Production Functions, Isoquants, Isoclines and Economic Optima in Corn Fertilization for Experiments With Two and Three Variable Nutrients

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CONTENTS

Production Functions, Isoquants, Isoclines and Economic Optima in Corn Fertilization for Experiments With Two and Three Variable Nutrients[1]

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This study deals with the basic agronomic and economic relationships of fertilizer use. It is the second in a series of methodological studies designed to predict production surfaces, isoquants isoclines, marginal products and marginal replacement rates between nutrients when two or more nutrients are used in promoting increased crop yields. These quantities, which are fundamental in obtaining a basic science knowledge of fertilizer-crop relationships, are then used to predict optimum levels of fertilization and optimum ratios of nutrients with profit maximization as the criterion of selection. While the major objectives of the study are of a methodological nature, illustrations are included to show how the basic relationships and principles can be adapted to simple forms for farmer and educational uses.

The logical foundations for research of the type reported in this bulletin are reported elsewhere.[2] The production functions and economic optima predicted in this study are for corn on three types of soils with two and three nutrients variable in quantity.

[1] Project 1294, Iowa Agricultural Experiment Station.

[2] Heady, Earl O., Pesek, John T. and Brown, William G. Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Exp. Sta. Res. Bul. 424. 1955.

SOURCE OF DATA

Data for this study are from three experiments conducted in 1953. Corn experiments were conducted on three soil types—Carrington, Moody and Haynie. Nitrogen, phosphorus and potash were varied on each experiment. All three experiments were of a factorial nature (i. e., every level of one nutrient was combined with every level of the other two nutrients). All other resources or inputs were held constant except that variable quantities of labor and machine services were used for fertilizer application and harvesting. Planting rates were constant in all three experiments. However, stands obtained did vary among plots at some of the locations. These variations have been taken into account in certain of the regression estimates which follow.

TABLE 1. AVERAGE CORN YIELDS PER ACRE IN 1953 FOR 60 FERTILIZER TREATMENTS ON CARRINGTON SOIL.*

Lbs.† P_2O_5	Lbs.† K_2O	Pounds of nitrogen†				
		0	40	80	160	240
0	0	97.95	103.20	106.95	109.50	102.40
0	40	106.35	115.10	101.65	116.75	104.40
0	80	112.20	120.25	113.55	118.15	110.95
40	0	97.95	107.40	108.05	93.95	116.20
40	40	109.10	116.35	112.55	118.05	110.80
40	80	111.20	110.10	115.65	113.10	116.65
80	0	94.70	100.35	89.60	108.85	111.20
80	40	109.15	115.95	109.70	113.80	106.00
80	80	126.35	120.75	124.55	119.80	122.80
120	0	99.65	112.35	95.05	99.30	93.80
120	40	120.05	118.75	107.75	115.95	114.90
120	80	101.00	111.05	122.55	119.90	131.05

* Each entry is the average of two observations, one from each randomized block.

†Per acre.

CARRINGTON EXPERIMENTAL DATA[3]

Yields of corn on Carrington silt loam for various fertilizer rates are presented in table 1. The factorial experiment providing the data consisted of two randomized blocks, each block having five levels of N, four levels of P_2O_5 and three levels of K_2O. Yields were high in this experiment; plots without fertilizer averaged almost 98 bushels per acre. Large yield responses for fertilizer were not expected since the soil was at a relatively high fertility level (i. e., high yields were obtained on the check plots). However, an average increase of 9.8 bushels per acre was obtained from 40 pounds of K_2O. Application of 80 pounds of K_2O resulted in an average increase of 14.2 bushels over the plots with no potash. The significant potassium effect (table 2) might have been antici-

[3] Howard Smith, farmer-cooperator, Fayette County, Iowa.

TABLE 2. ANALYSIS OF VARIANCE OF CORN YIELDS ON CARRINGTON SOIL, RANDOMIZED BLOCK DESIGN.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	119	18,570.37		
Blocks	1	6,336.53	6,336.53	106.39**
Treatments	59	8,719.703	147.80	2.48**
N	4	514.042	128.51	2.16
P	3	79.285	26.43	0.44
K	2	4,198.086	2,099.04	35.24**
N x P	12	523.262	43.61	0.73
N x K	8	630.862	78.86	1.32
P x K	6	870.996	145.17	2.44**
N x P x K	24	1,903.170	79.30	1.33
Error	59	3,514.137	59.56	

*P = 0.05

**P = 0.01

pated since the experimental plot was, according to soil tests, low in K_2O.

Average yields of the plots receiving 0, 40, 80 and 120 pounds of P_2O_5 do not differ greatly in table 1. Similarly, the analysis of variance in table 2 failed to indicate significant phosphorus treatment effects. Lack of response to P_2O_5 was surprising since the soil test for the experimental plot indicated a low level of P_2O_5 availability. Soil tests also showed a 3-ton lime requirement.

Plots receiving different levels of nitrogen behaved somewhat erratically, as shown in table 1. An average increase of 5.5 bushels was obtained for all plots receiving 40 pounds of N. However, at 80 pounds of N, the average yield increase was only 2.9 bushels. At 160 pounds of N, the average yield increase was 5.3 bushels over check plots; the yield increase declined to 3.8 bushels for 240 pounds of N. In table 2 the mean square for nitrogen is not significant.

Values of F in table 2 provide information for variables to be used in the estimating equation or production functions which follow. Potash should be included since K_2O gives a consistent and statistically significant increase in yield. Phosphorus can be dropped from consideration because, even if all of the mean square due to P were explained by one regression term, its F value would not be significant. Nitrogen is an intermediate case; there is some logical justification for including it even though it is not significant at the 0.05 level of probability. Phosphorus $\times$ potash interaction is significant at the 0.05 level. However, it was not included in the regression because no term was found which would significantly account for the variance in yield due to this term. An analysis of covariance indicated that stand had a highly significant effect on yield. Similar results were obtained when stand was included as a variable in the multiple regression.

Regression Analysis for Carrington Soil

The basic purpose of this study is to estimate crop yield production functions for fertilizer. Accordingly, information on the derivation of the regression equations is included in this section. In the preliminary analysis for each experiment, two general types of equations were used: (1) a quadratic equation with squared terms and (2) a square root transformation of a quadratic equation. In some cases, squared and square root terms have been included in a single predicting equation. These two general types of equations were used because they (1) allow specification of the one nutrient combination allowing maximum per-acre yields, (2) allow convergence of isoclines to the point of maximum yield and indication of changes in nutrient ratios required to attain higher yields, (3) do not require constant substitution rates between nutrients and (4) do not force constant elasticities of production. In the presentation which follows, only the type of equation which appeared to give the most efficient predictions is included.

The highly significant difference between the yields of the two randomized blocks (table 2) raised a question as to whether the response surface differed significantly between the two blocks. To test whether the response differed between blocks, regressions were calculated for each block separately as indicated in equations (1) and (2).

$$\text{(Block I)} \quad \hat{Y} = 57.97 + 0.3800K - 0.002711K^2 + 0.4365\sqrt{N} - 0.02638N + 0.002552S \quad (1)$$

$$\text{(Block II)} \quad \hat{Y} = 51.64 + 0.2702K - 0.001162K^2 + 0.6414\sqrt{N} - 0.02490N + 0.002081S \quad (2)$$

In the above equations, $\hat{Y}$ refers to predicted total yield in bushels per acre, K refers to pounds of K_2O per acre, N to pounds of elemental nitrogen per acre, and S refers to stalks per acre. The t values of the regression coefficients are given in the upper half of table 3. To help determine whether the two blocks should be pooled, t tests of the differences between corresponding regression coefficients were made (table 3).[4] The t values for the difference between corresponding regression coefficients of the two blocks are small. A value of t as large or larger than the t value of difference for K^2, $t = 0.834$, could occur by chance 40 percent of the time even though the population

[4] An analysis of variance was also computed to test for homogeneity of regression; the results were similar to those obtained from the t tests.

TABLE 3. VALUES FOR t FOR COEFFICIENTS OF INDIVIDUAL BLOCK REGRESSIONS AND TEST OF DIFFERENCE BETWEEN CORRESPONDING COEFFICIENTS OF THE TWO BLOCKS.

Coefficient	Values of t for equation (1)	Significance level*	Values of t for equation (2)	Significance level*	Values of t for difference between equations (1) & (2)†	Significance level*	Values of t for pooled regression equation (3)	Significance level*
K	3.515	0.001	2.006	0.06	0.739	0.47	4.118	0.0001
K^2	2.059	0.05	0.810	0.43	0.834	0.41	1.965	0.06
N	0.737	0.48	0.996	0.33	0.211	0.84	0.316	0.20
$\sqrt{N}$	0.719	0.48	0.620	0.55	0.045	0.92	1.030	0.31
S	3.757	0.001	2.554	0.02	0.731	0.46	4.556	0.00002
B	....	....	....	....	....	....	9.570	0.00001

* Probability of obtaining as large or larger value of t by chance, given the null hypothesis.

† These t's have been computed by subtracting each particular regression coefficient in equation (2) from the corresponding regression coefficient in equation (1) and dividing by the weighted standard error.

of K^2 coefficients was the same. The other coefficients had even smaller t values of difference. Since there was no evidence that the blocks had different response surfaces (different regression coefficients), a regression for the pooled data of the two blocks was computed as indicated in equation (3).

$$\hat{Y} = 77.866 + 0.3162K - 0.001813K^2 + 0.9190N - 0.04453N + 0.002241S - 13.497B \quad (3)$$

In the pooled regression above, B represents the particular block; B is 1.0 for Block I and 2.0 for Block II. Stand and block are used here as a method of adjustment similar to covariance. The experiment was not designed to include stand as a variable, but variation in stand did occur. This experiment cannot be used to determine optimum stand. However, precision of estimates is considerably improved by including stand in the regression (as shown by its t value of 4.56 in table 3).

Blocks were included in the regression to allow an estimating equation for either block and to increase precision of estimation. Including blocks in the regression is justified since it takes out the variability due only to the difference in stand and yield level of the two blocks. Predicting the actual yield is secondary to predicting the response of corn yield to fertilizer inputs. That is, more interest is in the slopes of the production surface rather than the absolute level of yield. The values of the N and K coefficients are important in determining the most profitable amount of nitrogen and potash to apply. Stand and blocks were introduced only to increase the precision of estimate of the N and K coefficients.

For an average stand and for Block I, the intercept ($N = O$; $K = O$) of equation (4) becomes 105.971. For an average stand and Block II, the intercept is 92.474. Equation (4) is the average of the two blocks with an average stand of around 18,000 stalks per acre and will be used in the later economic analysis.

$$\hat{Y} = 99.223 + 0.3162K - 0.001813K^2 + 0.9190\sqrt{N} - 0.04453N \quad (4)$$

The value of t (4.118) for the linear response of yield to potash in table 3 is highly significant. Accordingly, greater reliability can be placed in the potash response than in the nitrogen response. (The variance of the N response was also reflected in the average figures of table 1 and the analysis of variance in table 2.)

TABLE 4. ANALYSIS OF VARIANCE FOR REGRESSION OF CORN YIELD, CARRINGTON SOIL.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	119	18,570.37		
Due to regression equation (3)	6	12,013.09	2,002.18	33.04**
Deviation from regression	113	6,557.29	58.03	
Other treatment effects	55	3,043.15	55.33	
Error	58	3,514.14	60.59	

**$P \leq 0.01$

TABLE 5. PREDICTED YIELDS OF CORN PER ACRE FOR SPECIFIED NUTRIENT COMBINATIONS APPLIED ON CARRINGTON SOIL.

Lbs. K_2O	Pounds nitrogen per acre						
per acre	0	40	80	120	160	200	240
0	99.2	103.3	103.9	103.9	103.7	103.3	102.8
20	104.8	108.9	109.5	109.5	109.3	108.9	108.4
40	109.0	113.0	113.6	113.7	113.5	113.1	112.5
60	111.7	115.7	116.3	116.4	116.2	115.8	115.2
80	112.9	117.0	117.6	117.6	117.4	117.8	116.5
100	112.7	116.8	117.4	117.4	117.2	116.8	116.3

The analysis of variance for the regression estimates is presented in table 4. The F value of 43 indicates that the proportion of variance explained by the regression equation (3) is highly significant. However, only 65 percent of the total sum of squares is accounted for by equation (3) (i. e., the coefficient of determination is 0.647).

Production Surface for Carrington Soil

Equation (4) is used for the economic analysis of the experiment on Carrington soil. Prediction of the yields to be expected at various combinations of N and K_2O are presented in table 5. These yields, estimated from the production function, correspond to points on the production surface. Since the soil was fertile, yields are predicted to start at 99 bushels per acre with no fertilizer. A yield of almost 118 bushels per acre is predicted at 80 pounds each of N and K_2O. The figures indicate ranges of both increasing and decreasing total yields (i. e., positive and negative marginal products).

A geometric view of the predicted production surface is provided in fig. 1. The height of the surface represents yield while the horizontal axes represent inputs of N and K_2O. Points on the surface (located by the intersection of the "roof" trusses) correspond to the yields in table 5. The highest points on the "roof" are also the highest yields in table 5. The slope of the surface indicates the response to both N and K_2O. The slope is greater along the K_2O axis than along the N axis; the steeper slope corresponds to the greater response to K_2O as compared to N in tables 1 and 5.

A slice through the surface parallel to the potash axis in fig. 1 would represent response of corn to K_2O at a fixed level of N. Three individual yield response curves to potash are given in fig. 2 for 0-, 20- and 100-pound levels of nitrogen. The three K_2O response curves remain the same distance from each other. This lack of "interaction" between N and K_2O was probably a characteristic of the experimental site. Previous ex-

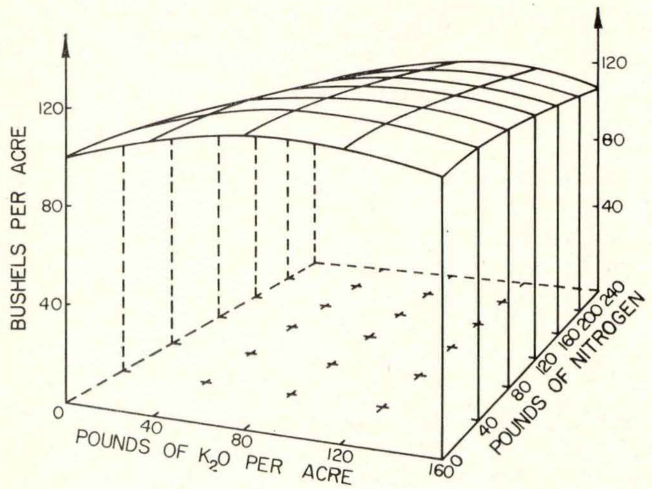


Fig. 1. Perspective view of predicted yield surface for corn on Carrington soil.

periments have provided production surfaces with important interactions between fertilizer nutrients.[5] However, N and K may interact less with each other than N does with P or P does with K.

Corn response to nitrogen at three levels of K_2O is shown by the three curves in fig. 3. Although the N response is strong for the first few pounds, it soon levels out and declines slightly.

[5] Earl O. Heady, John T. Pesek and William G. Brown. Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Exp. Sta. Res. Bul. 424. 1955.

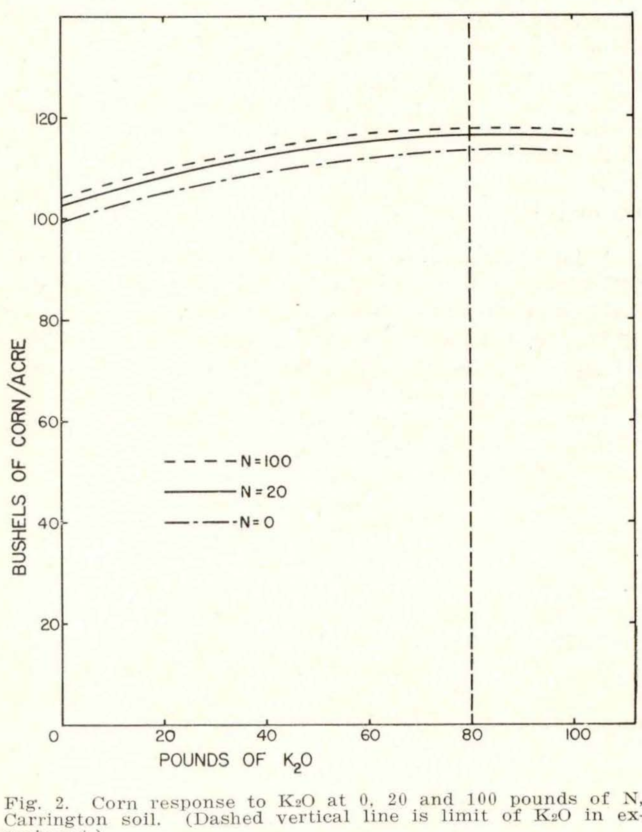


Fig. 2. Corn response to K_2O at 0, 20 and 100 pounds of N, Carrington soil. (Dashed vertical line is limit of K_2O in experiment.)

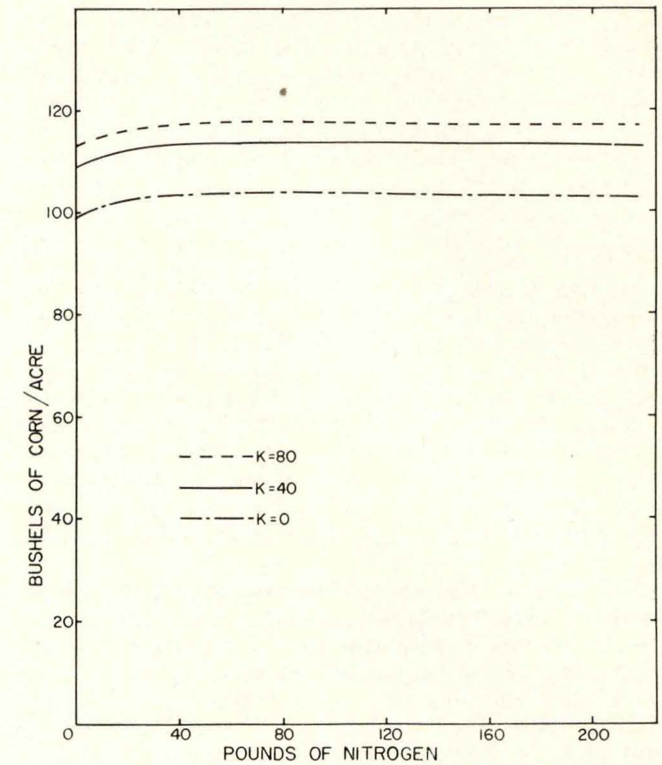


Fig. 3. Corn response to N at 0, 40 and 80 pounds of K_2O, Carrington soil.

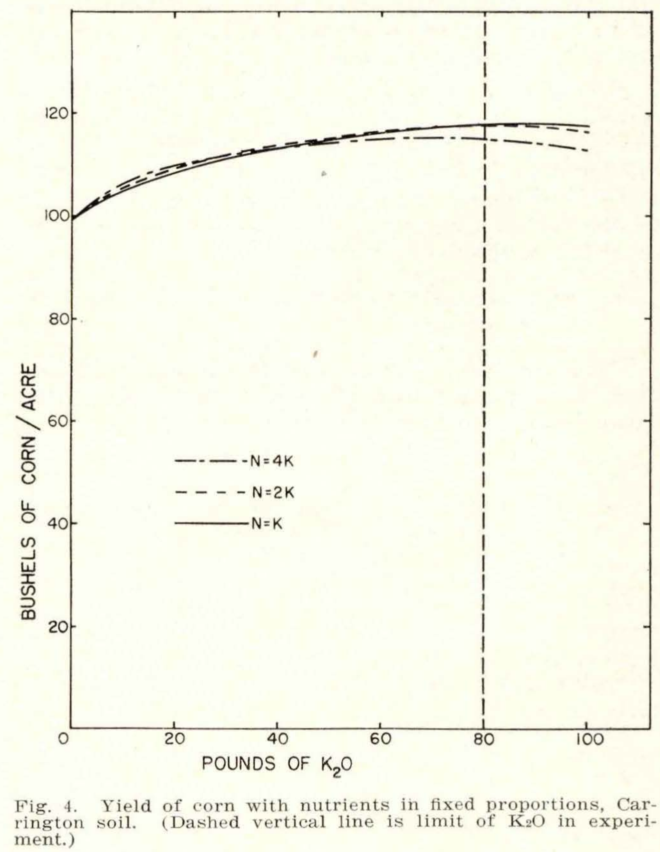


Fig. 4. Yield of corn with nutrients in fixed proportions, Carrington soil. (Dashed vertical line is limit of K_2O in experiment.)

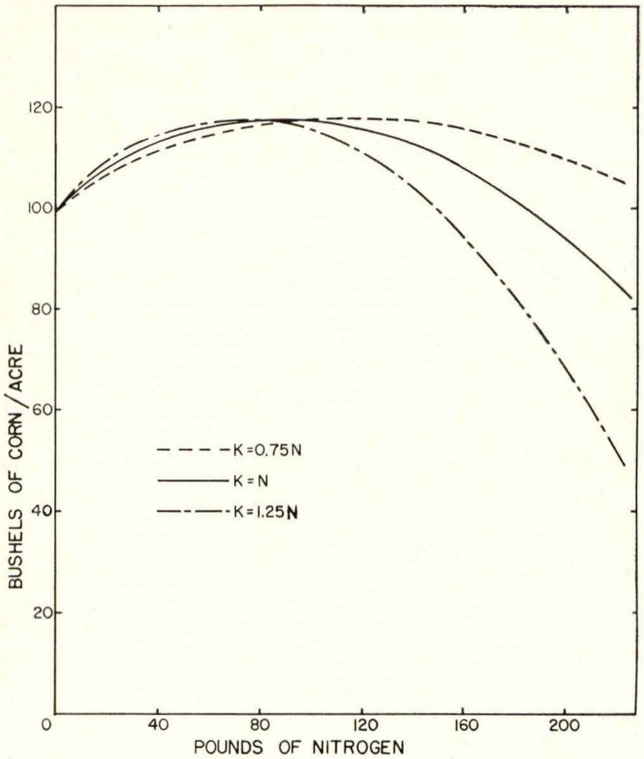


Fig. 5. Yield of corn with nutrients in fixed proportions, Carrington soil.

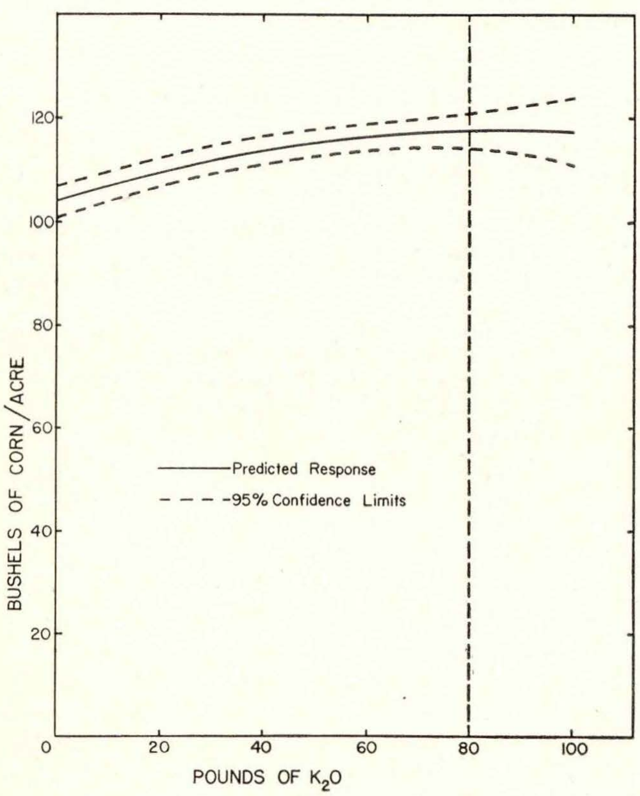


Fig. 6. Confidence limits for corn response to K_2O at 104 pounds of N, Carrington soil. (Dashed vertical line is limit of K_2O in experiment.)

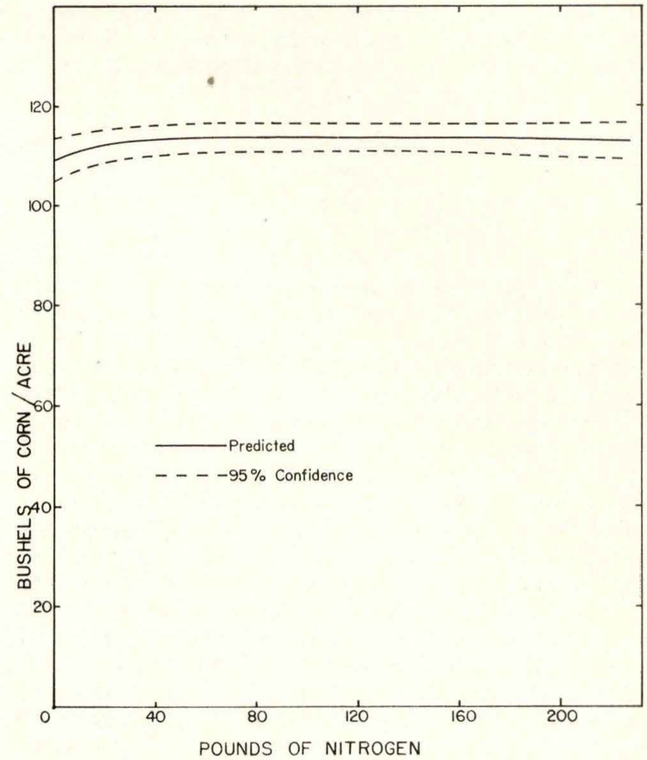


Fig. 7. Confidence limits for corn response to N at 40 pounds of K_2O, Carrington soil.

Figure 3 shows that the total increase in yield from N is less than half that for K_2O in fig. 2.

Predicted input-output or response curves when N and K_2O are held in fixed proportions are given in figs. 4 and 5. For the fixed ratio of nutrients in fig. 4, N is equal to one, two and four times the quantity of K_2O. The small effect of N on yield is indicated by the lack of spread of the three curves. Greater proportions of N cause a slight increase in yield at first, then a small decline at heavier inputs. Larger proportions of K_2O in fig. 5 have more effect because the initial increase is greater where K is 1.25 N. However, yield also declines more rapidly. Part of the large decline at heavier inputs of K_2O is probably due to extrapolations beyond the 80-pound K_2O limits of the experiment. If experimental inputs of K_2O had been extended to 160 pounds, a better estimate could have been made for the K_2O response.

Figure 6 shows the 95-percent confidence limits of the yield estimates for K_2O. The spread at the ends of the curve is due to the increased distance from the mean, as the response is extrapolated beyond the 80-pound limit of K_2O application in the experiment. Confidence intervals for the N response in fig. 7 are also relatively narrow, indicating some degree of precision in estimation.

Marginal physical products of N remain the same at all levels of K_2O because there is no interaction between N and K_2O in equation (4). Conversely, the marginal physical product of K_2O is not affected by the level of N. The marginal physical products of K_2O represented in equation (5) and of N in equation (6) were derived from the production function, (4). The partial derivative of yield was taken with respect to K to obtain equation (5) and with respect to N to obtain equation (6).

$$\frac{\partial \hat{Y}}{\partial K} = 0.3162 - 0.003626K \qquad (5)$$

$$\frac{\partial \hat{Y}}{\partial N} = -0.04453 + \frac{0.4595}{\sqrt{N}} \qquad (6)$$

The numerical values of the marginal products or yields from K (bushels per pound of K_2O) can be computed directly from equation (5). By inspecting equations (5) and (6) it can be seen that yield increases become smaller and smaller as fertilizer application is increased. The marginal yields from K correspond to the slope or incline of the "roof" in fig. 1 parallel to the N axis. At 0, 20, 40, 60, 80 and 100 pounds of K_2O, the marginal yields are 0.32, 0.24, 0.17, 0.10, 0.03 and —0.05 bushel, respectively. Similarly, marginal yields for N are computed from equation (6); for 1, 20, 40, 60, 80 and 120 pounds of N, the marginal products are 0.41, 0.06, 0.03, 0.01, 0.007, 0.001 and —0.003 bushel, respectively. It can be seen from the N marginal yields that while N returns a fairly large increase in yield at small inputs the response soon levels out. Negative marginal products for either N or K indicate that further inputs at the particular levels cause a decline in total per-acre yield.

YIELD ISOQUANTS FOR CARRINGTON SOIL

Yield isoquants in fig. 8 are another aspect of the basic yield surface. The general isoquant equation, (7), was derived from the production function, (4).

$$K = 87.23 \pm \frac{\sqrt{0.00666\sqrt{N} - 0.000323N + 0.8194 - 0.00725\hat{Y}}}{-0.003626} \qquad (7)$$

The isoquant curves in fig. 8 were computed from equation (7). The isoquant curves show the various combinations of N and K_2O which can be used to produce yields of 104, 107, 110, 113 and 116 bushels of corn per acre. As yields are increased by 3 bushels per acre, increasingly greater inputs of N and K_2O are required. The slopes of the isoquants show the change in amount of nitrogen required to maintain a given yield when another unit of potash is added. The substitution or "replacement" rates of N for K_2O are predicted to change since the isoquants in fig. 8 are curved.

Changes in substitution or replacement rates are shown in table 6 for yield isoquants of 104 and 113 bushels. At 13.37 pounds of K_2O and 1 pound of N for the 104-bushel yield, one small added unit of K_2O would replace only 0.61 unit of N in production. However, as N is increased to 10 pounds and K_2O is reduced to 7.66 pounds, one small added unit of K_2O would replace 2.57 units of N. The marginal rates of substitution of K_2O

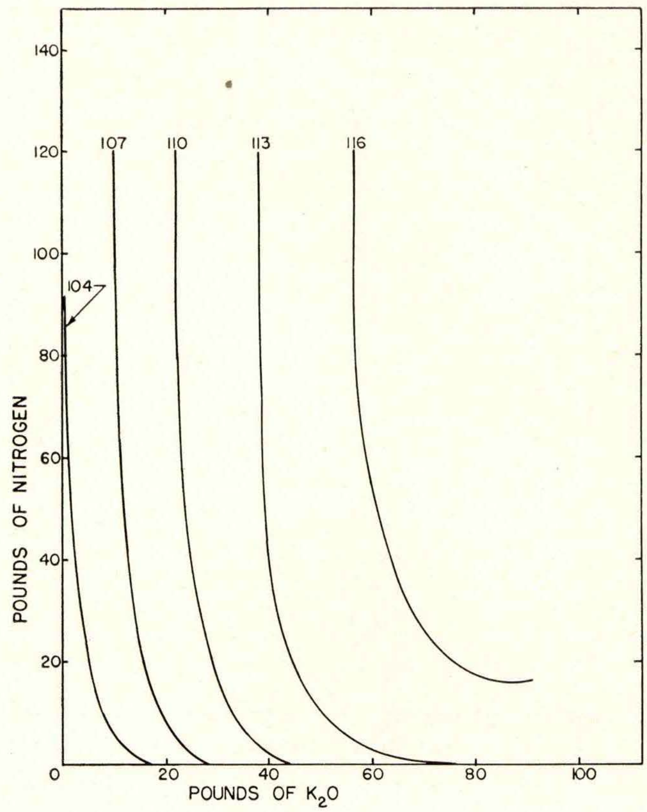


Fig. 8. Yield isoquants for corn on Carrington soil.

for N in table 6 correspond to the slopes of the isoquants in fig. 8.

YIELD ISOCLINES FOR CARRINGTON SOIL

Yield isoclines are directly related to isoquants: A particular isocline intersects all isoquants at points where the isoquants have the same given slope. For example, the middle isocline labeled $P_k = P_n$ in fig. 9 intersects all isoquants where their slope, dN/dK, is equal to 1.0. Along a par-

TABLE 6. COMBINATIONS OF NUTRIENTS TO PRODUCE SPECIFIED YIELDS PER ACRE AND CORRESPONDING MARGINAL RATES OF SUBSTITUTION (MRS), CARRINGTON SOIL.

104 bushels*			113 bushels†		
Lbs. of N	Lbs. of K_2O	MRS of K for N, dN/dK‡	Lbs. of N	Lbs. of K_2O	MRS of K for N, dN/dK‡
1	13.37	—0.61	1	65.08	—0.19
10	7.66	—2.57	10	50.27	—1.19
20	6.07	—4.49	20	44.99	—2.31
40	2.39	—9.10	40	39.98	—5.07
60	1.05	—16.07	60	37.62	—9.25
80	0.38	—28.98	80	36.46	—16.94
100	0.13	—222.84	100	36.03	—130.99

*Increase in yield from fertilizer is 4.78 bushels at a total yield of 104 bushels.

†Increase in yield from fertilizer is 13.78 bushels.

‡Change in N required to maintain yield when one unit of K_2O is added.

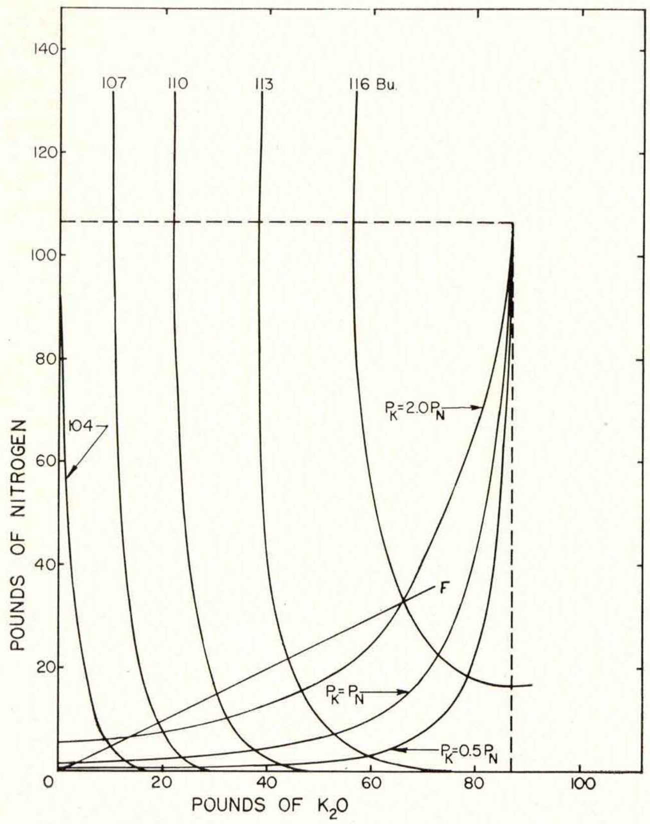


Fig. 9. Isoquants and isoclines with dashed ridgelines, Carrington soil.

ticular isocline, nitrogen and potash replace each other in production at a constant rate. Therefore, if N costs twice as much per pound as K_2O, that isocline should be chosen where 2 pounds of K_2O replace 1 pound of N. It represents the least-cost expansion path, indicating changes in nutrient ratios necessary to give maximum profits as higher yields are attained. Along a particular isocline, the marginal rate of substitution or "replacement rate" for nutrients corresponds to the price ratio of K_2O to N. Thus, the isoclines in fig. 9 can be thought of as the optimum fertilizer ratio curves for the specified prices of N and K_2O.

Isoclines, like isoquants, are derived from the basic production function. The isoclines in fig. 9 were computed from equation (8). Equation (8) was derived from equation (4) by dividing the partial derivative of yield with respect to K by the partial derivative of yield with respect to N and setting this ratio or equation equal to the K_2O/N price ratio, α. Then, K was expressed as a function of N.

$$K = 87.20 + 12.28\alpha - \frac{126.72}{\sqrt{N}}\alpha. \tag{8}$$

Under existing potash-nitrogen price relationships, nitrogen costs about twice as much per pound as K_2O. The appropriate isocline in fig. 9 then is the bottom curve labeled $P_K = 0.5 P_N$ For this price relationship, very little N would be used until almost 60 pounds of K_2O are applied. Beyond 60 pounds of K_2O per acre, the ratio of N to K_2O should be increased sharply, if production is to be expanded beyond 113 bushels per acre.

The dashed lines in fig. 9 represent ridgelines which denote the economic limits of the isoclines. The ridgelines define the portion of the production surface included between the extremes of zero (or infinite) substitution rates for nutrients. (In other words, they are isoclines with zero substitution ratios, indicating the extreme limits of nutrient substitution in obtaining specified yields). The ridgelines (isoclines of zero substitution rates) indicate the boundaries of the surface with positive slopes along both input axes; beyond the "ridges," one or both slopes are negative. If nitrogen were "free" in price but K_2O were not, it would pay to expand production along the top ridgeline, always applying 106 pounds of N and purchasing K_2O according to its cost and return. On the other hand if potash were "free" and nitrogen were not, production should be expanded along the right hand vertical ridgeline. Since N and K_2O were independent in basic surface equation (4), the ridgelines are straight and meet at a right angle. However, where nutrients interact (as in the two experiments presented later) the ridgelines have different characteristics.

All the isoclines (including ridgelines) converge and intersect at the point of maximum physical product. If both N and K_2O were free and cost nothing to apply, inputs should be extended to 87.2 pounds of K_2O and 106.5 pounds of N, the point of isocline convergence. A maximum physical yield of 117.76 bushels is predicted from these inputs of N and K_2O.

Economic Optima for Carrington Soil

Isoclines derived from the basic production function (4) provide the optimum combination of N and K_2O for any yield level. The point of intersection of the appropriate isocline with a specified isoquant in fig. 9 gives the optimum combination of N and K_2O for the given yield. The inputs of N and K_2O which minimize nutrient costs for specified yields are presented in table 7. For current price conditions, where N costs twice as much per pound as K_2O, the indicated amounts

TABLE 7. COMBINATIONS OF NITROGEN AND K_2O TO MINIMIZE FERTILIZER COSTS AT SPECIFIED YIELDS FOR DIFFERENT PRICE RATIOS, CARRINGTON SOIL.

Yield level (bu. per acre)	Price of N	Price of K_2O	N/K price ratio	Optimum lbs. N	Optimum lbs. K_2O
104	\$0.16	\$0.08	2.0	1	16
110	0.16	0.08	2.0	2	42
116	0.16	0.08	2.0	18	78
104	0.10	0.10	1.0	2	13
110	0.10	0.10	1.0	4	38
116	0.10	0.10	1.0	22	73
104	0.08	0.16	0.5	6	9
110	0.08	0.16	0.5	10	32
116	0.08	0.16	0.5	33	66

of N are small, except for the high yield of 116 bushels. When K_2O is assumed to be twice as expensive per pound as N, the optimum amounts of N are increased. For a yield of 116 bushels, 33 pounds of N would be used with 66 pounds of K_2O. Isoclines are optimum fertilizer ratio curves, but the relationship of fertilizer cost to crop price must also be considered to determine the most profitable rate of fertilizer application. The optimum level of application (and at the same time the optimum combination of nutrients) can be obtained by setting the partial derivatives of the production function, with respect to the various nutrients, equal to the respective nutrient-product price ratios. Optimum nutrient inputs in table 8 were computed by equating the marginal physical products (the partial derivatives of Y with respect to K and N) with their respective factor-product price ratios. For potash, the optimum input for the first price situation was obtained from equation (9).

$$0.3162 - 0.003625K = \frac{0.08}{2.00} \tag{9}$$

Solving equation (9) for K, an optimum input of 76.2 pounds is indicated. Similarily, the optimum input of N is found to be 14.8 pounds from equation (10). Because $N \times K$ interaction was lacking in the production function, (4), N does not appear in the partial derivative of yield with respect to K. Conversely, K does not appear in the marginal physical product of N (partial derivative of yield with respect to N). Consequently, the optimum inputs of N can be found independently of K and vice versa.

$$\frac{0.4595}{\sqrt{N}} - 0.04453 = \frac{0.15}{2.00} \tag{10}$$

By setting the inputs of N and K_2O of the basic equation (4) equal to the optima specified by equations (9) and (10), a predicted yield of 115.7 bushels is obtained. The optimum yield of 115.7 bushels exceeds the yield without fertilizer by 16.4 bushels. This additional yield at $1.50 per bushel has a value $16.59 greater than the $8.31 cost of the fertilizer. Thus, a return over cost of fertilizer of more than 100 percent appears possible under present price relationships for farmers with Carrington soil comparable to this experimental plot. This is true even though level of application is taken to the point where the last pound of fertilizer just pays for itself (i. e., marginal cost equals marginal revenue). At lower levels of fertilization, return per dollar of fertilizer would be even greater (because of diminishing marginal physical productivity of both nutrients). A more conservative gain in profit from fertilizer is given in the second price situation of table 8 where corn is priced at $1.00 per bushel, with N at $0.15 and K_2O at $0.08 per pound.

TABLE 8. OPTIMUM INPUTS OF FERTILIZER AND PREDICTED YIELDS OF CORN UNDER VARIOUS NITROGEN, POTASH AND CORN PRICE SITUATIONS; CARRINGTON EXPERIMENT, 1953.

Price of corn per bu.	Price of N per lb.	Price of K_2O per lb.	Optimum inputs in lbs. per acre		Predicted yield per acre	Gain from fertilizer*
			N	K_2O		
$2.00	$0.15	$0.08	14.8	76.2	115.7	$24.58
1.00	0.15	0.08	5.6	65.2	114.1	8.78
0.50	0.15	0.08	1.8	42.1	110.6	1.99
2.00	0.10	0.10	23.6	73.4	116.1	24.02
1.00	0.10	0.10	10.1	59.6	114.1	7.91
0.50	0.10	0.10	3.5	32.1	109.1	1.36
2.00	0.20	0.05	10.1	80.3	115.4	23.31
1.00	0.20	0.05	3.5	73.4	114.2	7.32
0.50	0.20	0.05	1.1	59.6	112.5	0.59

*Computed by multiplying increase in yield from use of fertilizer times price of corn and subtracting cost of fertilizer.

If technological progress should reduce the price of nitrogen to $0.10 per pound and the price of K_2O should rise to $0.10 per pound, the fourth, fifth or sixth price situation might be appropriate. In such an event, application of N should almost be doubled, whereas inputs of K_2O should be reduced slightly. For corn prices of $0.50 per bushel, figures in table 8 indicate that only a small return could be made from the use of fertilizer under the given prices of N and K_2O. Under such price conditions and with presence of risk and capital rationing, farmers probably would not apply any fertilizer. Some tenant farmers who bear all fertilizer costs and receive only half the crop might rationally refrain from fertilizer application, even under present price conditions.[6]

Another factor which would need to be considered by farmers is the greater uncertainty associated with the nitrogen response in this experiment. Values of t for N in table 3 are much smaller than for K_2O (i. e., the standard errors were larger). Therefore, a farmer who is short on capital (or who dislikes taking a chance of getting no return) would be more "sure" of profit by investing in potash.

If the basic estimating equation, (4), is assumed to be accurate, what would be the cost of "not bothering" to apply nitrogen? The loss in revenue can easily be computed by obtaining the predicted yield with only the optimum K_2O input. If corn is $1.00 per bushel and N is $0.15 and K_2O is $0.08 per pound, the gain from use of fertilizer in table 8 is $8.78 per acre. By using 65.2 pounds of K_2O and no N, a yield of 112.1 bushels is obtained or a loss of about 2 bushels over the "complete" optimum yield. However, the cost of 5.6 pounds of N per acre is saved so that the net loss from not using nitrogen is only about $1.08 per acre. While N results in a relatively unimportant response, as compared to K_2O, farmers might still use it in their hill or row fertilizer.

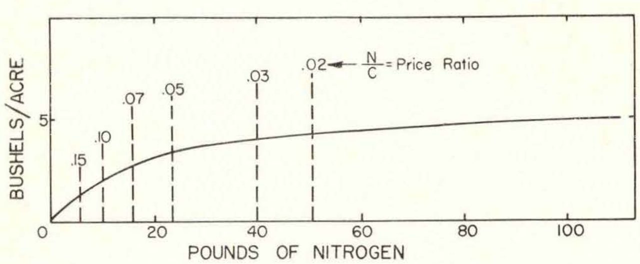
The extent to which profits are enhanced by following an isocline, rather than a fixed nutrient ratio, as level of yield is increased depends on the curvature of the isoclines and isoquants. One problem in agronomic and economic research is to determine yield surfaces where, for practical purposes, the same fixed nutrient ratios should or should not be recommended for farmers with

[6] All references to appropriate farmer action assume that the farmer has the same soil type and fertility conditions as for the experiment being discussed.

different capital levels (and who can attain different yield levels). In fig. 9, the line labeled F indicates the path of increasing yield when the ratio of nutrients is held fixed, with 2 pounds of K_2O for each pound of N (i. e., a fertilizer mixture such as 10-0-20). If this fixed ratio path were used to attain a yield of 113 bushels, with N at $0.16 and K_2O at $0.08 per pound, the cost of the nutrients would be about $7.04; with expansion along the $P_k = 0.5P_n$ isocline (where the 1 pound of K_2O substitutes for 0.5 pound of N), the cost of the 113 bushels (i. e., a 13.78-bushel increase) is $5.44. The difference would be smaller for a yield of 116 bushels. Differences, measured in the manner above, could be larger or smaller for other prices, other price and nutrient ratios and surfaces with more or less curvature in isoclines and isoquants.

Carrington Soil Presentation for Practical Use

While the main purpose of this study is that of dealing with certain basic or methodological aspects of fertilizer response and economics, it is useful to indicate how the results can be presented for farmers or extension personnel. Since N and K_2O effects were independent in the production function (9), the optimum rate for N can be selected without regard to the level of K_2O, and vice versa. To find the optimum input of either nutrient, divide the price (per pound) of the nutrient by the price of corn. Selection of



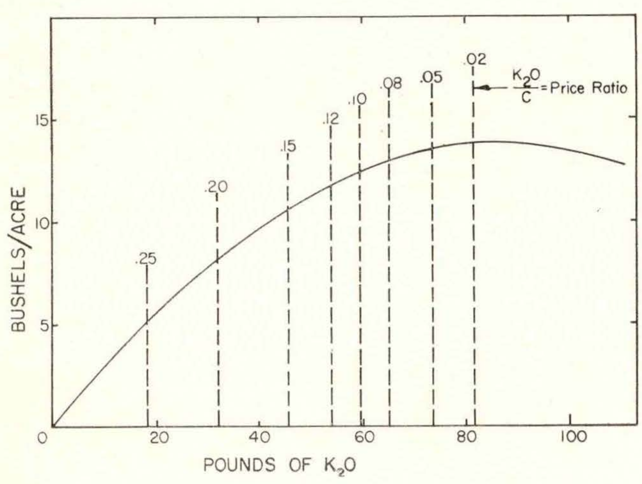


Fig. 10. Added bushels from K_2O and N and optimum rates for specified price ratios of fertilizer nutrients and corn, Carrington soil.

the corresponding ratio from one of the charts in fig. 10 then provides the optimum input.

Assuming N to be $0.15 and K_2O to be $0.08 per pound with corn at $1.00 per bushel, the appropriate N/C price ratio is 0.15 and the appropriate K/C price ratio is 0.08. Use of these ratios in fig. 10 indicates an optimum input of about 65 pounds of K_2O and 6 pounds of N. The gain in yield from these inputs also can be estimated. A gain of about 13 bushels per acre from use of K_2O and about 1.5 bushels from the N application is predicted. Of course, such a chart should be used only for a Carrington soil with fertility similar to the experimental field. Rainfall and biological conditions also would need to be as favorable as for the experimental field in 1953.

MOODY EXPERIMENTAL DATA[7]

The cropping history and soil tests indicated low availability of nitrogen and phosphorus, and high availability of potassium on this experimental plot. Large responses in corn yields were obtained by adding nitrogen; in fact, yield was more than doubled by applying 40 pounds of N (table 9). Further increases in yield were given by 80 and 160 pounds of N. However, with 240 pounds of N, a slight decline resulted. Potassium had little effect on yield. Phosphorus also seemed to have only a small effect since yield was increased by less than 8 bushels in rates ranging from 0 to 120 pounds. However, examination of the average response to P_2O_5 over all levels of N and K_2O hides part of the actual effect. Actually, as careful examination of the yields in table 9 indicates, there was a strong interaction between phosphorus and nitrogen. At zero level of N, P_2O_5 had a depressing effect on yield, but at 160 and 240 pounds of N it increased yield.

The analysis of variance in table 10 confirms the highly significant effect of nitrogen. The effect of P_2O_5 was significant at the 0.05 probability level. Interaction between N and P_2O_5 was highly significant. There also was a significant difference between the yield levels of the two ran-

[7] Percy Bylsma, farmer-cooperator, Sioux County, Iowa.

TABLE 9. AVERAGE CORN YIELDS PER ACRE IN 1953 FOR 60 FERTILIZER TREATMENTS ON MOODY SOIL.*

Lbs. P_2O_5	Lbs. K_2O	Pounds of nitrogen				
		0	40	80	160	240
0	0	26.05	56.70	65.25	63.70	51.30
0	40	32.00	49.85	65.50	68.25	61.20
0	80	26.45	48.70	76.45	62.65	53.95
40	0	16.65	52.80	59.30	75.25	76.35
40	40	32.25	49.55	61.55	74.75	66.90
40	80	24.80	52.90	52.85	76.40	69.15
80	0	23.40	46.50	60.40	72.65	83.15
80	40	23.25	48.55	62.65	88.15	78.45
80	80	23.25	49.85	76.45	71.95	77.00
120	0	26.20	50.60	58.95	78.20	82.00
120	40	20.20	50.75	69.80	86.80	81.90
120	80	22.45	56.20	70.25	90.00	81.15

*Each entry is the average of two observations, one from each randomized block.

TABLE 10. ANALYSIS OF VARIANCE OF CORN YIELDS ON MOODY SOIL, RANDOMIZED BLOCK DESIGN.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	117	53,164.96		
Blocks	1	432.82	432.82	4.57*
Treatments	59	47,330.08	802.20	8.46**
N	4	41,000.49	10,250.12	108.15**
P	3	1,051.22	350.41	3.70*
K	2	123.05	61.53	0.65
N x P	12	3,333.88	277.82	2.93**
N x K	8	455.83	56.98	0.60
P x K	6	99.55	16.59	0.18
N x P x K	24	1,266.06	52.75	0.56
Error	57	5,402.06	94.77	

*$P \leq 0.05$
**$P \leq 0.01$

domized blocks. Potash and the remaining interactions accounted for no significant portion of yield variance. The lack of potash response was expected since the experimental plots tested high in K. Since the soil test for P was low, negative phosphorus response at low levels of N in table 9 was unexpected. However, other evidence suggests that this can happen as a result of aggravating the nitrogen deficiency.

Regression Analysis for Moody Soil

Several algebraic forms of the yield predicting equation were tried before equation (11) was selected. Equation (11) had an R^2 of 0.827 and was logically more acceptable than certain other forms; it gave diminishing returns to fertilizer application (predicted gains in yield become smaller and smaller as equal increments of a fertilizer mix are added). Diminishing returns is a generally accepted condition for the fertilizer-yield function.

$$\hat{Y} = 13.543 + 0.5340N - 0.001743N^2 - 0.0003549P^2 + 0.001069NP + 0.000873S \quad (11)$$

In equation (11), $\hat{Y}$ again refers to total yield in bushels per acre, N refers to pounds of nitrogen per acre, P refers to pounds of P_2O_5 per acre and S refers to stalks per acre. Values of t for the coefficients in the order that they appear in equation (11) are 12.56, 14.47, 1.68, 5.44 and 1.50. The preceding t values for N, N^2 and NP are significant at the 0.00001 level of probability. The terms for P^2 and S are significant at 0.10 and 0.14 probability levels, respectively, and are retained for logical reasons. The negative P^2 term is important because it "forces" diminishing returns to fertilizer inputs. Some of the functions fitted to the data, or the particular function without this term, did not have this characteristic. For example, the full five-term square root or regular quadratic functions gave increasing returns for part of the production surface; increasing returns make it difficult to secure determinate economic solutions. The five-term quadratic equation was as follows:

$$Y = 30.277 + 0.533N - 0.00175N^2 - 0.623P + 0.000066P^2 + 0.00116NP \quad (12)$$

Equation (12)* was rejected in favor of equation (11) since (11) gave diminishing returns and a determinate predicted maximum yield. The stand variable was included in equation (11) to increase the precision of fit of the nutrient response curves; equation (13) has been adjusted to an average stand and was used for the subsequent economic analysis. Equation (13) is the same as equation (11) except that average plot stand is fixed at 18,000 stalks per acre. With the coefficient for S significant at the 0.14 probability level, the writers adopted this procedure as being more efficient than the conventional procedure of adjusting individual plot yields for stand.

$$\hat{Y} = 29.248 + 0.5340N - 0.001743N^2 - 0.0003549P^2 + 0.001069NP \quad (13)$$

The analysis of variance of the basic regression, equation (11), is given in table 11. The F value of 91.20 for the over-all regression is highly significant. The mean square for deviations from regression is smaller than the within-plot estimate of experimental error.

Production Surface for Moody Soil

Estimated yields in table 12 were predicted from equation (11). These yields correspond to points on the corn production surface for the Moody soil. The predicted yields parallel the original yield observations in table 9 in the respect that yields tend to decline with inputs of phosphorus for a zero level of nitrogen. With higher levels of N, application of phosphorus results in predicted yield increases.

The interaction of nitrogen and phosphorus can best be seen from the surface drawing in fig. 11. Yields increase sharply as nitrogen is applied at the zero level of phosphorus. However, even higher yields are obtained from N as P_2O_5 is increased. Yield at zero level of P_2O_5 but for different rates of N is represented by the edge of the surface directly above the nitrogen axis. A second line over the surface parallels the first and shows yield response to N at 40 pounds of P_2O_5. Thus, the strong positive interaction or comple-

TABLE 11. ANALYSIS OF VARIANCE FOR REGRESSION OF CORN YIELD ON MOODY SOIL.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	117	53,164.96		
Blocks	1	432.82	432.82	4.49*
Due to regression, equation (11)	5	43,991.51	8,798.30	91.20**
Deviations from regression	111	9,173.45	82.64	
Other treatment effects	55	3,771.39	68.57	
Error	56	5,402.06	96.47	

*$P \leq 0.05$
**$P \leq 0.01$

TABLE 12. PREDICTED PER-ACRE YIELDS OF CORN FOR SPECIFIED NUTRIENT COMBINATIONS APPLIED ON MOODY SOIL.

Lbs. P_2O_5	Pounds nitrogen per acre						
	0	40	80	120	160	200	240
0	29.2	47.8	60.8	68.2	70.1	66.3	57.0
20	29.1	48.5	62.4	70.7	73.3	70.5	62.0
40	28.7	49.0	63.7	72.8	76.3	74.3	66.7
60	28.0	49.1	64.7	74.6	79.1	77.9	71.1
80	27.0	49.0	65.4	76.2	81.5	81.2	75.3
100	25.7	48.5	65.8	77.5	83.6	84.2	79.1
120	24.1	47.8	66.0	78.5	85.5	86.9	82.7

mentarity of N and P can be seen from the high center ridge of the surface at large inputs of N and P.

Marginal physical products corresponding to the total yields in table 12 are given in table 13; they are the counterparts of the slopes of the vertical slices through fig. 11. For example, at 40 pounds of both N and P in table 13, the marginal product for P is 0.01. Thus, the "incline" or slope of the surface parallel to the P axis in fig. 11 is nearly level at this point. At the 40-pound combination of N and P in table 12, yields "leveled out" for small additional increases in P_2O_5. At the heavier rates of N in table 13, marginal products are larger at the higher levels of P_2O_5. These figures again illustrate that the marginal productivity of one nutrient depends on the amount of the other with which it is combined. Negative marginal products in table 13 indicate a diminishing total yield from further inputs of fertilizer.

Vertical slices through the surface parallel to the phosphorus axis in fig. 11 are equivalent to the N response of corn at fixed levels of P_2O_5. The corn response to nitrogen with no phosphorus application is considerably below the N responses at 40 and 120 pounds of P_2O_5 in fig. 12. Further-

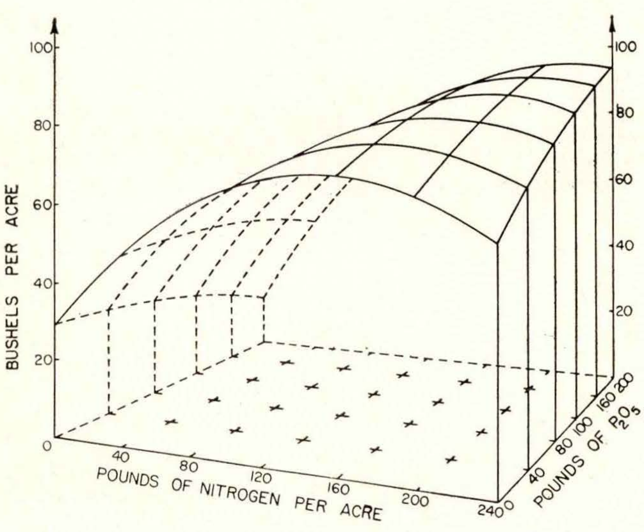


Fig. 11. Perspective view of predicted yield surface for corn on Moody soil.

TABLE 13. MARGINAL PRODUCT OR YIELD (BUSHELS PER POUND OF FERTILIZER NUTRIENT) FOR COMBINATIONS INDICATED IN ROWS AND COLUMNS. UPPER FIGURES ARE FOR NITROGEN, LOWER FIGURES FOR P_2O_5, MOODY SOIL.

Lbs. P_2O_5 per acre	Pounds nitrogen per acre						
	0	40	80	120	160	200	240
0	0.53 0.00	0.39 0.04	0.26 0.09	0.12 0.13	—0.02 0.17	—0.16 0.21	—0.30 0.26
20	0.56 —0.01	0.42 0.03	0.28 0.07	0.14 0.11	—0.00 0.16	—0.14 0.20	—0.28 0.24
40	0.58 —0.03	0.44 0.01	0.30 0.06	0.16 0.10	0.02 0.14	—0.12 0.19	—0.26 0.23
60	0.60 —0.04	0.46 0.00	0.32 0.04	0.18 0.09	0.04 0.13	—0.10 0.17	—0.24 0.21
80	0.62 —0.06	0.48 —0.01	0.34 0.03	0.20 0.07	0.06 0.11	—0.08 0.16	—0.22 0.20
100	0.64 —0.07	0.50 —0.03	0.36 0.01	0.22 0.06	0.08 0.10	—0.06 0.14	—0.20 0.19
120	0.66 —0.09	0.52 —0.04	0.38 0.00	0.24 0.04	0.10 0.09	—0.03 0.13	—0.17 0.17

more, the maximum yield on the nitrogen response curve comes at higher levels of N as more P is applied because of the positive N $\times$ P interaction term in equation (13). With a zero phosphorus application, the highest yield obtainable from nitrogen is about 70 bushels with 150 pounds of N. With 40 pounds of P_2O_5 per acre, over 76 bushels of corn are predicted from 170 pounds of

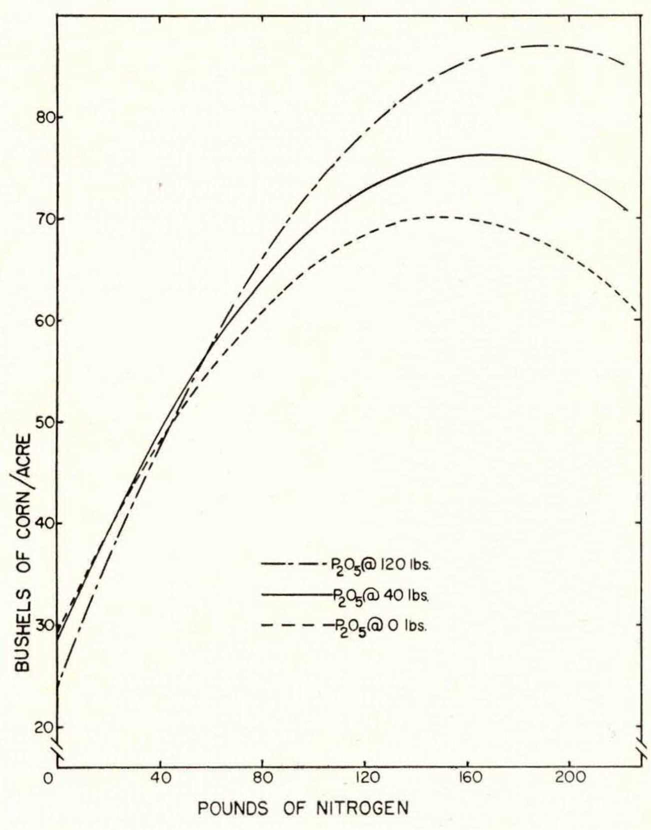


Fig. 12. Corn response to N at 0, 40 and 120 pounds of P_2O_5, Moody soil.

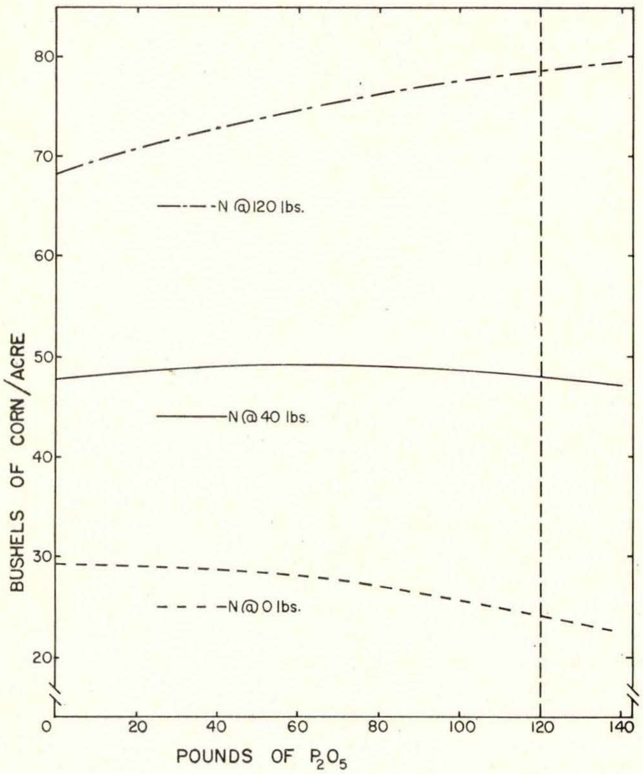


Fig. 13. Corn response to P_2O_5 at 0, 40 and 120 pounds of N, Moody soil. (Dashed vertical line is limit of P_2O_5 in experiment.)

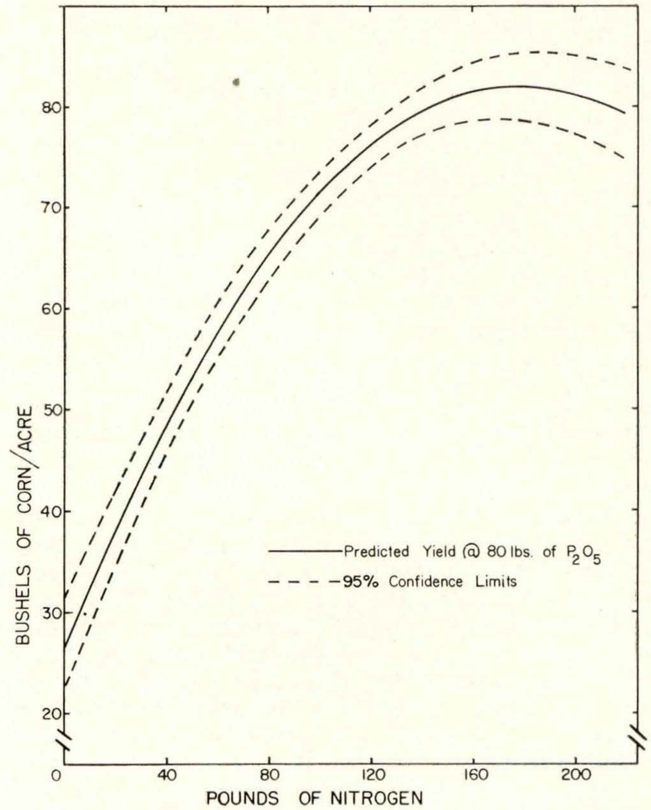


Fig. 15. Confidence limits of corn response to N at 80 pounds of P_2O_5, Moody soil.

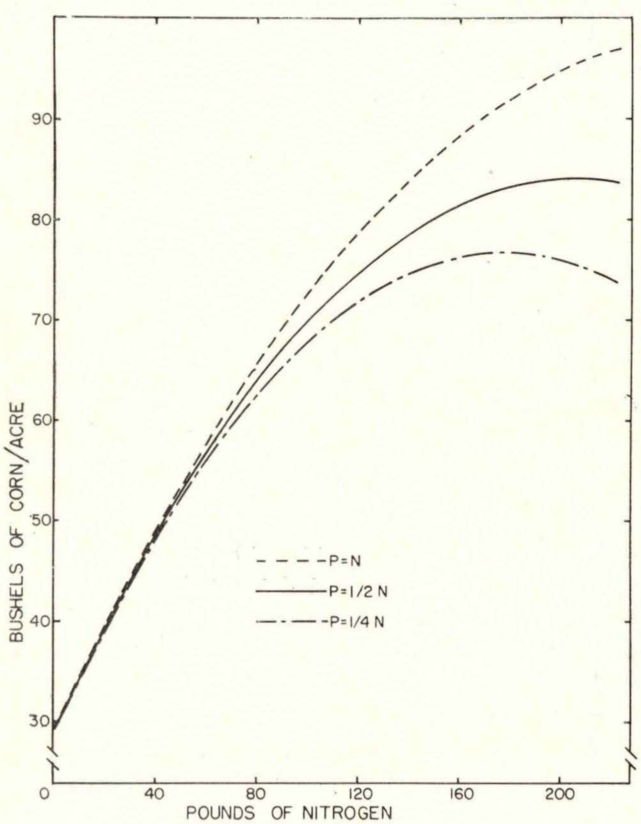


Fig. 14. Corn yields with nutrients in fixed proportions, Moody soil.

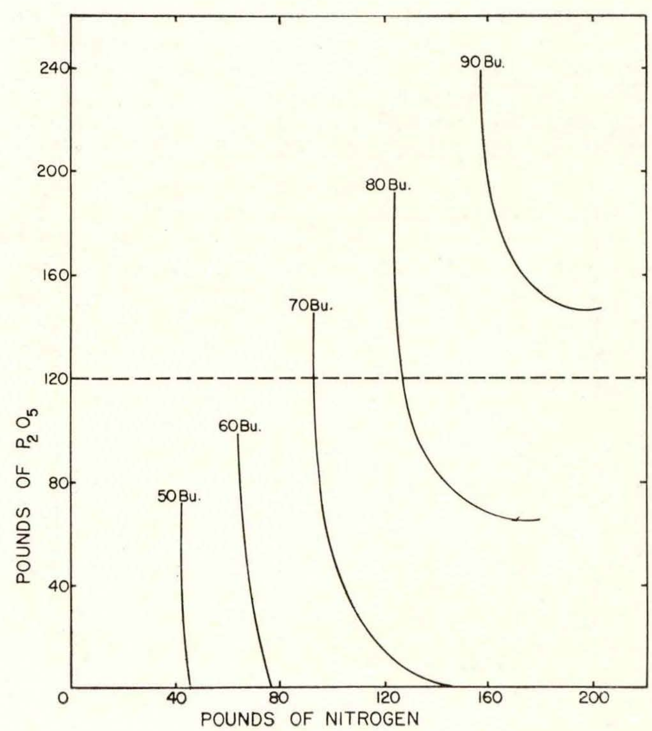


Fig. 16. Yield isoquants for corn on Moody soil. (Dashed horizontal line is limit of P_2O_5 in experiments.)

N. Approximately 87 bushels per acre are estimated at 190 pounds of N and 120 pounds of P_2O_5.

The predicted corn response to P_2O_5 is negative in fig. 13 when no nitrogen is applied. With 40 pounds of N, P_2O_5 gives little response. However, at 120 pounds of N, there is an increase of more than 10 bushels per acre from 120 pounds of P_2O_5. Thus, strong $N \times P$ interaction can be seen in both the N and P_2O_5 response curves.

Predicted input-output curves with N and P_2O_5 in fixed proportions in fig. 14 show that yields continue to increase at high levels of N when P_2O_5 is equal to N (i. e., a 1:1 ratio of the two nutrients). When P_2O_5 is applied at only half the N rate, yields start to decline with 200 pounds of N. However, greater uncertainty is involved in predictions of the surface for high inputs of phosphorus: The basic experimental rates for phosphorus went only as high as 120 pounds of P_2O_5.

Ninety-five-percent confidence limits of the yield response to nitrogen in fig. 15 are fairly narrow, especially from zero to 160 pounds of N. At 120 pounds of N the limits are closest, being only 2 bushels on either side of the mean predicted value. The limits then widen to 5.5 bushels at 240 pounds of N. Within the range of 40 to 150 pounds, representing economic application, the confidence limits are 3 bushels or less from the predicted values. However, the limits would be somewhat wider for P_2O_5 applications greater than 80 pounds.

YIELD ISOQUANTS FOR MOODY SOIL

In the yield isoquant equation (14), derived from the basic regression equation (13), P_2O_5 is expressed as a function of N. The yield isoquants in fig. 16 are based on equation (14).

$$P = \frac{1.506N \pm \sqrt{0.0007581N - 0.000001332N^2 + 0.0415 - 0.00142\hat{Y}}}{-0.0007098} \quad (14)$$

For yields as low as 50 or 60 bushels in fig. 16, isoquants are nearly vertical. These steep slopes for lower yields mean that many pounds of P_2O_5 are necessary to replace 1 pound of nitrogen in attaining the constant yield (or, practically, that added P_2O_5 does not substitute for N in attaining these yields when N input is low). As yield is increased to 70 bushels per acre and more N is used, the isoquant becomes more curved as it approaches the N axis. The 70-bushel isoquant intersects the N axis; 70 bushels per acre are predicted from the equation with all nitrogen and zero of P_2O_5. However, the 80-bushel isoquant requires P_2O_5 in addition to N; a yield this high requires the complementary effect of P_2O_5 with N.

Since the slopes of the isoquants in fig. 16 show the change in amount of P_2O_5 required to maintain a given yield when another unit of N is added, the curvatures of the isoquants indicate the change in the rate of substitution of N for P_2O_5. Substitution or replacement rates predicted in table 14 for the 70-bushel yield show that at 100 pounds of N, an additional unit of N replaces about 3.5 units of P_2O_5. However, at 140 lbs. of N, an additional pound of N replaces only about one-third of a pound of P_2O_5, if an 80-bushel yield is to be retained.

TABLE 14. COMBINATIONS OF NUTRIENTS TO PRODUCE SPECIFIED YIELDS PER ACRE AND CORRESPONDING MARGINAL RATES OF SUBSTITUTION (MRS), MOODY SOIL.

70 bushels*			80 bushels†		
Lbs. of N	Lbs. of P_2O_5	MRS of N for P, dP/dN‡	Lbs. of N	Lbs. of P_2O_5	MRS of N for P, dP/dN‡
100	54.64	—3.58	130	106.73	—3.09
110	28.89	—1.87	140	84.98	—1.53
120	14.36	—1.11	150	73.50	—0.83
130	5.74	—0.64	160	67.53	—0.39
140	1.02	—0.32	170	65.22	—0.08

*Increased yield from fertilizer is 40.75 bushels.
†Increased yield from fertilizer is 50.75 bushels.
‡Change in P_2O_5 required to maintain yield when unit of N is added. Predicted by computing the derivatives of equation (14) for the nutrient combinations shown.

Since the slopes of the isoquants in fig. 16 change along a scale line (fixed nutrient combination) the combination of nutrients or fertilizer ratio which gives lowest cost for one yield level is not the same fertilizer ratio which gives lowest cost for another yield level. For example, the least cost combination of N and P_2O_5 will not be the same for 70- and 80-bushel yields.

YIELD ISOCLINES FOR MOODY SOIL

Each isocline in fig. 17 intersects every isoquant at a point of specified slope on the isoquant. For example, the isocline labeled $P_n = 3.0P_p$ goes through each of the 70-, 80- and 90-bushel isoquants at points where the slope (i. e., the marginal rate of substitution) is 3:1. On the isocline labeled $P_n = 0.33P_p$, each isoquant is intersected where the slope is 1:3. On this isocline, each pound of P_2O_5 would replace 3 pounds of N. Therefore, if the price of N were one-third the price of P_2O_5 per pound, production should be expanded along the isocline labeled $P_n = 0.33P_p$, if the path of fertilizer ratios for least-cost yields is to be traced out.

Under current prices, the isocline labeled $P_n = 1.5P_p$ is the optimum fertilizer ratio line. This isocline starts at about 105 pounds on the N axis; it would be most profitable to apply 105 pounds of N before any P_2O_5 is used. Since 105 pounds of N results in a predicted yield of about 66 bushels per acre, any yield less than 66 bushels per acre could be obtained at lowest cost by using all N and no P_2O_5 (with $P_n = 1.5P_p$).

Isoclines and all other features of the production surface were derived from the basic yield estimating equation, (11). The equations of isoclines in fig. 17 were found by setting the ratio of the marginal physical products (partial deriva-

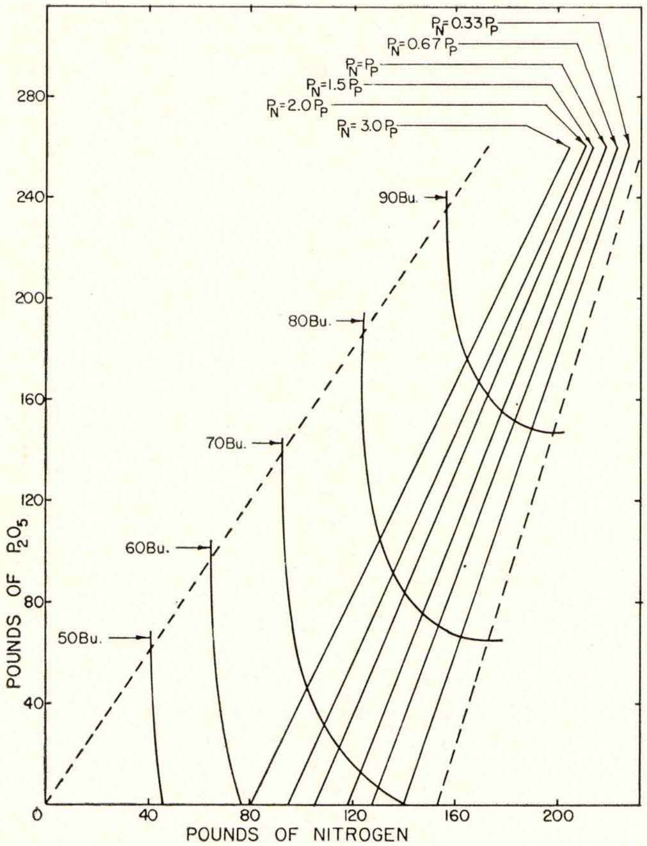


Fig. 17. Yield isoquants and isoclines with dashed ridgelines, Moody soil.

tives of yield) equal to the phosphorus-nitrogen price ratio. Letting a equal the P_2O_5/N price ratio, the isocline equation is (15).

$$P = \frac{(0.001069 + 0.003486a)N - 0.5340a}{0.001069a + 0.0007098} \quad (15)$$

The dotted isoclines in fig. 17 are those where the marginal rates of substitution, dN/dP at the upper end and dP/dN at the lower end are zero. Hence, they are ridgelines defining the technical limits of replacing one nutrient with the other to attain a given yield. They indicate the points on the isoquants where the two nutrient resources become technical complements. Isoquants become vertical along the upper ridgeline; along the lower ridgeline the isoquants are horizontal. It would never be more profitable to apply a fertilizer ratio falling outside the ridgelines than one falling within them, even if one or both nutrients were free.

All the isoclines (including the ridgelines) in fig. 17 are predicted to converge and intersect at 284.6 pounds of N and 428.4 pounds of P_2O_5. The intersection at these indicated inputs represents the one level and ratio of nutrients which results in a maximum physical yield (105.2 bushels). This last prediction must be "held with great uncertainty," however, because indicated input of P_2O_5 is far beyond the P_2O_5 levels in the experiment. If P_2O_5 had been applied at 0-, 50-, 100- and 200-pound intervals, the point of maximum physical product (and the level and ratio of nutrients which give it) could have been estimated with more certainty.

Economic Optima for Moody Soil

Isoclines in fig. 17 indicate the optimum combination of nutrients for given fertilizer prices. But, again, to determine where to stop on the isoclines (i. e., the rate of fertililization) the corn price also must be considered. For example, assume corn is $1.00 per bushel and the price of N is $0.15 and P_2O_5 is $0.10 per pound. Optimum inputs are found by setting the partial derivative of $\hat{Y}$ with respect to N in equation (13) equal to the nitrogen-corn price ratio and solving it simultaneously with the partial derivative of $\hat{Y}$ with respect to P in equation (17) set equal to the P_2O_5-corn price ratio.

$$0.5340 - 0.003486N + 0.001069P = \frac{0.15}{1.00} \quad (16)$$

$$-0.0007098P + 0.001069N = \frac{0.10}{1.00} \quad (17)$$

Solving equations (16) and (17), optimum inputs are predicted to include 124.4 pounds of N and 46.4 pounds of P_2O_5. For these inputs, a yield of 74 bushels is predicted from equation (13). In table 15 the value of net gain from using fertilizer has been computed. For the preceding price situation with corn at $1.00 per bushel, the value of the gain in yield ($44.86) less the cost of fertilizer ($23.30) leaves a net gain of $21.56 from using the optimum quantity of fertilizer. A farmer in this case would receive almost a $2 return per $1 expended on fertilizer, even though the return on the last $1 invested in fertilizer gives a return of just $1. If corn were $2.00 per bushel, the net gain would be increased to almost $82.00 per acre from the use of $54.42 of fertilizer per acre, or a return of $2.50 per $1.00 expended for fertilizer.

TABLE 15. OPTIMUM RATES AND COMBINATIONS OF FERTILIZER PER ACRE FOR SPECIFIED CROP AND NUTRIENT PRICES, MOODY SOIL.

Price per unit			Optimum inputs			
Corn per bu.	N per lb.	P_2O_5 per lb.	Lbs. N	Lbs. P_2O_5	Estimated yield	Net gain per acre from fertilizer*
$2.00	$0.15	$0.10	204.5	237.4	97.4	$81.98
1.00	0.15	0.10	124.4	46.4	74.1	21.56
0.50	0.15	0.10	67.1	0.0	57.2	3.93
2.00	0.20	0.10	191.2	217.4	95.3	72.09
1.00	0.20	0.10	97.7	6.3	65.4	16.01
0.50	0.20	0.10	38.4	0.0	47.2	1.29
2.00	0.10	0.10	217.8	257.5	99.3	92.54
1.00	0.10	0.10	151.0	86.5	81.4	28.45
0.50	0.10	0.10	95.8	0.0	64.4	8.00

*Computed by multiplying increase in yield from use of fertilizer times price of corn and subtracting cost of fertilizer.

It becomes more profitable to apply P_2O_5 as N becomes cheaper (table 15) because of strong complementarity or $N \times P$ interaction in the basic experiment. How economically important is the complementary effect of the P_2O_5? If corn is \$1.00 per bushel, N is \$0.15 per pound and P_2O_5 is \$0.10 per pound, the optimum solution where no P_2O_5 is used is given by equation (18). This equation is the partial derivative of $\hat{Y}$ with respect to N where P has been set equal to zero.

$$0.5340 - 0.003486N = \frac{0.15}{1.00} \qquad (18)$$

$$N = 110.2 \text{ lbs.}$$

Introducing $N = 110.2$ pounds back into equation (13), a yield of 66.93 bushels is predicted. Net gain from use of fertilizer is then \$21.15 per acre, or only \$0.40 less per acre than obtainable where both P_2O_5 and N were used. However, if N and P_2O_5 are both assumed to be \$0.10 per pound, with corn at \$1.00 per bushel in table 15, then optimum inputs of 151.0 pounds of N and 86.5 pounds of P_2O_5 result in a net gain of \$28.45 per acre from use of fertilizer. If P_2O_5 is not used, the optimum N input of 124.5 pounds results in a net gain of \$27.02, or \$1.43 less per acre than when P_2O_5 is used with N.

If corn is \$2.00 per bushel when N and P_2O_5 are \$0.10 per pound, optimum inputs of 217.8 pounds of N and 257.5 pounds of P_2O_5 result in a predicted net gain from fertilizer of \$92.54 per acre. If no P_2O_5 is used, 138.9 pounds of N is the optimum input. The resulting margin over fertilizer cost with no P_2O_5 is about \$25 per acre less than that obtainable through use of P_2O_5 with N. However, it should be remembered that 257.5 pounds of P_2O_5 is far beyond the 120-pound P_2O_5 limit of the experiment; large inputs of P_2O_5 would probably not be as profitable as indicated. It is concluded that P_2O_5 could be ignored for low or medium yields, but it appears profitable to include P_2O_5 if (1) product prices should be unusually high, (2) nitrogen prices should be low and (3) large amounts of N are applied and high yields are sought.

The relative advantage in equating substitution ratios to price ratios, in specifying the least-cost fertilizer ratio for a particular yield level, is greater for the Moody experiment than for the Carrington experiment. With a price of N 1.5 times the price of P_2O_5 (\$0.15 per pound for N and \$0.10 for P_2O_5), the least-cost ratio for an 80-bushel yield is roughly 88 pounds of P_2O_5 and 135 pounds of N (isocline $P_n = 1.5P_p$ in fig. 17) with a per-acre cost of \$29.05. If a "fixed" nutrient ratio of 1N:1P were used, the 80-bushel yield could be attained with approximately 125 pounds of N and 125 pounds of P_2O_5, with a per-acre cost of \$31.25—an increase of \$2.35 per acre over the optimum. If a "fixed" nutrient ratio of 2N:1P were used, approximately 148 pounds of N and 74 pounds of P_2O_5 would be required; the per-acre cost of \$29.60 would be only \$0.55 greater than for the optimum nutrient combination denoted along the isocline, $P_n = 1.5P_p$ in fig. 17. However, the difference in cost between use of a fixed ratio and an "isocline optimum" would be greater with lower yield levels. This is true because the distance between the points where (1) a straight line through the origin (denoting a "fixed" nutrient ratio) and (2) the relevant isocline intersect a given isoquant, increases for successively lower yield levels. For a 70-bushel yield level, the "isocline optimum," for the prices above ($P_n = \$0.15$, $P_p = 0.10$) includes 115 pounds of N and 25 pounds P_2O_5 for a total cost of \$19.75.

A 1N:1P fertilizer ratio would require about 95 pounds each of N and P_2O_5, with a per-acre cost of \$23.75, or a difference of \$4.00 from the "isocline optimum." Hence, it is again obvious that the relative gain in using an "isocline optimum" nutrient ratio, rather than a "fixed" ratio for all yield levels, depends on the slope of the isoclines and isoquants, the "fixed" ratios under consideration, the prices of the nutrients and the yield to be attained. The difference can be small for one of these situations, but large for another.

Moody Soil Presentation for Practical Use

Individual nutrient response curves again are used to determine optimum inputs of nutrients for this experiment. The optimum N input depends upon the level of P_2O_5 application and vice versa. It is possible to locate these simultaneously determined optimum inputs from charts such as fig. 18. As an example, assume corn to be \$1.00 per bushel, N to be \$0.15 per pound and P_2O_5 to be \$0.10 per pound. Since the price of N is 1.5 times the price of P_2O_5, the straight line (isocline), leading from the bottom axis to the upper right, labeled $P_n = 1.5P_p$ is chosen. This line gives the optimum $N:P_2O_5$ combination for all levels of production when N is 1.5 times as expensive as P_2O_5. To find how far to go on this line (isocline) it is necessary to determine the nitrogen-corn price ratio. In the above case, $P_n/P_c = 0.15$. Therefore, the line labeled $P_n = 1.5P_p$ is followed until the dashed line labeled 0.15 is reached. Then by dropping straight down from this point, a reading of about 124 pounds of N is obtained. Likewise, by reading straight to the left from the same point on the fertilizer ratio line, about 46 pounds of P_2O_5 are indicated. The approximate predicted corn yield for the optimum inputs of N and P_2O_5 can be estimated from the isoquants. For 124 pounds of N and 46 pounds of P_2O_5, a yield of about 74 bushels is indicated.

Nitrogen-corn price ratios in fig. 18 implicitly include P_2O_5-corn price relationships. The points of intersection of the dashed lines with the isoclines were found from the simultaneous optimum solutions such as given by equations (16) and (17).

It can be seen from fig. 18 that for nitrogen-

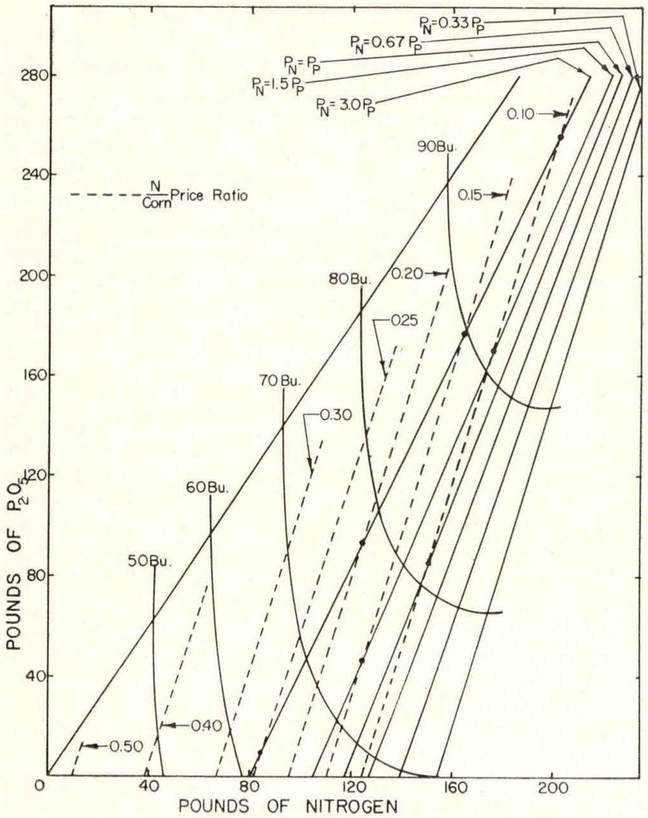


Fig. 18. Optimum inputs of N and P_2O_5 for various price ratios, Moody soil.

corn price ratios greater than 0.25, it would seldom pay to use P_2O_5. For example, if N is \$0.15 and P_2O_5 is \$0.10 per pound and corn drops to \$0.75 per bushel, the most profitable input is indicated by the intersection of the dashed line labeled 0.20 with the N axis at about 95 pounds of N and no P_2O_5. For indicated optimum inputs greater than 120 pounds of P_2O_5 in fig. 18, caution must be exercised since inputs of P_2O_5 did not exceed 120 pounds in the basic experiment.

HAYNIE EXPERIMENTAL DATA[8]

Yields of corn on a Haynie silt loam soil testing very low in N and P, and medium in K, are given in table 16 for three levels of N, P_2O_5 and K_2O application. Rates of application included 0, 40 and 80 pounds of each nutrient. Improved estimates of the N and P_2O_5 response would have resulted if N and P_2O_5 levels had extended higher. Average yields of the nitrogen responses in table 16 are 8 bushels for the last 40 pounds of N applied. Similarly, 80 pounds of P_2O_5 gave almost 5 bushels more corn per acre than did 40 pounds. To estimate the N $\times$ P interaction, nitrogen and phosphorus inputs should go far enough to cause a decline, or at least a leveling out, of total yield. Additional increases in yield might have been obtained at heavier N and P_2O_5 combinations.

The analysis of variance in table 17 emphasizes the level of N response; simple N effects accounted for over 60 percent of the total treatment sum of squares. Direct P_2O_5 response also was highly significant. Direct potash effects fell slightly short of the 0.05 probability level, although potash was later included in the yield estimating equation. One justification for retaining potash was the highly significant P $\times$ K interaction detected by analysis of variance in table 17. Significant N $\times$ P interaction might also have occurred if N and P inputs had gone higher; to detect positive interaction such as between N and P it seems necessary to extend inputs higher than in the case of negative interaction as between P and K.

TABLE 16. AVERAGE CORN YIELDS PER ACRE IN 1953 FOR 27 FERTILIZER TREATMENT COMBINATIONS ON HAYNIE SOIL.*

Lbs. P_2O_5	Lbs. K_2O	Pounds of nitrogen		
		0	40	80
0	0	43.30	55.25	54.60
0	40	44.95	57.70	71.45
0	80	48.10	75.80	78.50
40	0	57.90	73.85	83.10
40	40	49.50	80.55	82.85
40	80	47.00	65.85	76.45
80	0	47.60	75.40	78.25
80	40	58.50	83.50	99.35
80	80	60.30	71.25	86.65

*Each entry is the average of two observations, one from each randomized block.

Analysis of covariance indicated a highly significant effect of stand on yield. Similarly, when stand was included as a variable in the yield estimating equation, its *t* value was highly significant. The positive effect of stand might have been expected since stalk numbers averaged only 9,000 per acre. The low stand probably limited yields in table 16, especially at the heavier fertilizer rates.

Regression Analysis for Haynie Soil

Of several possible algebraic forms of the yield estimating regression tried, equation (19) was selected for predictions. Equation (19) fit the data best, with an R^2 of 0.778. It was selected

TABLE 17. ANALYSIS OF VARIANCE OF CORN YIELDS ON HAYNIE SOIL, RANDOMIZED BLOCK DESIGN.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	53	13,876.57		
Blocks	1	125.74	125.74	1.83
Treatments	26	12,094.09	465.16	6.79**
N	2	7,619.01	3,809.50	55.57**
P	2	1,981.71	990.86	14.45**
K	2	406.34	203.17	2.96*
N x P	4	162.99	40.75	0.59
N x K	4	202.02	50.50	0.74
P x K	4	1,218.14	304.54	4.44**
N x P x K	8	503.88	62.98	0.92
Error	26	1,782.48	68.5569	.

$^{**}P \leq 0.01$
$^{*}\ 0.10 < P < 0.05$

[8] Gene Hinze, farmer-cooperator, Fremont County, Iowa.

over the square root function partly because it gave a determinate surface maximum. If inputs of N and P_2O_5 had been extended to higher levels, the square root function might have given better results. Higher levels of N and P_2O_5 might have also revealed a significant $N \times P$ interaction.

$$\hat{Y} = -0.9751 + 0.7126N - 0.004352N^2 + 0.5255P - 0.003103P^2 + 0.2546K - 0.001624K^2 - 0.002255PK + 0.003863S \quad (19)$$

The term, S, again refers to stalks per acre while N, P and K refer to pounds of N, P_2O_5 and K_2O per acre. The values of t for the regression coefficients are given in table 18. They show that the coefficients for the N variables are significant in explaining yield variance, as does the analysis of variance in table 17. The value of t for K^2 is only 1.06. A value this large could occur by chance in about one-third of the time where K^2 had no real effect. Nevertheless, the K^2 term is retained for logical reasons. Without the negative K^2 term, an unlimited linear response to K would be implied.

The analysis of variance of regression in table 19 shows the over-all regression to be highly significant. The deviations from regression mean square are about the same as the estimate of experimental error from within plots.

In equation (20), stand is fixed at 9,220, the average stalk count for all experimental plots. If stand were included as a controlled variable in the original experiment, the optimum level of stand could be determined by economic analysis. However, none of the experiments analyzed in this study was so designed, and stand is used only to improve the precision of estimate of the fertilizer response.

$$\hat{Y} = 35.0587 + 0.7126N - 0.004352N^2 + 0.5255P - 0.003103P^2 + 0.2546K - 0.001624K^2 - 0.002255PK \quad (20)$$

TABLE 18. VALUES OF t FOR INDIVIDUAL REGRESSION COEFFICIENTS OF EQUATION (19).

Variable	N	N^2	P	P^2	K	K^2	PK	S
t value	5.67	2.89	3.72	1.96	1.90	1.06	2.10	2.99
P level	0.00001	0.005	0.0004	0.05	0.07	0.30	0.04	0.004

TABLE 19. ANALYSIS OF VARIANCE FOR REGRESSION OF CORN YIELD ON HAYNIE SOIL.

Source of variation	Degrees of freedom	Sum of square	Mean square	F
Total	53	13,876.57		
Due to 8-term regression equation (19)	8	10,794.66	1,349.33	18.92**
Deviations from regression	45	3,081.91	68.49	
Other treatment effects	20	1,299.43	64.97	
Error	25	1,782.48	71.30	

**$P \leq 0.01$

Production Surface for Haynie Soil

Equation (20) is not easily illustrated as a three-dimensional geometric surface, as for the preceding two experiments, because four variables are included. However, since the effects of N were independent of P and K, the phosphorus-potassium surface retains the same shape at different levels of N. In fig. 19 the K-P surface shows a greater rise in yield from inputs of P_2O_5 than from K_2O. Also, the surface is relatively flat over the top, indicating that yields do not change greatly for many combinations of P_2O_5 and K_2O. For a line stretched diagonally over the surface from the zero corner to the opposite corner, a sharp increase in yield is followed by a decrease. The "dropping off" at the opposite corner for high levels of both P_2O_5 and K_2O is due to the negative $P \times K$ interaction. The decline at high levels of P_2O_5 and K_2O is in contrast to the high ridge at high levels of N and P_2O_5 in fig. 11, caused by positive $N \times P$ interaction.

Predicted yields under the zero N column in table 20 correspond to points on the production surface in fig. 19. Yields are predicted to increase by about 21 bushels as nitrogen inputs are increased to 40 pounds. For 80 pounds of N, predicted yields are increased 29 bushels over corresponding P-K treatments receiving no nitrogen. These relationships between nitrogen and P-K responses for a particular angle of the P-K yield surface are shown in figs. 20A, 20B and 20C. (The view is more to the front of the P_2O_5 axis than in fig. 19.) In fig. 20A, the P-K yield surface is shown with a zero level of N. Figure 20B gives P-K yields with 40 pounds of N while fig. 20C shows P-K yields with 80 pounds of N. The surface of fig. 20B is 21 bushels higher than for fig. 20A because of the response of the 80-pound application of N. The shapes of the three surfaces are exactly the same, but one may see more

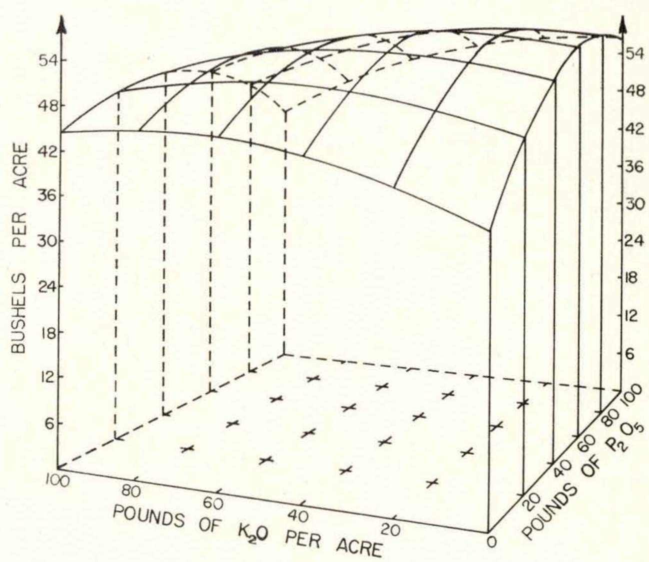


Fig. 19. Perspective view of predicted yield surface for corn on Haynie soil with no application of N.

TABLE 20. PREDICTED YIELDS OF CORN PER ACRE FOR SPECIFIED NUTRIENT COMBINATIONS APPLIED ON HAYNIE SILT LOAM.

Lbs. P_2O_5	Lbs. K_2O	Pounds of nitrogen 0	20	40	60	80
0	0	35.1	47.6	56.6	62.2	64.2
0	20	39.5	52.0	61.0	66.6	68.7
0	40	42.6	55.2	64.2	69.7	71.8
0	60	44.5	57.0	66.0	71.6	73.6
0	80	45.0	57.5	66.6	72.1	74.2
20	0	44.3	56.8	65.9	71.4	73.5
20	20	47.9	60.4	69.4	75.0	77.0
20	40	50.1	62.6	71.6	77.2	79.3
20	60	51.0	63.6	72.6	78.1	80.2
20	80	50.7	63.2	72.2	77.8	79.8
40	0	51.1	63.6	72.6	78.2	80.3
40	20	53.8	66.3	75.3	80.8	82.9
40	40	55.1	67.6	76.6	82.2	84.2
40	60	55.1	67.6	76.7	82.2	84.3
40	80	53.9	66.4	75.4	81.0	83.0
60	0	55.4	67.9	77.0	82.5	84.6
60	20	57.2	69.7	78.7	84.2	86.3
60	40	57.6	70.1	79.1	84.7	86.8
60	60	56.7	69.2	78.3	83.8	85.9
60	80	54.6	67.1	76.1	81.7	83.7
80	0	57.2	69.7	78.8	84.3	86.4
80	20	58.1	70.6	79.6	85.2	87.2
80	40	57.6	70.1	79.1	84.7	86.8
80	60	55.8	68.4	77.4	82.9	85.0
80	80	52.8	65.3	74.3	79.9	81.9

of the underside of the declining surface in the higher structures. If the P-K yield surface for 120 pounds of N were shown it would be of the same height as fig. 20B, since predicted yields start to decline around 82 pounds of N.

Marginal physical products of yields from P and K are presented in table 21. The equation of the marginal product for phosphorus is derived from the production function equation (20) by taking the partial derivative of $\hat{Y}$ with respect to P. Similarly, the marginal products for K and N are found by taking the partial derivatives of

TABLE 21. MARGINAL PRODUCT OR YIELD (BUSHEL PER ADDED POUND OF FERTILIZER NUTRIENT) FOR COMBINATIONS INDICATED IN ROWS AND COLUMNS. UPPER FIGURES ARE FOR K_2O, LOWER FIGURES FOR P_2O_5, HAYNIE SOIL.

Lbs. K_2O per acre	Pounds of P_2O_5 per acre 0	20	40	60	80
0	0.25 0.53	0.21 0.40	0.16 0.28	0.12 0.15	0.07 0.03
20	0.19 0.48	0.14 0.36	0.10 0.23	0.05 0.11	0.01 —0.02
40	0.12 0.44	0.08 0.31	0.03 0.19	—0.01 0.06	—0.06 —0.06
60	0.06 0.39	0.01 0.27	—0.03 0.14	—0.08 0.02	—0.12 —0.11
80	—0.01 0.35	—0.05 0.22	—0.10 0.10	—0.14 —0.03	—0.19 —0.15

$\hat{Y}$ with respect to K and N. Marginal product equations for P, K, and N are given by equations (21), (22) and (23), respectively.

$$\frac{\partial \hat{Y}}{\partial P} = 0.5255 - 0.006206P - 0.002255K. \qquad (21)$$

$$\frac{\partial \hat{Y}}{\partial K} = 0.2546 - 0.003248K - 0.002255P. \qquad (22)$$

$$\frac{\partial \hat{Y}}{\partial N} = 0.7126 - 0.008704N. \qquad (23)$$

Lack of interaction between N and the other nutrients in equation (20) is reflected in the marginal physical product of N in equation (23) ; the marginal yield per pound of N depends only on the level of N. The predicted increase in yield or marginal product of N at 0, 20, 40, 60, 80 and 100 pounds of N is 0.71, 0.54, 0.36, 0.19, 0.02 and —0.16 bushel, respectively. As greater inputs of N are applied, marginal products grow smaller and finally become negative at about 82 pounds.

Equation (21), representing the marginal yield

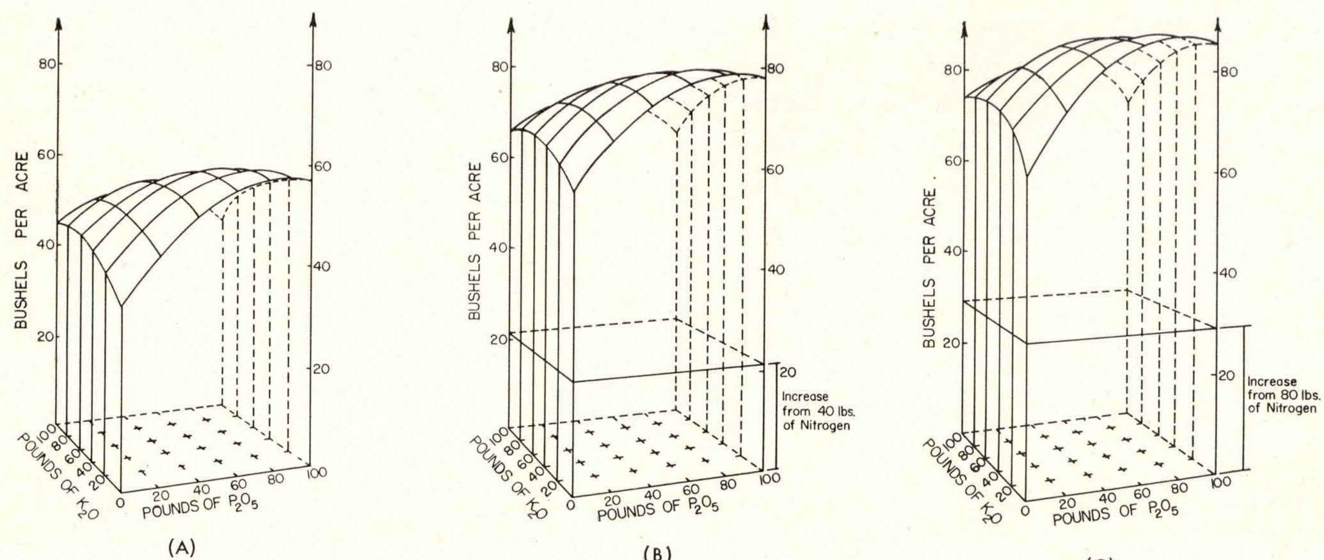


Fig. 20. (A) Predicted yield surface for corn on Haynie soil with no N application; (B) Predicted yield surface at 40 pounds of N; (C) Predicted yield surface at 80 pounds of N.

of P, contains a negative K term. This negative K term indicates that marginal yields from P_2O_5 will be lower for higher levels of K_2O. Accordingly, marginal yields of P_2O_5 in table 21 are lower at high levels of K_2O than at low levels; the decrease in marginal productivity of P_2O_5 is about 0.18 bushel as K_2O is increased from 0 to 80 pounds. Similarly, marginal yields from K_2O decline about 0.18 bushel when K_2O is held constant and P_2O_5 is increased from 0 to 80 pounds. The strong effect of K_2O on the productivity of P_2O_5, and vice versa, indicates that the optimum economic level of P_2O_5 or K_2O cannot be determined independently of each other.

The predicted increase in yield from a particular input of nitrogen again is independent of the levels of P_2O_5 and K_2O in fig. 21; while the response curve is higher with greater levels of P and K, it has the same slope in each case. The independence of the N response with P_2O_5 and K_2O in this experiment may have been because (1) P_2O_5 and K_2O levels in the soil were high enough before application so that they did not limit nitrogen response, and/or (2) P_2O_5 and N levels did not go high enough to allow interaction to be detected. Single-line response curves for P_2O_5 and K_2O, with various levels of the other, are shown in figs. 22 and 23.

Corn yields for fixed proportions of K_2O and P_2O_5 in fig. 24 show that the greatest yields are obtained when K_2O is equal in amount to half of

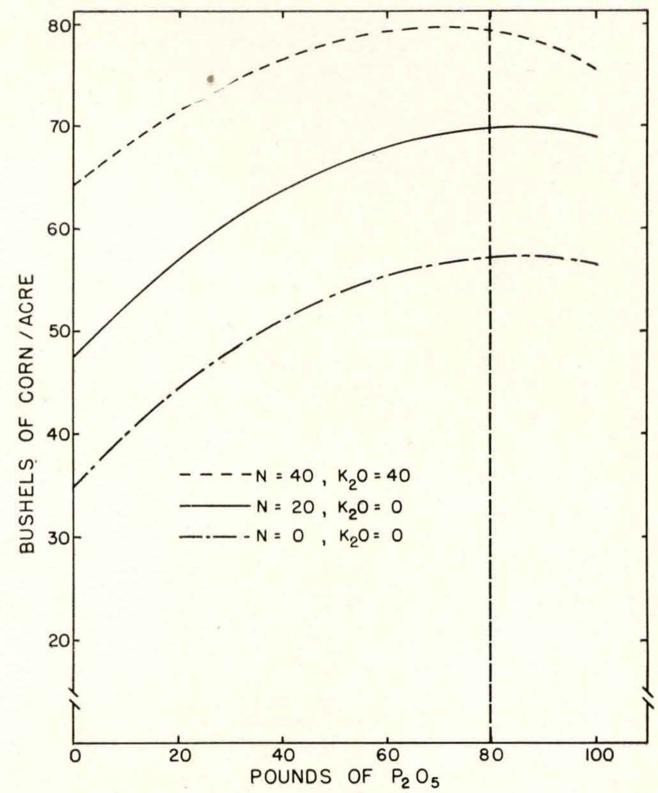


Fig. 22. Corn response to P_2O_5 at three levels of N and K_2O, Haynie soil. (Dashed vertical line is limit of P_2O_5 in experiment.)

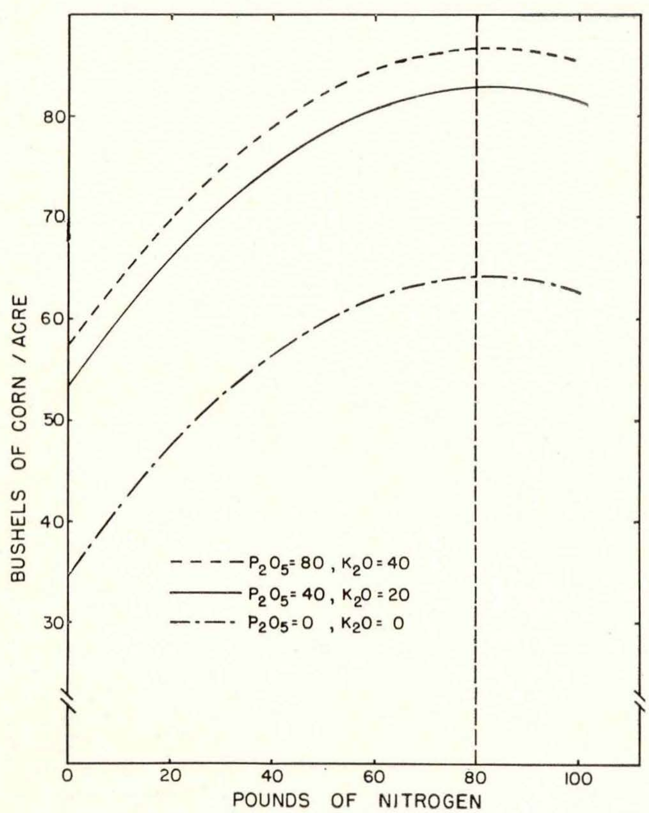


Fig. 21. Corn response to N at three levels of P_2O_5 and K_2O, Haynie soil. (Dashed vertical line is limit of N in experiment.)

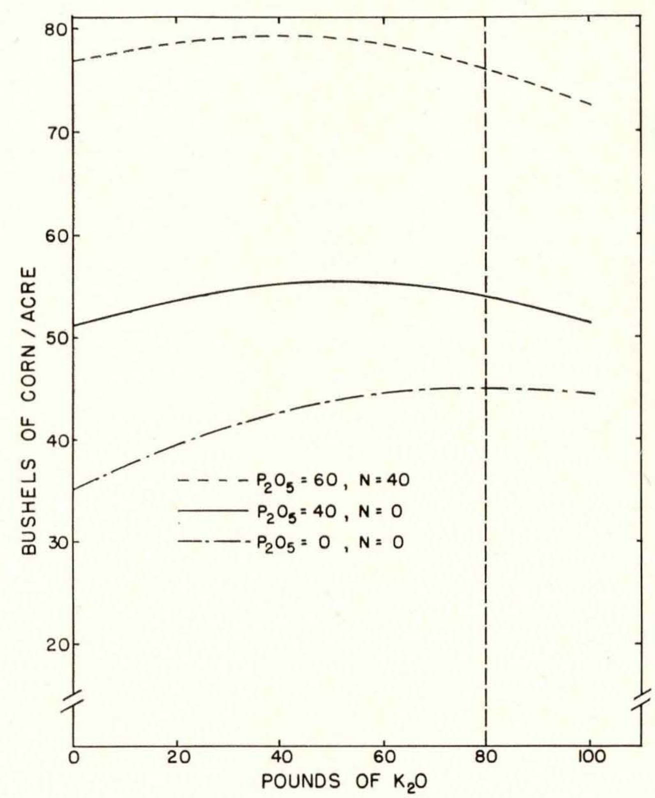


Fig. 23. Corn response to K_2O at three levels of N and P_2O_5, Haynie soil. (Dashed vertical line is limit of K_2O in experiment.)

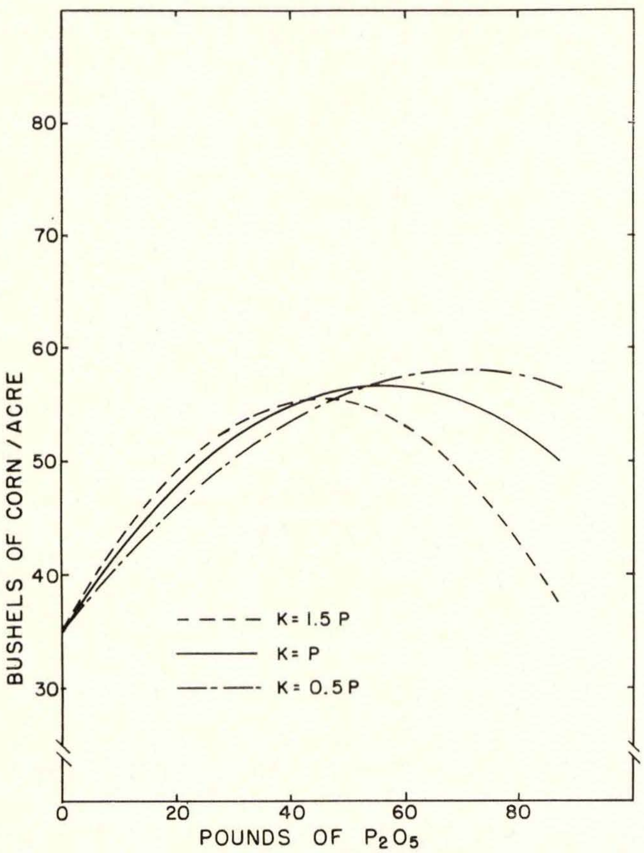


Fig. 24. Corn yields with K_2O and P_2O_5 in fixed proportions and no application of N, Haynie soil.

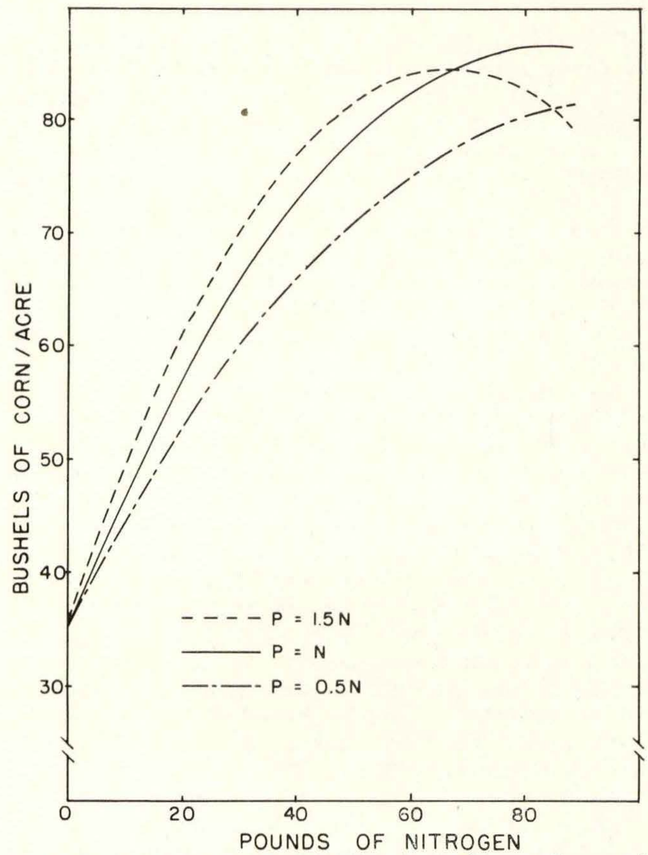


Fig. 26. Corn yields with N and P_2O_5 in fixed proportions and no K_2O application, Haynie soil.

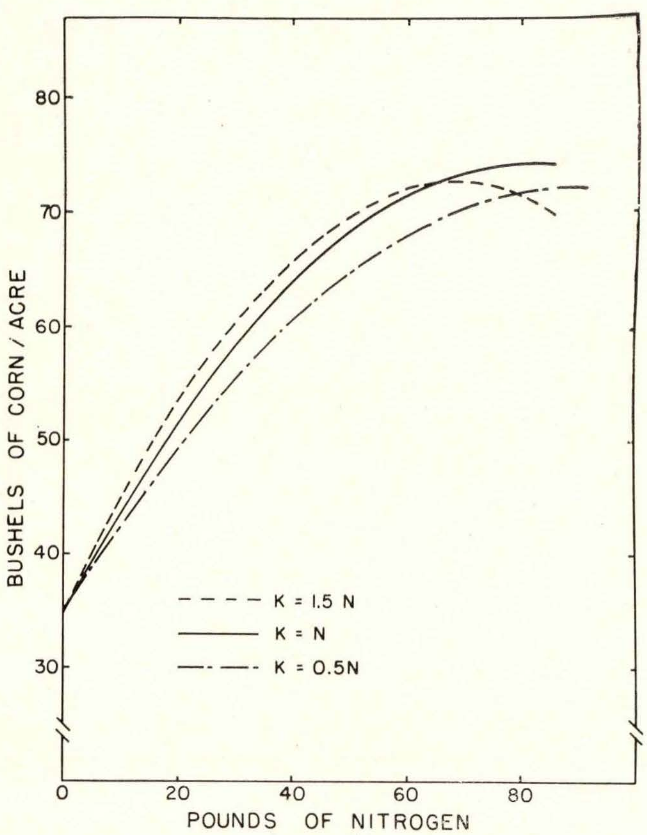


Fig. 25. Corn yields with N and K_2O in fixed proportions and no application of P_2O_5, Haynie soil.

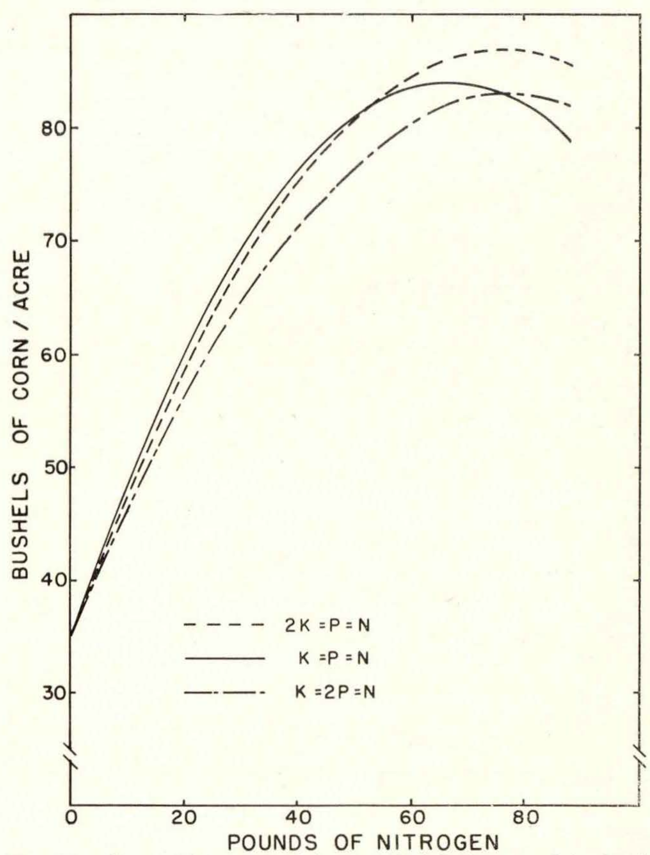


Fig. 27. Corn yields with K_2O, N and P_2O_5 increased in fixed proportions, Haynie soil.

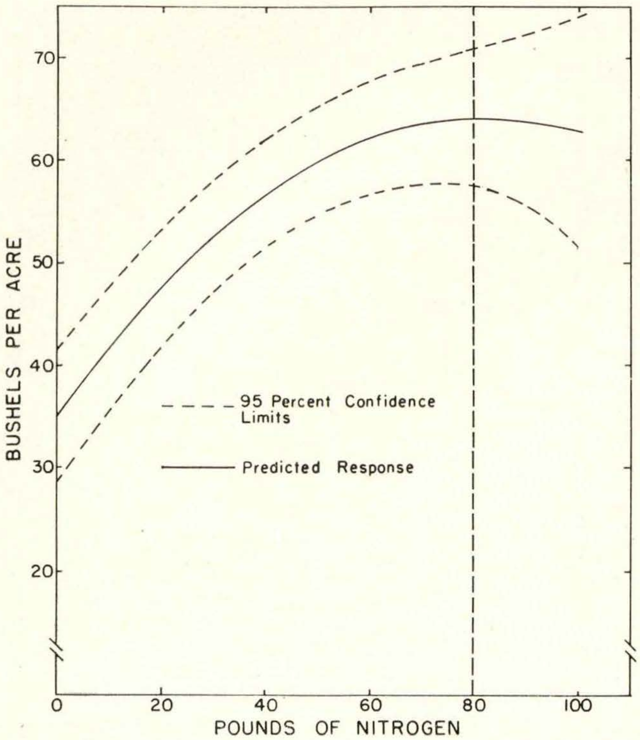


Fig. 28. Confidence limits for corn response to N with no application of P_2O_5 and K_2O, Haynie soil. (Dashed vertical line is limit of N in experiment.)

the P_2O_5 input. However, higher yields occur for initial inputs when K_2O is a larger proportion of the fertilizer. Similarly, larger inputs of K_2O in relation to N application give higher yields for initial inputs in fig. 25 but soon cause yields to decline. The fertilizer ratio with equal parts of N and K_2O results in the greatest yield in fig. 25. The greatest corn yields in fig. 26 also occur when N and P_2O_5 are combined in equal parts. The reason for highest yields being obtained with equal parts of N and P_2O_5 in equation (20) is that N and P_2O_5 responses are independent in the basic regression equation, and both nutrients have a maximum yield at about 80 pounds. Similarly, the K_2O response curve reaches a maximum at about 80 pounds (when P_2O_5 is at zero).

Scale line responses with N, P_2O_5 and K_2O all increased in fixed proportions are given in fig. 27. The greatest yield of over 86 bushels is obtained when K_2O is restricted to half the N and P_2O_5 inputs. Highest yields are obtained with less K_2O because of the negative interaction between K_2O and P_2O_5. The additional yield obtainable from further K_2O inputs is more than offset by the reduction in yield from the negative interaction.

YIELD ISOQUANTS FOR HAYNIE SOIL

Various combinations of N and P_2O_5 can be used to produce given yields as shown in fig. 29. Yields of 50 bushels can be obtained by applying 24.7 pounds of N and no P_2O_5 or by using about 27 pounds of P_2O_5 and no nitrogen. For the 60-bushel yield, 50 pounds of N and no P_2O_5 can be applied. However, P_2O_5 alone cannot be used to produce the 60-bushel yield; some nitrogen also must be applied. At 70 bushels, nitrogen alone becomes insufficient; some P_2O_5 must be added. At higher yields the isoquants are spaced further apart, indicating diminishing returns to fertilizer applications.

Isoquants in fig. 29 were computed from equation (24) [which in turn was derived from the yield regression, equation (20)], where K_2O was set equal to zero.

$$N = 81.87 \frac{\pm \sqrt{0.5078 + 0.01741\,(0.5255P - 0.003103P^2 + 35.06 - \hat{Y})}}{-0.008704} \qquad (24)$$

Isoquants in fig. 30 were plotted from equation (25) which was also derived from the basic regression equation (20).

$$N = 81.87 \frac{\pm \sqrt{0.5078 + 0.01741\,(0.2546K - 0.001624K^2 + 35.06 - \hat{Y})}}{-0.008704} \qquad (25)$$

The isoquants in fig. 30 again show that yield is more responsive to nitrogen than to potassium since the slopes are nearly horizontal for lower yields. For example, 45 bushels of corn per acre could be produced by applying 16 pounds of N or by using nearly 80 pounds of K_2O. In other words, 1 pound of nitrogen will produce as much as several pounds of K_2O at the lower yields. However, if yield is to be increased to 65 or 70 bushels, some potassium must be applied; the isoquants do not intersect the N axis, indicating that N and K_2O do not serve as alternatives in attaining these yield levels. The sharp curvature and greater

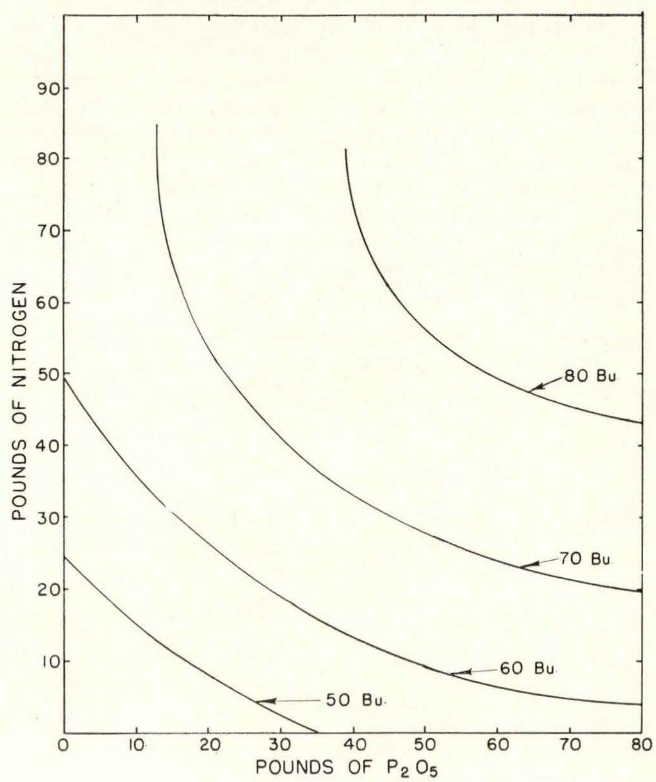


Fig. 29. Yield isoquants for corn on Haynie soil with no K_2O application.

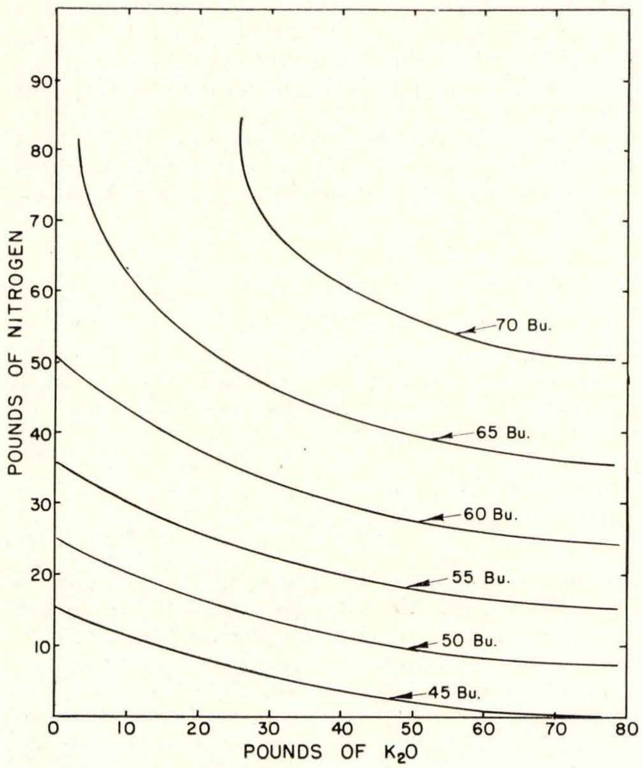


Fig. 30. Yield isoquants for corn on Haynie soil with no P_2O_5 application.

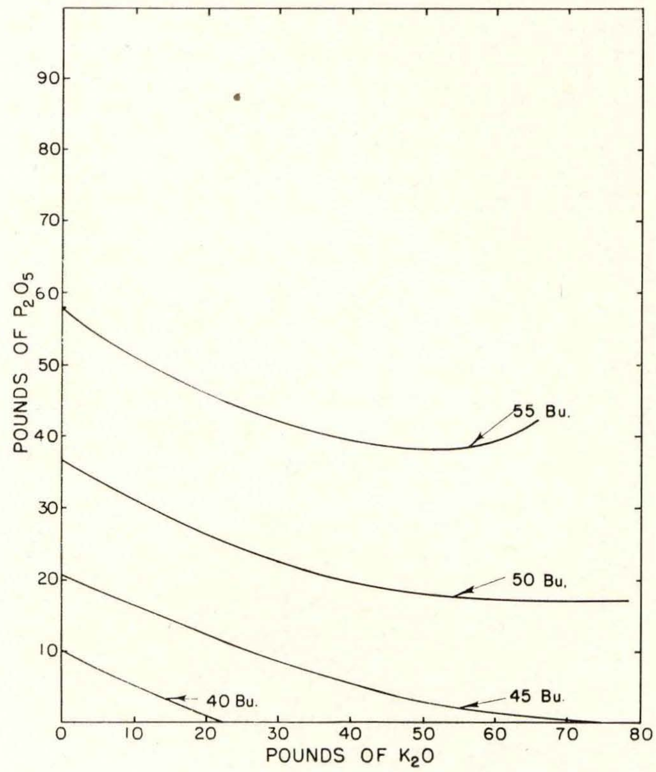


Fig. 31. Yield isoquants for corn on Haynie soil with no application of N.

slope of the upper end of the 65- and 70-bushel isoquants indicate that a large amount of nitrogen replaces only a small amount of K_2O, in attaining a given yield, at these yield levels.

The gentle slopes of the isoquants in fig. 31 indicate that P_2O_5 also increased yield more per pound than did K_2O. Yield isoquants in fig. 31 are calibrated at zero level of N. However, the level of N will not change the shape of the isoquants in fig. 31; only the level of yield rises as N is added. Nitrogen does not affect the rate at which P_2O_5 and K_2O substitute for each other because there is no interaction between N and P or K in the basic production function, (20).

Isoquants in fig. 31 were computed from equation (26) which also was obtained from the basic regression equation, (20).

$$P = -0.3633K + 84.66 \pm \frac{\sqrt{0.0007906K - 0.00001508K^2 + 0.7113 - 0.01241\hat{Y}}}{-0.006206} \quad (26)$$

TABLE 22. ISOQUANT COMBINATIONS OF NUTRIENTS FOR PRODUCING SPECIFIED YIELDS PER ACRE AND CORRESPONDING MARGINAL RATES OF SUBSTITUTION, HAYNIE SOIL.

45 bushels*			55 bushels†		
Lbs. of P_2O_5	Lbs. of K_2O	Marginal rate of substitution, dP/dK‡	Lbs. of P_2O_5	Lbs. of K_2O	Marginal rate of substitution, dP/dK‡
21.70	0	—0.526	57.42	0	—0.740
16.76	10	—0.462	50.89	10	—0.574
12.45	20	—0.401	45.84	20	—0.440
8.74	30	—0.341	42.05	30	—0.317
5.64	40	—0.280	39.52	40	—0.187
3.16	50	—0.216	38.39	50	—0.260
1.33	60	—0.148	39.10	60	—0.193

*Increased yield from fertilizer is 9.94 bushels.
†Increased yield from fertilizer is 19.94 bushels.
‡Change in units of P_2O_5 required to maintain a given yield when another unit of K_2O is added.

Marginal rates of substitution for several points on two isoquants from fig. 31 are given in table 22. For the 45-bushel yield and at 21.70 pounds of P_2O_5, the addition of 1 pound of K_2O will replace about ½ pound of P_2O_5. It would, therefore, be profitable to use all phosphorus and no potash to obtain a yield of 45 bushels, unless K_2O costs only half as much as P_2O_5. At the 55-bushel yield, 1 pound of K_2O will replace 0.74 pound of P_2O_5. However, it still is not profitable to use K_2O unless the price of K_2O is less than 74 percent of the P_2O_5 price.

YIELD ISOCLINES FOR HAYNIE SOIL

Equations of the isoclines were derived from the basic regression or production function, (20). The general form of the N-P_2O_5 isoclines is given by equation (27) where a represents the N/P_2O_5 price ratio and is graphed in fig. 32.

$$N = 81.87 + a\,(0.7131P - 60.37) \quad (27)$$

Yield isoclines in fig. 33 for N and K_2O reflect the greater productivity of N. With prices where N is about 50 percent more expensive per pound than K_2O, nearly 40 pounds of N should be applied

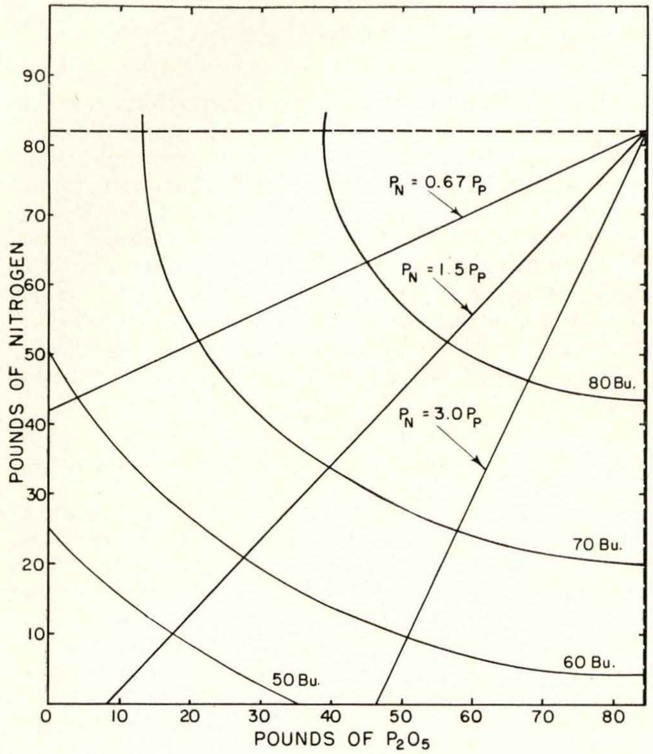


Fig. 32. Yield isoquants and isoclines with dashed ridgelines at zero level of K_2O, Haynie soil.

before any K_2O is used. In fig. 32, with a nitrogen price 50 percent greater than the P_2O_5 price, about 10 pounds of P_2O_5 would be used before any nitrogen were applied.

Equations of the isoclines in fig. 34 were derived from equation (20) by dividing the partial derivative of $\hat{Y}$ with respect to P by the partial derivative of $\hat{Y}$ with respect to K and setting this ratio equal to the P_2O_5/K_2O price ratio. For any P_2O_5/K_2O price ratio, a, K can be expressed in terms of P as in equation (28).

$$K = \frac{0.2546\,a - 0.5255 + (0.006206P - 0.002255\,a)}{0.003248\,a - 0.002255} \qquad (28)$$

All isoclines (including ridgelines) in fig. 34 intersect at 75 pounds of P_2O_5 and 26 pounds of K_2O. The intersection point indicates the inputs of phosphorus and potassium which give a maximum total product of about 58 bushels (at $N = 0$). If the yield-maximizing input of 82 pounds of N is applied, a total yield of 87 bushels is predicted from equation (20).

Ridgelines (i. e., the dotted lines where substitution rates are zero) in figs. 32 and 33 meet at right angles which is a characteristic feature when two nutrients are economic "independents," that is, when the level of one nutrient does not affect the profitable amount of the other. Where negative interaction between nutrients exists, as for P_2O_5 and K_2O in fig. 34, ridgelines meet at an angle greater than 90 degrees: Negative $P \times K$ interaction gives the production surface a comparatively flat top; economic limits of nutrient combination are wide. For close complementarity and positive interaction (i. e., between N and

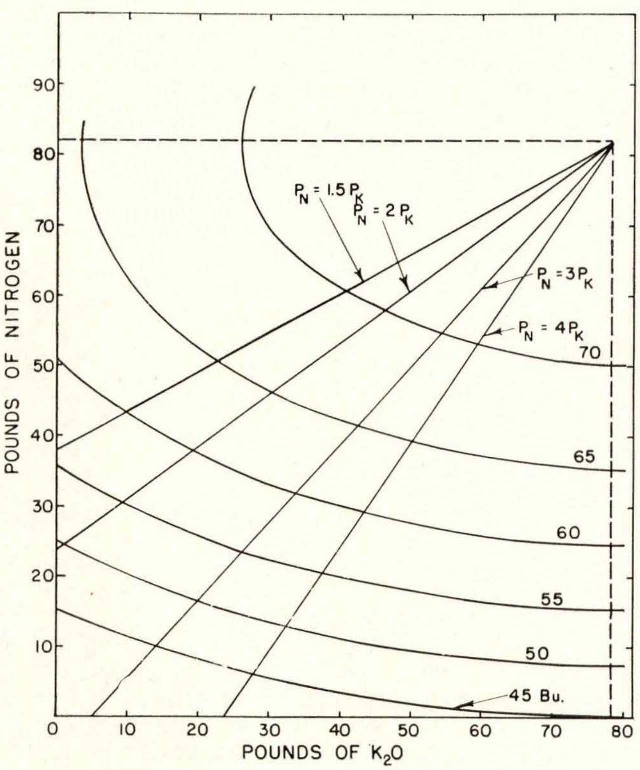


Fig. 33. Yield isoquants and isoclines with dashed ridgelines at zero level of P_2O_5, Haynie soil.

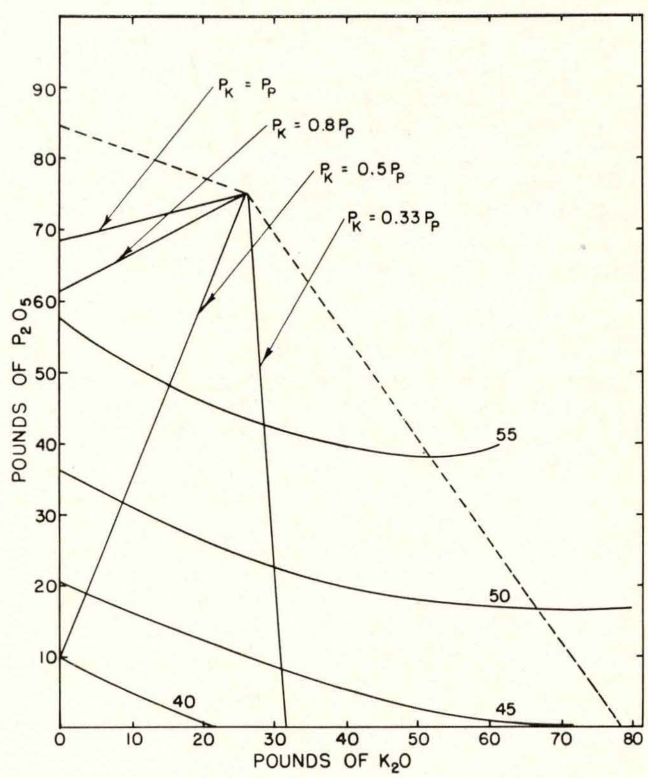


Fig. 34. Yield isoquants and isoclines with dashed ridgelines at zero level of N, Haynie soil.

P_2O_5 for corn in fig. 17), the ridgelines are close together and meet at an angle of less than 90 degrees. With positive $N \times P$ interaction, as in fig. 17 where ridgelines are close together, a non-optimum nutrient combination could be very costly. However, to deviate slightly from the optimum fertilizer ratio line (isocline) in fig. 34, if the price ratio differs only slightly from the substitution ratio indicated by the isocline, may depress profits only slightly since rates of substitution change slowly along the isoquants.

Under prices where the price of K_2O is 80 percent of the P_2O_5 price, over 60 pounds of P_2O_5 should be applied before any K_2O is used (fig. 34). However, with K_2O at one-third the price of P_2O_5 it would pay to apply over 30 pounds of K_2O before any P_2O_5 is used.

Economic Optima for Haynie Soil

Since N is independent of P and K in equation (20), the optimum level of N can be found independently of P_2O_5 and K_2O. With corn at $1.00 per bushel and elemental N at $0.15 per pound, the optimum input of N is found by setting the partial derivative of $\hat{Y}$ with respect to N in equation (20) equal to the nitrogen/corn price ratio.

$$\frac{\partial Y}{\partial N} = 0.7126 - 0.008704N = \frac{0.15}{1.00} \quad (29)$$

Solving equation (29), an optimum input of about 65 pounds of N is indicated. Optimum inputs of P_2O_5 and K_2O are found in the same way except that P_2O_5 and K_2O must be solved simultaneously since they interact with each other. If corn is $1.00 per bushel, P_2O_5 is $0.10 per pound and K_2O is $0.08 per pound, the solutions are found from equations (30) and (31), indicating an optimum of 8.24 pounds of K_2O and 65.56 pounds of P_2O_5. Several fertilizer and corn price situations are presented in table 23, with the maximum profit level of fertilization indicated.

$$\frac{\partial Y}{\partial P} = 0.5255 - 0.006206P - 0.002255K = \frac{0.10}{1.00} \quad (30)$$

TABLE 23. OPTIMUM RATES AND COMBINATIONS OF FERTILIZER FOR SPECIFIED CROP AND NUTRIENT PRICES, HAYNIE SOIL.

Price per unit				Optimum inputs per acre				
Corn per bu.	N per lb.	K_2O per lb.	P_2O_5 per lb.	Lbs. N	Lbs. K_2O	Lbs. P_2O_5	Estimated yield	Net gain per acre from fertilizer*
$2.00	$0.20	$0.15	$0.15	70.4	6.6	70.2	85.8	$75.80
1.00	0.20	0.15	0.15	58.9	0.0	60.5	82.4	26.44
0.50	0.20	0.15	0.15	36.0	0.0	36.3	70.1	4.85
2.00	0.15	0.08	0.10	73.2	17.2	70.3	86.7	83.81
1.00	0.15	0.08	0.10	64.6	8.2	65.5	84.8	32.79
0.50	0.15	0.08	0.10	47.4	0.0	52.4	78.1	9.15
2.00	0.10	0.10	0.10	76.1	13.1	71.8	86.8	87.30
1.00	0.10	0.10	0.10	70.4	0.0	68.5	85.1	36.11
0.50	0.10	0.10	0.10	58.9	0.0	52.4	80.9	11.77

*Computed by multiplying increase in yield from use of fertilizer times price of corn and subtracting cost of fertilizer.

$$\frac{\partial Y}{\partial K} = 0.2546 - 0.003248K - 0.002255P = \frac{0.08}{1.00} \quad (31)$$

Under the least favorable price situations for fertilizer application in table 22, some application of N and P_2O_5 still is indicated. For example, with corn at $0.50 per bushel, N at $0.20 per pound and P_2O_5 at $0.15 per pound, inputs of 36 pounds per acre of both N and P_2O_5 would maximize profit from fertilizer. However, the net gain per acre from applying fertilizer is less than $5.00. Since cost of fertilizer in this instance is about $12.65 per acre, because of risk and uncertainty many farmers might not apply any fertilizer under this price situation.

Presentation for Practical Use on Haynie Soil

Since the yield response to nitrogen was independent of potassium and phosphorus in the basic regression equation, (20), optimum inputs of N are calculated independently of P_2O_5 and K_2O and are presented in fig. 35. The increase in yield from nitrogen application is given by the response curve. Dashed vertical lines again represent optimum points of input under various nitrogen-corn price relationships (see earlier discussion).

Simultaneous optimum solutions for P_2O_5 and K_2O can be located in fig. 36: To find optimum inputs when K_2O is $0.08 per pound, P_2O_5 is $0.10 per pound and corn is $1.00 per bushel, the isocline labeled $P_k = 0.8P_p$ is selected. The "place

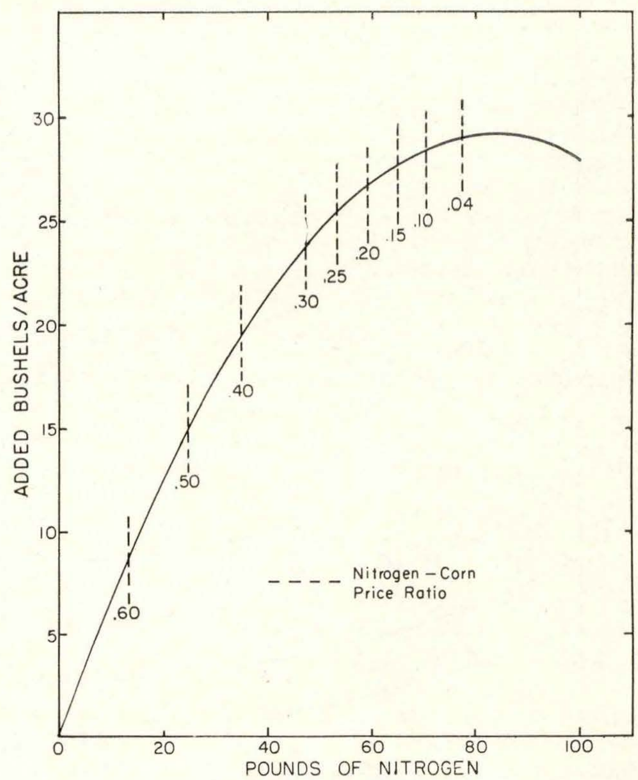


Fig. 35. Increase in yield from N and optimum inputs for various N/corn price ratios, Haynie soil.

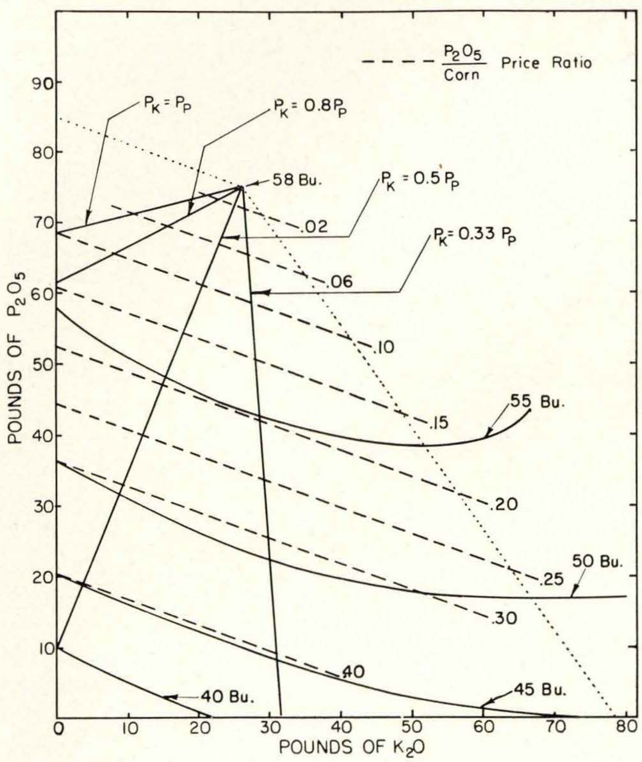


Fig. 36. Yield isoquants and isoclines with optimum rates indicated by dashed lines representing P_2O_5/corn price ratios. (Dotted lines are ridgelines.)

to stop" on the isocline is denoted by the dashed line labeled 0.10, which is the P_2O_5/corn price ratio. The dashed lines also take the K_2O price into account since they intersect the isoclines on which P_2O_5 and K_2O are in a fixed price ratio. For the indicated optimum inputs of 66 pounds of P_2O_5 and 8 pounds of K_2O, a yield of about 57 bushels can be estimated from isoquants in fig. 36. Since yield with no fertilizer is 35 bushels, a gain in yield of about 22 bushels per acre is predicted.

As another example, suppose a tenant farmer bears all the cost of fertilizer but receives only half of the crop. For the same price situation with corn at \$1.00 per bushel and N at \$0.15, P_2O_5 at \$0.10 and K_2O at \$0.08 per pound, the optimum inputs for the tenant can also be found in figs. 35 and 36. The effective corn price for the tenant is \$1.00/2, since he receives only half of the increase from fertilizer. The tenant's nitrogen/corn price ratio would be $\frac{0.15}{0.50} = 0.30$ in fig. 35 and would indicate an input of around 47 pounds of N. Similarly, the P_2O_5/corn price ratio would be 0.20 in fig. 36 and would indicate an optimum input of about 52 pounds of P_2O_5 and no K_2O. From applying 52 pounds of P_2O_5 and 47 pounds of N, a total yield of about 78 bushels per acre is predicted from figs. 35 and 36. Since yield with no fertilizer would be 35 bushels, a gain of about 43 bushels per acre is predicted from use of fertilizer. The financial result to the tenant is estimated to be the following:

Increased value of crop (tenant's share)	\$21.50
Cost of 52 lbs. of P_2O_5	5.20
Cost of 47 lbs. of N	7.50
Tenant's margin over cost of fertilizer	\$9.25

Net gain from use of fertilizer by the tenant is only \$9.25 per acre as compared to the gain of \$32.66 for a non-renter under the same price situation. The tenant would realize a 176-percent return. However, an owner-operator would realize a 294-percent return on his expenditure.

Use of Production Functions Under Limited Capital

In the preceding sections, the optimum levels of fertilization were specified for given prices where capital is unlimited. The most profitable level of fertilization is defined by equation of the marginal product and the nutrient/crop price ratio. However, under limited capital, the optimum level of fertilization is determined by the return from this investment versus other investments in the farm business. As an example, suppose that a farmer is operating under limited capital and/or uncertainty to the extent that the last dollar invested in fertilizer, livestock or machinery must return twice the cost before he will risk the expenditure. If he has Haynie silt loam soil similar to that outlined above, the "restricted optimum" input of fertilizer can still be found from figs. 35 and 36. If corn is \$1.00 per bushel and N is \$0.15, P_2O_5 is \$0.10 and K_2O is \$0.08 per pound, a 200-percent return on the last unit of fertilizer is obtained if he applies 47.4 pounds of N, 52.4 pounds of P_2O_5 and no K_2O. The "restricted optimum" input of N is located in fig. 35 by doubling the N/corn price ratio ($2 \times 0.15/1.00 = 0.30$). Similarly, in fig. 36 the appropriate P_2O_5/corn price ratio is doubled ($2 \times 0.10/1.00 = 0.20$). The restricted inputs result in an estimated gain over fertilizer cost of \$30.65 per acre which is about \$2.00 less than estimated gain for unrestricted "optimum" application. However, investment in fertilizer is reduced from \$16.91 to \$12.35 by restricting the inputs. Also, returns per \$100 of fertilizer are increased from \$294 to \$348 per acre.

Information from production functions for crops or livestock can also be integrated into the overall farm plan with fertilizer so as to allow selection of the most profitable combination of investments and enterprises. By so doing, the amount of land and labor and the farmer's capital position can be taken into account, along with marginal returns from fertilizer, feed or other expenditures. These steps are not included in this study, however, because of space limitations.

SUMMARY

This study includes predictions of production functions, isoquants, isoclines and economic optima for fertilization of corn on three soil types. The data are for the 1953 production year, with one experiment each on Carrington, Moody and Haynie soil types. Estimates include only 1953 responses and do not consider residual effects of fertilizer.

The production function, isoquant and isocline equations used for Carrington silt loam are provided, respectively, in equations (a), (b) and (c) below. Nitrogen (N) and K_2O (K) are the variable nutrients, and average stand is denoted by (S). The price ratio, P_k/P_n, is denoted by a.

$$Y = 99.223 + 0.3162K - 0.001813K^2 + 0.9190\sqrt{N} - 0.04453N \qquad (a)$$

$$K = 87.23 \pm \frac{\sqrt{0.00666\sqrt{N} - 0.000323N + 0.8194 - 0.00725Y}}{-0.003626} \qquad (b)$$

$$K = 87.23 + 12.28a - \frac{126.72}{\sqrt{N}}a \qquad (c)$$

The production surface for this function is relatively flat, with both positive and negative marginal yields for K_2O. The isoquants are nearly vertical over most of their range, indicating rigid limits of nitrogen response when the level of K_2O is low. Yield isoclines trace a path along the vertical, or K_2O, axis then curve sharply in nitrogen distance of the nutrient plane.

The production function, isoquant and isocline equations used in predictions for Moody soil are (d), (e) and (f), respectively, where N refers to nitrogen and P refers to P_2O_5. The price ratio, P_p/P_n, is represented by a.

$$Y = 29.248 + 0.534N - 0.001743N^2 - 0.0003549P^2 + 0.001069NP \qquad (d)$$

$$P = 1.506N \pm \frac{\sqrt{0.0007581N - 0.000001322N^2 + 0.0415 - 0.00142Y}}{-0.0007098} \qquad (e)$$

$$P = \frac{(0.001069 + 0.003486a)N - 0.5340a}{0.001069a + 0.0007098} \qquad (f)$$

The production surface for this function has definite ridges in both the nitrogen and P_2O_5 spaces of the nutrient plane. Complementarity or positive interaction also is denoted by high yield responses when both nutrients are increased together. The isoclines are linear with origin on the nitrogen axis. Ridgelines, denoting limits of nutrient replacements, are quite close together. The isoclines also indicate that at most price relationships nitrogen applications would be carried to quite high levels before any P_2O_5 would be used.

Production function, P_2O_5-K_2O isoquant, nitrogen-P_2O_5 isocline and P_2O_5-K_2O isocline equations for Haynie soil are given in (g), (h), (i) and (j), respectively, where the terms

$$Y = 35.0587 + 0.7126N - 0.004352N^2 + 0.5255P - 0.003103P^2 + 0.2546K - 0.001624K^2 - 0.002255PK \qquad (g)$$

$$P = 84.66 - 0.3633K \pm \frac{\sqrt{0.00079K - 0.0000151K^2 + 0.7113 - 0.0124Y}}{-0.006206} \qquad (h)$$

$$N = 81.87 + a(0.7131P - 60.37) \qquad (i)$$

$$K = \frac{0.2546a - 0.5255 + (0.006206P - 0.002255a)}{0.003248a - 0.002255} \qquad (j)$$

have the meaning indicated above. Because N does not interact with P_2O_5 and K_2O, the surface for the latter two nutrients is independent of the first. Surfaces for P_2O_5 and K_2O have been predicted with various levels of nitrogen. They reach peaks in the center with lower yield levels at all four "corners." Isoquants for nitrogen and K_2O have a relatively small slope for low yield levels, indicating a greater return from nitrogen. At higher yield levels, the curvature of the isoquants indicates complementarity of nutrients. Isoclines for nitrogen and P_2O_5 or K_2O are linear with convergence at the point of maximum yield. Ridgelines meet at an angle of 90 degrees because of the lack of interaction between nitrogen and the other elements. Ridgelines for P_2O_5 and K_2O converge at an angle of more than 90 degrees because of negative interaction. The equations listed above, and others, are used in predicting optimum nutrient ratios and levels for various price ratios of (1) nutrients in relation to each other and (2) nutrients in relation to corn.

Simple graphs have been devised to illustrate possible use of the basic data by farmers and educators. These have been arranged to allow use of various prices and to specify optimum nutrient ratios and fertilization levels. Modifications for rented farms and limited capital situations also are explained.