just discussed are curves 4 and 5 showing the actual water table observed for the 8- and 16-meter spacings, respectively. To avoid confusion, only the center points (which coincide) are shown for the 10- and 12-meter spacings. In agreement with the statements of the previous paragraph, the values for y/a at the midpoint all fall within the range $0.090 < y/a < 0.106$. However, contrary to expectations, the curves for the different spacings do not coincide. One possible explanation is based on the lack of homogeneity of the experimental field; there was a tight peat layer at 180 cm. depth below the soil surface (86 cm. below the drain centers) and sand beneath the peat.

If the position of the point T is calculated from eq. [89], using γ = 25.4, L = 1 and K = 100, it is found that h/a = –0.5337. It has been pointed out that the effect of the limiting streamline PT may be taken equivalent to that of a horizontal impermeable layer midway between P and T. Hence, an impermeable layer a distance h/2 or more below the x-axis will have a negligible effect on the flow pattern above this layer. In the present case, h/2a = –0.2669 and the depth, –h/2, corresponding to a spacing of 8 meters is 1.07 m.; for 2a = 16 m., there corresponds –h/2 = 2.14 m. These figures show that the less permeable peat layer at y = –0.86 meters does not greatly affect the accuracy of the Van Deemter analysis for the 8-meter spacing but does have a pronounced effect on the 16-meter spacing. Curves 4 and 5, therefore, should not be expected to coincide.

Furthermore, curves 4 and 5 are distinctly flatter than curves 1, 2 and 3. A possible explanation for this difference in shape lies in the choice of β as $\beta = \gamma$. The use of a smaller value for β would result in a flatter water table and as such might give better agreement between theory and observation.

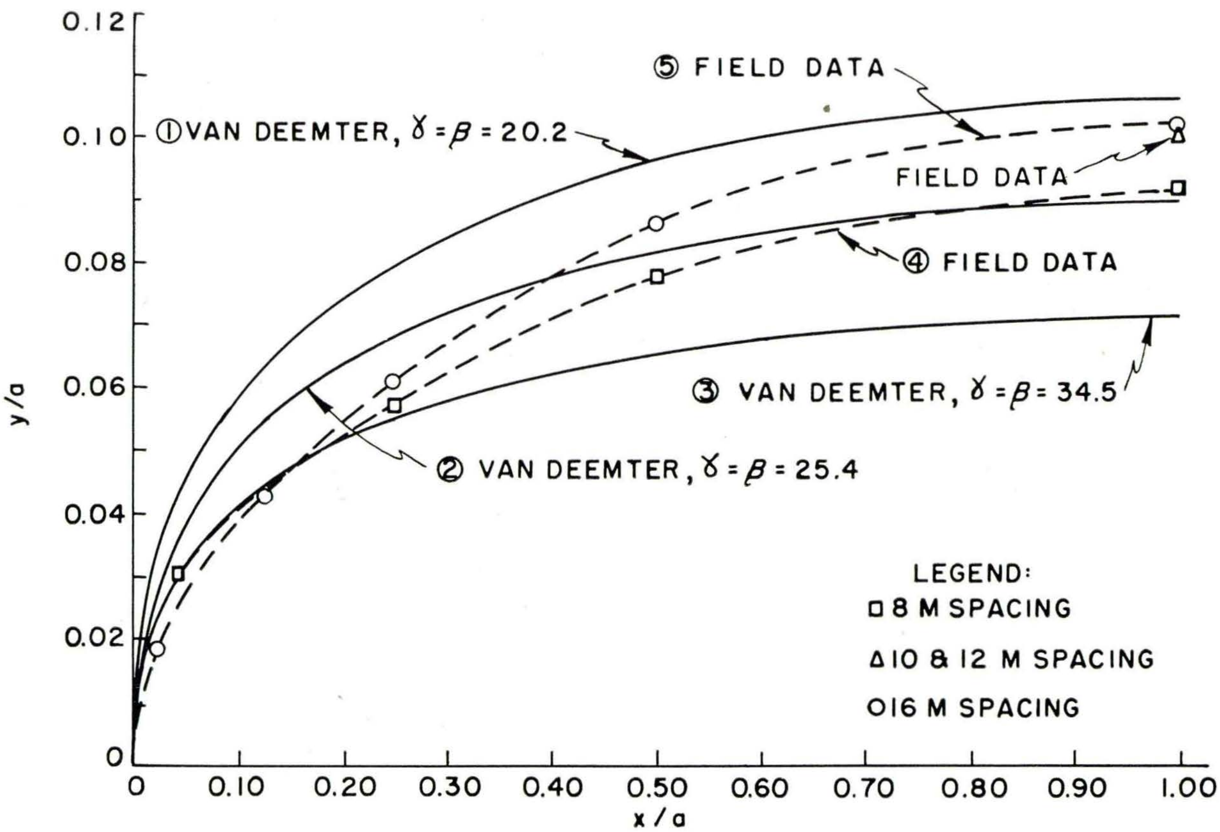

Fig. 22. Comparison of Van Deemter's equations with actual water table shape observed by Kirkham and De Zeeuw.

Nonsteady State Data (The Falling Water Table)

Contrary to the steady state case, a variety of observations have been reported concerning the nonsteady states of falling water tables. The present discussion will be restricted to (a) certain data of Kirkham and De Zeeuw mentioned earlier, (b) a series of observations made by Manson (44) on the Gibbs farm near LeSueur, Minnesota and (c) some data collected by the Iowa Agricultural Experiment Station (28). The equations of Walker and Glover now will be compared with these data.

walker's equation

The analysis of Walker yielded eq. [34] for the determination of the proper spacing. To check this equation against the field data of Kirkham and De Zeeuw, one may consider their observations of Dec. 2-4 and Dec. 7-9. For other periods, frequent showers make their data unsuitable. One has, then, $K = 100$ mm./day, $f = 0.025$ and depth to the center of the tile drains, 94 cm. One also has, from their data, for times t_1 and t_2, the heights y_1 and y_2 above the drain centers of the water table midway between drains. Therefore, one has, for use in eqs. [33] and [34], the quantities $t_2 - t_1 = t$, $y_1 - y_2 = \triangle y$ and $(y_1 + y_2)/2 = \bar{y}$, so that Θ and S can be calculated. The values of S thus obtained are given in table 6 and are for the times 9:00 and 12:00 of Dec. 2 and 4 and 17:00 and 10:00 of Dec. 7 and 9 and for the corresponding y-values shown in fig. 3 of Kirkham and De Zeeuw (39). The values in the second and third columns of table 6 are to be compared with those in the first column. The calculated values are 3 to 30 times as large as the actual values. It has been shown previously that Walker's theory would lead to larger values than one would observe.

Since there is uncertainty as to the values of K and f given by Kirkham and De Zeeuw, K and f should be eliminated from tests of Walker's equation. This elimination can be accomplished by observing that eqs. [33] and [34] yield

$$S = (4K/f)\ (t\,\bar{y}\sin\Theta/\triangle y)$$

so that, if S_{10} represents the calculated spacing corresponding to the actual spacing 10 meters and S_x the calculated spacing corresponding to another spacing $x(= 8, 12, 16$ meters), one has

TABLE 6. COMPARISON OF WALKER'S (59) EQUATION WITH DATA OF KIRKHAM AND DE ZEEUW (39) FOR TILE DRAINS.

Actual spacing, meters	Spacing in meters calculated for:* Dec. 2-4	Dec. 7-9	S_x/S_{10}† Dec. 2-4	Dec. 7-9
8	41.6	29.8	0.935	0.986
10	44.5	30.2	1.000	1.000
12	48.2	30.6	1.083	1.012
16	465.0	54.8	10.45	1.818

*For $K = 100$ mm./day, $f = 0.025$, depth of drain centers $= 94$ cm.
†S_x/S_{10} = ratio calculated spacing corresponding to actual spacing x to calculated spacing coresponding to actual spacing of 10 m., values to be compared with 8/10, 10/10, 12/10 and 16/10.

$$\frac{S_x}{S_{10}} = \frac{(t\,\bar{y}\sin\Theta/\triangle y)_x}{(t\,\bar{y}\sin\Theta/\triangle y)_{10}}$$

in which K and f do not appear. In table 6 the fourth and fifth columns give values of S_x/S_{10}. They depart widely from the correct values 0.8, 1.0, 1.2 and 1.6. The closest agreement occurs for Dec. 7-9 for the 16-meter spacing, 1.818 as compared with 1.6. The worst agreement occurs for Dec. 2-4, at the same 16-meter spacing, where the Walker procedure gives 10.45 instead of 1.6. Again it is clear that the Walker theory, unless modified, is incorrect.

A similar comparison can be made using the Gibbs farm data (table 7) of Manson. Before making the comparison, however, some remarks are in order. The tile spacing experiment on the Gibbs farm consisted of seven replicates at 25-foot spacing, eight at 50 feet, four at 100 feet and one at 300 feet. The soil was described as Webster silty clay loam, but no further information concerning the physical characteristics of the soil or the cropping practices is known. All tile lines were placed at a nominal depth of 4 feet. The spacings were grouped from narrow to wide across the field without randomization. The presence of a deeper main near the first two tile lines 25 feet apart affected the rate of drawdown in that area, and the 300-foot wide plot was drained on one side by a shallow open ditch rather than a tile drain. These factors no doubt affect the results, but it is believed that by excepting the unreplicated widest spacing and by eliminating some of the data for the 25-foot spacing a sound comparison can be made. Drawdown curves are available for the water table following three rains in 1946. The average height of the water table above the tile drain bottoms (using the average elevation of adjacent drains) midway between drains at a series of days after each of these rains was determined from these records. These data are presented in table 7.

Because the hydraulic conductivity and the porosity are not known for table 7, the spacings cannot be calculated explicity. The product fS/K, however, is tabulated in table 8, together with the values S_x/S_{100}. These ratios of S_x/S_{100}, which theoretically should be (reading downward in the table) 0.25, 0.50, 1.00 and 3.00, show that the actual and calculated spacings are ordered similarly, but little more can be said about them. If it is assumed that the hydraulic conductivity of the Webster silty clay loam was about 10 ft./day and its drainable porosity about 5 percent, the values of fS/K would yield spacings varying from 800 to 10,000 feet, compared to the actual range from 25 to 300. So again it is found that Walker's equation results in spacings that are far too wide and that the relative effect of spacings cannot be predicted from it with any certainty.

The same type of analysis has been applied to the data from the spacing experiments in Iowa (tables 9, 10, 11 and 12). The results, summarized in table 13, are very similar to those of tables 7 and 8. It is interesting to note, although probably not significant, that the calculated ratios S_x/S_{100} in one case, the McCormick farm, are nearly identical to the ratios of the actual spacings. The several tests of eq. [34] have been made because original, but limited, tests with the equation as reported by Walker indicated its agreement with field practice and hence its validity.

TABLE 7. AVERAGE HEIGHT IN FEET OF WATER TABLE ABOVE TILE BOTTOMS MIDWAY BETWEEN DRAINS ON GIBBS FARM FOLLOWING THREE RAINS IN 1946.*

Date	Spacing, feet 25	50	100	300
June 24	(1.5 inches of precipitation)			
25	0.64	0.93	2.54	3.64
26	0.29	0.44	1.47	3.36
27	0.24	0.20	0.90	3.00
28	0.10	0.11	0.68	2.54
29	0.08	0.07	0.56	2.50
July 28	(3.3 inches of precipitation)			
29	0.70	1.55	3.08	3.72
30	0.68	0.89	2.18	3.55
31	0.58	0.67	1.42	3.25
Aug. 1	0.49	0.63	1.25	2.73
2	0.40	0.58	1.08	2.44
3	0.40	0.52	0.90	2.44
4	0.37	0.49	0.81	2.33
5	0.33	0.46	0.72	2.08
6	0.28	0.43	0.72	1.93
Sept. 8	(4.2 inches of precipitation)			
9	1.30	1.54	3.24	3.52
10	0.80	1.08	2.18	3.28
11	0.27	0.38	1.18	3.04
12	0.15	0.18	0.79	2.76
13	......	......	0.34	1.84

*From Manson (44).

TABLE 8. COMPARISON OF WALKER'S EQUATION WITH DATA FROM GIBBS FARM (44).

Actual spacing, ft.	June 25-27 fS/K days	June 25-27 S_x/S_{100}	July 29-30 fS/K days	July 29-30 S_x/S_{100}	July 29-Aug. 2 fS/K days	July 29-Aug. 2 S_x/S_{100}	Sept. 9-11 fS/K days	Sept. 9-11 S_x/S_{100}
25	4.00	0.57	78.00	7.53	13.9	0.94	4.16	0.55
50	4.00	0.57	5.57	0.54	12.6	0.89	4.89	0.64
100	7.08	1.00	10.35	1.00	14.2	1.00	7.61	1.00
300	19.80	2.80	78.60	7.60	35.4	2.49	49.48	6.50

TABLE 9. WATER TABLE DRAWDOWN DATA FROM SPACING EXPERIMENT ON R. K. GOODWIN FARM, CHICKASAW COUNTY, IOWA.*

Spacing, feet	Depth water table below surface, feet 5/3/53	5/4/53	5/5/53	5/6/53
50	2.32	2.37	2.64	3.05
75	1.73	1.89	2.02	2.14
100	1.46	1.58	1.76	1.88
125	0.92	1.22	1.51	1.67
150	0.40	0.63	1.08	1.19
200	0.19	0.45	0.81	0.94

*Carrington silt loam; approximate K= 9 ft./day; nominal depth of tile, 4 feet.

TABLE 10. WATER TABLE DRAWDOWN DATA FROM SPACING EXPERIMENT ON RAYMOND KNEER FARM, JEFFERSON COUNTY, IOWA.*

Spacing, feet	Depth water table below surface, feet 6/8/53	6/9/53	6/10/53	6/11/53	6/12/53
33	1.29	1.37	1.50	1.57	1.66
50	0.88	1.01	1.19	1.28	1.39
66	0.85	0.93	1.05	1.13	1.21

*Haig soil type; K unknown; nominal depth of tile, 3 feet.

TABLE 11. WATER TABLE DRAWDOWN DATA FROM SPACING EXPERIMENT ON HOWARD COUNTY EXPERIMENTAL FARM, HOWARD COUNTY, IOWA.*

Spacing, feet	Depth water table below surface, feet					
	5/4/53	5/5/53	5/6/53	5/7/53	5/8/53	5/9/53
50	2.29	2.36	2.44	2.54	2.65	2.74
100	0.76	1.08	1.39	1.57	1.74	1.87

*Plastic till phase of Carrington-Clyde complex; approximate K = 10 ft./day; nominal depth of tile, 4 feet.

TABLE 12. WATER TABLE DRAWDOWN DATA FROM SPACING EXPERIMENT ON WM. McCORMICK FARM, WEBSTER COUNTY, IOWA.*

Spacing, feet	Depth water table below surface, feet					
	5/1/53 17:00	5/2/53 13:30	5/3/53 8:00	5/4/53 7:15	5/5/53 7:15	5/6/53 7:15
50	1.32	1.83	2.15	2.37	2.59	2.77
75	0.75	1.12	1.34	1.57	1.82	1.98
100	0.66	0.94	1.14	1.33	1.46	1.68

*Webster silty clay loam; approximate K = 20 ft./day; nominal depth of tile, 4 feet.

TABLE 13. COMPARISON OF WALKER'S EQUATION WITH DATA FROM IOWA SPACING EXPERIMENTS.

Actual Spacing, ft.	Actual S_x/S_{100}	Goodwin farm May 3-6		Howard Co. farm May 4-9		McCormick farm May 3-6		Actual S_x/S_{50}	Kneer farm June 8-12	
		fS/K days	S_x/S_{100}	fS/K days	S_x/S_{100}	fS/K days	S_x/S_{100}		fS/K days	S_x/S_{50}
33	......	......	....	......	....	......	....	0.66	57.3	1.098
50	0.50	16.7	0.288	52.6	1.228	23.4	0.469	1.00	52.2	1.000
66	......	......	....	......	....	......	....	1.33	78.6	1.507
75	0.75	51.7	0.891	......	....	37.4	0.750	......	......	....
100	1.00	58.0	1.000	42.9	1.000	49.8	1.000	......	......	....
125	1.25	48.9	0.843	......	....	......	....	......	......	....
150	1.50	44.2	0.761	......	....	......	....	......	......	....
200	2.00	50.2	0.865	......	....	......	....	......	......	....

GLOVER'S SOLUTION

The spacing equation proposed by Glover also can be compared with field data. One comparison can be made by applying eq. [17] to the data of Kirkham and De Zeeuw. Considering the height of the water table at 16:00 on Dec. 7 as the initial condition, the spacing necessary to lower the water table to the position recorded for 8:00, Dec. 8, can be calculated. Using again K = 100 mm./day, f = 2.5 percent and $D = 86 + y_0/2$ cm. (drain axes at 94 cm. below the surface and impermeable layer at 180 cm. depth), such calculations yield the comparison shown in table 14.

Here, the calculated spacings are all too small and, as in the case of the ellipse equation, the theory results in a slower rate of change in spacing than was observed in the field. Since Glover's equation is based on the same assumptions as the ellipse equation, it is probable that here also the slow increase of spacing observable in the second and fourth columns of table 14 is caused by Glover's omission of the convergence effect in his analysis.

As noted previously, there is some doubt about the proper value of K, but the value K = 100 mm./day is on the high side. Decreasing it would simply decrease

TABLE 14. COMPARISON OF GLOVER'S (12) EQUATION WITH DATA OF KIRKHAM AND DE ZEEUW (39) FOR WATER TABLE HEIGHTS AT 16:00 DEC. 7 AND 8:00 DEC. 8.

Actual spacing, meters	Calculated spacing,* meters	Percent difference	S_x/S_{10}
8	6.26	—21.8	0.97
10	6.44	—35.6	1.00
12	6.78	—43.5	1.05
16	9.48	—40.7	1.47

*Based on K = 100 mm./day, f = 0.025.

the calculated spacings and increase the percentages of difference in table 14. It could be argued further that the value of D should be decreased because of the existence of artesian pressure in the Kirkham-De Zeeuw case (which is not accounted for in the Glover theory) —but decreasing D would result in still narrower spacings.

Apart from the uncertainties already mentioned, another objection could be raised against the treatment leading to table 14. The initial condition used applied only to the centerpoint of a curved water table, whereas the initial condition leading to eq. [17] was given as a flat water table over the whole range $0<x<S$ (fig. 3). However, exactly the same solution will be found by Glover's procedure if the initial condition is given as $y = y_0$ at $x = S/2$ when $t = 0$. Moreover, Dumm himself, in reporting Glover's work, proposed the use of the equation in this manner (12, p. 730).

Despite the foregoing remarks, it is of interest to compare the Glover equation with the data of Kirkham and De Zeeuw without introducing the initial condition as described above. One proceeds as follows: Writing Glover's equation in the form of eq. [16], the spacing can be considered as fixed; the initial height of the water table can be taken to be at the surface; and the height reported for 16:00, Dec. 7 can be taken as an intermediate position at time t_1. Then the predicted position of the water table at time $t_1 + 16$ hours can be determined from eq. [16]. This calculation was carried out, and the results are tabulated in table 15.

Somewhat better agreement between theory and field data is obtained by this method of approach than was found by the previous method. This may be at least partly because of the more realistic assumption concerning an initial condition. The percentage columns of tables 14 and 15 cannot be compared directly because in table 14 spacings are in question whereas in

TABLE 15. DROP IN WATER TABLE BETWEEN 16:00, DEC. 7 AND 8:00, DEC 8 OBSERVED BY KIRKHAM AND DE ZEEUW (39) COMPARED WITH CORRESPONDING DROP CALCULATED FROM GLOVER'S (12) EQUATION.

Spacing, meters	Water table drop, cm. Actual	Calculated*	Difference, percent
8	17.5	19.2	+ 9.71
10	25.0	27.3	+ 9.20
12	29.3	15.8	—46.1
16	11.8	9.6	—18.7

*Based on K = 100 mm./day, f = 0.025.

table 15 it is distances of fall of water tables. The logarithmic form of eq. [17] would cause a great difference in water table drop compared to the difference in spacing. Hence, one cannot account for the decrease in percentage of difference in the two methods on the basis of this relationship. Much of the difference is caused by the choice of D. For table 14, it was assumed that $D = 86 + y_0/2$, with y_0 the height of the water table at 16:00, Dec. 7. For table 15, y_0 was taken as 94, the height of the surface above the drain axes. Whereas this choice of D is in accordance with the procedure outlined by Dumm, it is highly unrealistic since, in effect, it assumes that flow will take place above the water table. Decreasing D would make the percent differences found in the second analysis of the same order as those of the first method.

When one attempts to compare Glover's equation with the field observations made on the Gibbs farm, one encounters the difficulty that the depth to an impermeable layer is infinite. Considering the decreasing effect of the impermeable layer with increasing depth, the spacings have been calculated corresponding to the rates of drop encountered in the field for several arbitrarily chosen different values of d, namely, 4, 8 and 12 feet. The resulting calculated spacings, made with

TABLE 16. COMPARISON OF GLOVER'S (12) EQUATION WITH DATA (44) COLLECTED ON THE GIBBS FARM FOR SEVERAL ASSUMED DEPTHS d TO AN IMPERMEABLE LAYER.

Actual spacing, feet	Spacing, ft., calculated* for field data of: June 25-27, depth d (ft.) 4	8	12	July 29-Aug. 2, depth d (ft.) 4	8	12	Sept. 9-12, depth d (ft.) 4	8	12
25	118	159	199	207	286	348	107	146	177
50	99	137	167	175	237	287	109	148	178
100	127	169	202	184	241	286	142	186	221
300	230	298	354	264	342	406	266	347	411

*Assuming K = 10 feet/day, f = 0.05.

TABLE 17. S_x/S_{100} RATIOS FOR SPACINGS PER GLOVER APPLIED TO GIBBS DATA. RATIOS ARE COMPUTED FROM DATA IN TABLE 16.

Actual spacing, feet	S_x/S_{100} ratios corresponding to field data of: June 25-27, depth d (ft.) 4	8	12	July 29-Aug. 2, depth d (ft.) 4	8	12	Sept. 9-12, depth d (ft.) 4	8	12
25	0.93	0.94	0.99	1.13	1.19	1.22	0.76	0.79	0.80
50	0.78	0.81	0.83	0.95	0.99	1.00	0.77	0.80	0.81
100	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
300	1.81	1.76	1.75	1.43	1.42	1.42	1.88	1.87	2.00

K taken as 10 ft. per day and f = 5 percent, give an indication (tables 16 and 17) of the agreement of the theory with field observations and also show the effect of the impermeable layer on the theoretical results.

Inspection of the first of these tables shows that the calculated spacings are considerably higher than the actual, but this could be due to an improper choice of K and f. The actual ratio K/f could have varied by a factor 2 from the value of 200 used. The spacings, which in the Glover theory vary as the square root of this ratio, thus may be said to be of the right order of magnitude, although somewhat high. Changing the depth to the assumed impermeable layer changed the magnitude of the calculated spacings considerably. In fact, in view of the findings of Hooghoudt and Van Deemter that the effect of an impermeable layer is important if it is located at less than one-fourth the spacing below the drain axes, it is likely that even at 12 feet an impermeable layer has an effect that cannot be neglected. If that is the case, the spacings are definitely too wide, even if a factor of 1.4 is allowed in recognition of the unknown values of K and f.

As brought out by the S_x/S_{100} ratios of table 17, the calculated spacings also vary far less than the corresponding ones in the field case. For example, in the second column of the table the values 0.93, 0.78, 1.00 and 1.81 are found instead of 0.25, 0.50, 1.00 and 3.00. The choice of the depth of the impermeable layer has little effect on these ratios.

This comparison with field data bears out the earlier conclusion that the neglect of the effect of convergence of flow toward the drains in Glover's analysis causes spacings calculated by his theory to be generally too wide and results in an insufficient response to water table drawdown rate.

DISCUSSION OF COMPARISONS OF THEORY WITH FIELD DATA

In interpreting data by methods such as those used in the foregoing discussion, one must keep in mind that the theory only takes into account the movement of water due to gravitational forces through those regions that are completely water saturated and above atmospheric pressure. In practice, however, there will be some flow through the capillary fringe, and the evapo-transpiration processes also will cause water removal. Either or both of these processes could be of considerable magnitude.

For the experiment of Kirkham and De Zeeuw, seeded to clover, it was reported that evapo-transpiration losses were negligibly small. In the other cases, no information is available beyond the fact that the various fields were planted to different crops. The area on the Goodwin farm was in meadow during 1953, the field on the Howard County farm in corn, that on the McCormick farm in oats and that on the Kneer farm in corn. The 1946 crop on the Gibbs experiment was not reported. Considering the limited knowledge concerning evapo-transpiration rates, no efforts were made to take them into account. Similarly, lack of data concerning the height of the capillary fringe made it impossible to apply a correction for its effect.

Had corrections been made for these effects, then the calculated spacings would have been greater than those based solely on gravitational flow below the water table.

Furthermore, there is the possibility of error due to poor response of the observation wells, used in obtaining the experimental data, to changes in the water table. There is also the inevitable heterogeneity of the soil, here assumed to be homogeneous. Nevertheless, the large number of observations that were averaged to give the data used for the analysis and the consistency of these averaged results are a convincing indication that these types of error need not be of undue concern.

Notwithstanding the limitations that must be put on the validity of the comparisons, several conclusions can be drawn: When only the relationship of the height of the water table midway between drains and the spacing is considered, the analyses based on the Dupuit-Forchheimer assumptions, such as Donnan's ellipse equation and Glover's equation for the changing water table, have two serious shortcomings. First, they generally resulted in greater spacings than required in the range of practical significance. Secondly, they showed a smaller effect of water table behavior on spacing than was found in the field. Both deviations can be explained by the fact that these analyses fail to take into account the effect of the convergence of flow into the drains.

For the steady state problem (continual steady rainfall discharged continually and steadily by the drains) the semi-empirical combination of radial flow and horizontal flow as proposed by Hooghoudt seemed to represent the actual condition fairly well. In view of the uncertainties involved in determining the soil characteristics, there is little need for a better approximation. Van Deemter's analysis gave results very much like those of Hooghoudt.

In the case of a falling water table, no better solution has as yet been proposed [except for the numerical solution of Kirkham and Gaskell (40) which by sufficient work can be carried to any degree of exactness] than that offered by Glover. This statement is not an acceptance of Glover's equation, but rather recognition that no adequate solution is available at present. The objections raised to Walker's analysis on theoretical grounds were substantiated by comparison with field data. In all cases studied, representing a wide range of soil conditions and climates, the spacings calculated by Walker's equation were so much greater than the actual spacings that one must conclude that the equation cannot be used even as a rough approximation. The inconsistency in the ratios S_x/S_{100} and S_x/S_{10}, when compared to the actual cases, also makes it doubtful whether the equation is valuable for a study of the relative effects of different spacings.

As to the agreement between Van Deemter's equations for the shape of the water table and field observations, it may be said that the theory resulted in higher water tables near the drains than were actually observed. This is in accordance with the theoretical observation made in an earlier section that lowering β at constant γ will cause a flattening of the water table. The comparison of fig. 22 was based on the assumption $\beta = \gamma$. Somewhat better agreement would be expected if a lower value of β had been used. Only by trial could the proper value of β be ascertained.

V. SUMMARY AND CONCLUSIONS

Various approaches to a rational theory of the drainage problem have been analyzed. In particular, the assumptions of horizontal and radial flow, as they apply to the theoretical treatment of the problem of seepage flow to drain tubes and ditches for drainage of nonponded water, have been evaluated. Solutions based on these assumptions have been analyzed and compared both among themselves and with field data. Special attention has been given to the solutions of Hooghoudt, Van Deemter, Glover and Walker.

It has been shown that a judicious combination of the horizontal and radial flow assumptions can lead to a valuable and reliable approximation of the actual steady state problem, that is, the problem of the steady removal of steady rain (or its equivalent) by drains. For routine applications, it was found that the ellipse equation of Hooghoudt, in which he introduces an "equivalent depth" for those cases where an impermeable layer is not present near the bottom of the drains, offers one of the most satisfactory existing ways to solve field problems of design for the steady state. This method requires the availability of Hooghoudt's tables of "equivalent depths" (Appendix B), but enables rational design with a minimum of calculations. The use of the ellipse equation by itself (that is, without Hooghoudt's tables of equivalent depths to the impermeable layer) tends to overdesign or underdesign, depending on whether the flow region is taken as the region above the drain axes or the region above the impermeable layer. This is true because of the omission of a part of the flow region in the first case and the neglect of convergence of flow toward the drains in the second.

A nomographic solution published by Visser and based on a series of relaxation solutions is even more convenient than Hooghoudt's.

In the more prevalent nonsteady state problem of a falling water table (rainfall having ceased), it was found that either the use of the horizontal flow assumption, as proposed by Glover and by Ferris, or the application of radial flow approximations, as used by Walker, leads to serious inconsistencies both theoretically and with respect to field data. Whereas neither the approach of Ferris nor Walker could be advocated, Glover's solutions did appear to have limited value for design purposes. It is shown that care must be used in applying his equations to cases where the impermeable layer is either absent or at relatively great depth.

The analysis of steady state drainage problems by the hodograph method was investigated, and Van Deemter's hodograph solution was verified. It was shown, using this solution, that the water table in the field seldom, if ever, will stand above the tile drains but that generally the water table will intersect the tile drains. For the water table to stand over the drains, the relative values of drain size, hydraulic conductivity and rainfall rate must be so as to make the case of limited practical use. A procedure was presented by which the shape and position of the water table can be determined for a given drain size running just full if the water table just touches the top of the drain. The possibility was suggested of extending the applicability of the Van Deemter solution to ditch drainage. Childs' electric analogue studies, based on the same assumptions as Van Deemter's work and restricted to cases where the water table stands above the drains, were found to be limited in usefulness, as predicted from the theory.

The field data used for testing the above theories were gathered at different times and locations by different investigators. The tests with the field data substantiated the theoretical findings that solutions based on the assumption of horizontal flow alone would yield results deviating from observed field measurements by an amount attributable to the omission of the effect of convergence of flow toward the drains. The rate of change of the spacings calculated on the horizontal flow theory was less than that of the actual spacings. When convergence was taken into account by a combination of the assumptions of radial flow and horizontal flow, the rates of change of the spacings in the field and those determined theoretically were nearly equal.

The equations of Van Deemter deviated from the field observations because of failure to correctly account for the effect of drain size and pressure and for the effect of an impermeable layer. Walker's radial flow equation for the falling water table resulted in spacings far greater than found in the field. This again was in accordance with the theoretical discussion.

APPENDIX A

Proof of Equivalence of Eqs. [36] and [37]

It must be shown that

$$\sum_{m}^{M} (Q/M\pi K) \left(\sum_{n} \ln r_{Amn} - \sum_{n} \ln r_{Bmn}\right) = 0 \qquad [91]$$

$$n = 0, 1, -1, 2, -2, \ldots$$

$$m = 1, 2, 3 \ldots, M, \ (M = \text{even})$$

In fig. 23, M is taken as 10. Any other even number could have been used as well.

Considering the first sum on n,

$$\sum_{mn}\sum \ln r_{Amn} = \ln \prod_{m}\prod_{n} r_{Amn}$$

$$= \ln (r_{A,1,0} \cdot r_{A,10,-1} \cdot r_{A,2,0} \cdot r_{A,9,-1} \ldots)$$

$$= \ln \left(\frac{S}{2M} \cdot \frac{S}{2M} \cdot \frac{3S}{2M} \cdot \frac{3S}{2M} \cdot \frac{5S}{2M} \cdot \frac{5S}{2M} \ldots\right)$$

$$= 2 \ln \prod_{p=1}^{\infty} \frac{(2p-1)S}{2M}.$$

Similarly, the second sum on n can be written

$$\sum_{mn}\sum \ln r_{Bmn} = \ln \prod_{m}\prod_{n} r_{Bmn}$$

$$= \ln (r_{B,5,0} \cdot r_{B,6,0} \cdot r_{B,4,0} \cdot r_{B,7,0} \ldots)$$

$$= \ln \left(\frac{S}{2M} \cdot \frac{S}{2M} \cdot \frac{3S}{2M} \cdot \frac{3S}{2M} \cdot \frac{5S}{2M} \cdot \frac{5S}{2M} \ldots\right)$$

$$= 2 \ln \prod_{q=1}^{\infty} \frac{(2q-1)S}{2M}.$$

Substituting these two identities into eq. [91], there results

$$\sum_{m}\sum_{n} (\ln r_{Amn} - \ln r_{Bmn}) = 2 \ln \prod_{p=1}^{\infty} \frac{(2p-1)S}{2M} - 2 \ln \prod_{q=1}^{\infty} \frac{(2q-1)S}{2M} = 0.$$

Hence, eq. [91] has been proved to be correct.

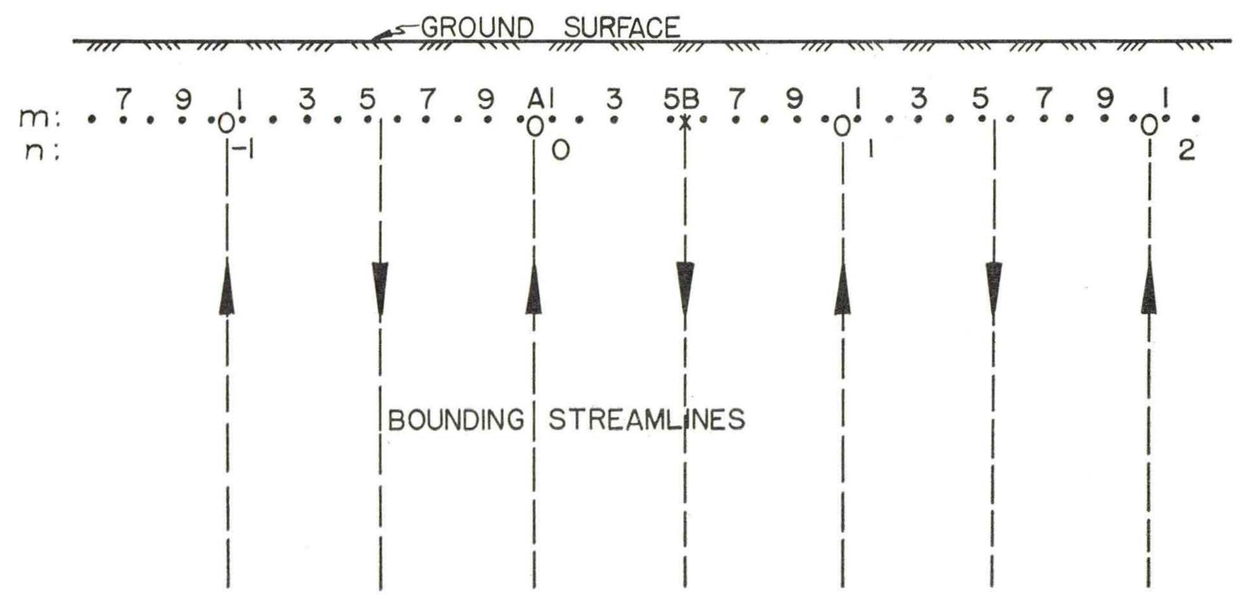

Fig. 23. Geometry and symbols used in showing the equivalence of eqs. [36] and [37].

APPENDIX B

Illustration of Use of Hooghoudt's Tables for Computation of Drain Spacing

GENERAL PROCEDURE

Designating the height of the water table above the drain axes midway between drains as m, and immediately over the drains as m_0, eq. [11] may be rewritten with the substitutions $H_0 = d + m$ and $h_0 = d + m_0$ as [compare (27, p. 593) where $Q = sl$; s is the rainfall rate and l the spacing]

$$S = (8Kd/Q)\ (m - m_o) + (4K/Q)\ (m^2 - m^2_o)\ .$$

In the case of open ditches, d represents the height of the water level in the ditch above the impermeable layer, making $m_0 = 0$. In the case of tile drainage, it generally is safe to assume $m_0 = 0$ as well. Where the horizontal flow assumption fails, d may be replaced by d_e, the depth to an "equivalent impermeable layer." Thus, in all cases, the appropriate equation may be written as

$$S = 8Kd_em/Q + 4Km^2/Q\ . \qquad [92]$$

Hooghoudt's table 5 (27, pp. 656-694) lists values of d_e for given S, d and r_0, where r_0 represents the drain radius or equivalent ditch dimension. In Hooghoudt's notation, S is replaced by l, d by H and d_e by d.

Given the values of r_0, K, d, Q and m, one assumes the proper spacing S, determines with the aid of r_0, d and the assumed S the corresponding d_e and calculates the resulting S from eq. [92]. If the assumed and calculated S-values do not agree, another trial is made.

EXAMPLES

(1) Given a semi-infinite, homogeneous soil with $K = 0.24$ m./day; $r_0 = 4$ cm.; desired depth of drains 80 cm.; precipitation to be removed, $-N = 5$ mm./day; minimum permissible distance of water table below surface 50 cm. Determine the proper spacing.

Assuming, in Hooghoudt's notation, $l = 10$ m., then his table 5.2 gives, with $H = \infty$ and $r_0 = 0.04$, the equivalent depth $d = 0.90$ m. $Q = lN = 0.05$ m.^{3}/day per meter of drain. Thus

$$l = (8)\ (0.24)\ (0.90)\ (0.80 - 0.50)/(0.05)$$
$$+ (4)\ (0.24)\ (0.09)/(0.05)$$
$$= 10.37 + 1.73 = 12.10 \text{ m.} \neq 10 \text{ m.}$$

For a second trial, assume $l = 11.5$ m. Then $Q = 0.0575$ m^3./day/m. and $d = 1.00$ m. Thus

$$l = 8(0.24)\ (1.00)\ (0.30)/(0.0575)$$
$$+ 4(0.24)\ (0.09)/(0.0575)$$
$$= 10.00 + 1.50 = 11.50 \text{ m.}$$

Since assumed and calculated spacings are identical, the proper spacing here is 11.50 meters.

(2) Given a homogeneous soil with an impermeable layer 2 m. below the surface; $K = 3.0$ m./day; rainfall 5 mm./day; desired drain depth 1.00 m.; minimum allowable distance of water table below surface 50 cm.; $r_0 = 10$ cm. Determine the proper spacing.

Assume $l = 40$ m. From table 5.8, with $H = 2.0 - 1.0 = 1.0$ m., $d = 0.96$ m. Also, $Q = 40 \times 0.005 = 0.20$ m.^{3}/day/m. Hence

$$l = (8)\ (3)\ (0.96)\ (0.50)/(0.20)$$
$$+ (4)\ (3)\ (0.25)/(0.20)$$
$$= 57.6 + 15.0 = 72.6 \text{ m.} \neq 40 \text{ m.}$$

Next try $l = 50$ m. with corresponding $d = 0.96$ m. and $Q = 0.25$ m.^{3}/day/m. Then

$$l = (8)\ (3)\ (0.96)\ (0.50)/(0.25)$$
$$+ (4)\ (3)\ (0.25)/(0.25)$$
$$= 46.0 + 12.0 = 58.0 \text{ m.} \neq 50 \text{ m.}$$

Finally, try $l = 54$ m. with $Q = 27$ and $d = 0.96$. Then

$$l = 42.7 + 11.1 = 53.8\text{m.}$$

This is near enough to the assumed 54 m. Thus, the spacing required is about 54 meters.

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