Milk Production Functions, Hay/Grain Substitution Rates and Economic Optima in Dairy Cow Rations

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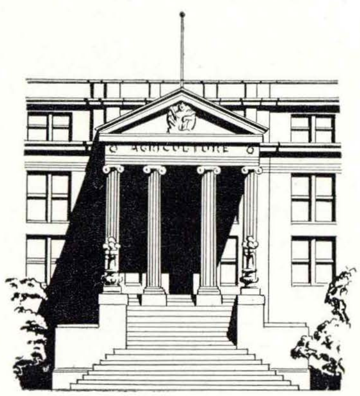

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CONTENTS

SUMMARY

1. This study was conducted to allow prediction of the milk production surface under particular conditions with respect to breeds, feeds, temperature and other environmental characteristics. Prediction of the production function allows derivation of milk isoquants, feed isoclines, input-input curves, marginal substitution coefficients between grain and forage, and marginal feed-milk transformation ratios. Derivation of these physical quantities allows specification of (1) the least-cost ration of grain and hay, given the prices or costs of these two feeds, and (2) the optimum level of feeding, given the price of milk.

2. The basic experiment was conducted with 36 Holstein cows over a period of 17 months. Each cow was put on the same adjustment ration for a preliminary period of 60 days. Cows then were randomly assigned to grain/hay rations varying from 15 to 75 percent hay. They were kept on the assigned rations over an experimental period of 182 days. Cows were fed at three levels of feed intake for each ration. Three cows representing low, average and high inherent production ability were assigned to each of the 12 ration-level treatments. Milk output and feed intake were measured over the 182-day experimental period. Cows were fed in drylot.

3. Production functions were derived from the basic experimental data. These several production functions represented different algebraic forms of equations, alternative variables and different time periods. Some of the functions have been used for evaluation of the customary ENE and TDN feed transformations. The function serving best to predict the milk production surface appeared to be equation a, where M is milk output, G is grain intake, H is hay intake, A is cow ability measured by production of 4 percent FCM in a preliminary period and T is time or month in the experimental period. The function explained 81.3 percent of the variance in milk production over the experimental period.

$$\text{(a)}\quad M = 1.6302H + 3.1309G + 0.1497A + 14.2243T - 0.000388H^2 - 0.001192G^2 + 4.3792T^2 - 0.00105HG - 0.1570GT - 0.0865HT - 731.76$$

An alternative function which gives similar predictions of the milk production function is equation b.

$$\text{(b)}\quad M = 0.6601H + 1.4276G + 0.1553A - 0.000054H^2T - 0.000152G^2T - 2.0752T^2 - 157.24$$

4. The isoquant and substitution equations ($T = 1$, A = mean of 36 cows) for equation a are derived in equations c and d.

$$\text{(c)}\quad H = 1{,}989.36 - 1.3608G \pm (-1{,}288.66)\sqrt{1.8553 + 0.001355G - 0.00000073G^2 - 0.001552M}$$

$$\text{(d)}\quad \frac{dH}{dG} = \frac{2.9740 - 0.002384G - 0.001056H}{1.5437 - 0.000776H - 0.001056G}$$

The isoquants and substitution equations ($T = 1$, A = mean of 36 cows) for equation b are derived in equations e and f.

$$\text{(e)}\quad H = 6{,}106.28 \pm (-9{,}250.69)\sqrt{0.4850 + 0.000309G - 0.000000033G^2 - 0.000216M}$$

$$\text{(f)}\quad \frac{dH}{dG} = \frac{1.4276 - 0.000304G}{0.6601 - 0.000108H}$$

5. Milk isoquants and feed substitution rates are predicted, using equation a, in the table below for a cow of average ability and for the mean month of the experimental period ($T = 3.5$). Similar quantities are provided for other levels of cow ability and other points of time in the text of this bulletin.

Lbs. grain	Lbs. hay for milk output in 28 days of:				Marginal rates of substitution $\Delta H/\Delta G$ for milk level of:			
	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.
150	815	1,036			2.54	3.09		
200	692	891			2.37	2.74		
250	577	760	1,032		2.24	2.50	3.41	
300	468	640	877		2.12	2.32	2.85	
350	365	528	744		2.02	2.17	2.53	
400	266	423	623	973	1.93	2.05	2.31	4.00
450	172	323	513	806	1.85	1.94	2.13	2.90
500		229	410	674		1.85	1.99	2.45
550		139	314	559		1.76	1.87	2.17
600			223	455			1.76	1.97
650			138	361			1.66	1.80
700				275				1.66
750				194				1.53
800				122				1.41

6. Quantities are derived in the text which predict the optimum ration and feeding levels for different price ratios. Production surface maps and isoquants are predicted to illustrate feed transformation and substitution relationships for cows of different abilities, for different rations and for different months of the lactation period. Input-output quantities are presented in Appendix F, and comparisons are made of milk productivity when feed input is measured by ENE standards.

Milk Production Functions, Hay/Grain Substitution Rates and Economic Optima in Dairy Cow Rations[1]

BY EARL O. HEADY, JOHN A. SCHNITTKER, NORMAN L. JACOBSON AND SOLOMON BLOOM

Interest in possibilities of forage-grain substitution in the dairy cow ration has been increased by recent agricultural developments. One development is acreage control which allows farmers to grow forage as a replacement crop for grain. Another is the continuing interest in conservation: Erosion control plans ordinarily require an increased acreage of grasses and legumes and fewer acres of grains and row crops. Both of these developments increase the supply of forages relative to grains and give rise to questions of using forage profitably. One possibility is the substitution of forage for grain in rations of ruminants. The feasibility of this adjustment depends, however, on the rate at which the various classes of feeds substitute for each other.

Recent changes in price structures, with dairy product prices depressed relative to feed and labor costs, also have caused farmers to examine substitution possibilities as a means of lowering costs and increasing profits. Then, too, yearly and geographic differentials in the costs of concentrates relative to forages and to the price of milk give rise to questions of the most profitable ration under particular economic circumstances. To what extent should the grain-forage ration be varied as the price of grain changes relative to the price of forage at particular locations? To what extent should the most profitable ration differ between grain surplus and grain deficit areas or other areas where concentrates are priced at different levels? These questions can be answered only if information is available on substitution ratios. The optimum ration, in terms of profit maximization, can be determined only by relating substitution ratios to price ratios. Finally, determining the nature of the milk production surface with its expression of feed/milk transformation ratios and feed substitution coefficients is a central problem in dairy cow nutrition.

OBJECTIVES

The experiment on which this report is based was designed to provide estimates of the milk production function and feed substitution ratios in alternative dairy cow rations. The experiment provides predictions of the milk production surface and milk isoquants indicating marginal rates of substitution between the two classes of feeds—concentrates and hay.

The primary objective is to establish (1) the rates at which grains and forages substitute under specific technical conditions and (2) the rate at which feeds are transformed into milk for various production levels and rations. An auxiliary objective is to investigate the economic potential of substituting forage for grain. Hence, details are provided for (a) explaining the models which serve as a basis for the experimental design, (b) illustrating the procedure used in predicting feed substitution and transformation rates and (c) determining the particular ration and level of grain feeding which results in the least-cost ration and the most profitable level of production per cow. Before empirical results are presented, basic logic of the production and economic relationships involved is explained. This material is included since it provides the fundamental models underlying the design of this study. Too, the concepts should prove useful to nutritionists and others who are concerned with predicting the outcome of rations but are unacquainted with the particular models.

The study reported was restricted in magnitude because of limitations in funds, cows, barn space and other facilities. Because of its limited magnitude, it should be looked upon as an exploratory study, to be supplemented by later investigations. The over-all objectives of this study are of a methodological nature. The central predictions revolve more nearly around estimation of the milk production function and feed transformation and substitution coefficients, than around use of the particular principles in determining economic optima in dairy rations. It is hoped that this fundamental study will provide the basis for and encourage other studies which allow more refined predictions of the milk production function and of the economic gains in using the profit-maximizing principles outlined.

BASIC PRINCIPLES IN NUTRITION AND ECONOMICS

Milk production is a complex process involving many resources, of which feeds represent but one class. The milk production function is of the general form:

$$(1)\quad M = f(C, F, X_1, X_2, X_3, X_4 \ldots\ldots\ldots\ldots X_n).$$

Here, M refers to milk production per cow in a specified time period, C refers to concentrate intake, F refers to forage intake, X_1 refers to body size, X_2 refers to inherent breed qualities of the cow, X_3 refers to labor used and X_4 through X_n refer to unspecified resources or inputs. While all of these resource or input categories are variable, most nutrition studies (and this investigation specifically) are carried on in the frame-

[1]Projects 1135 and 1195 of the Iowa Agricultural Experiment Station.

work of a production relationship such as that represented by equation 2.

$$(2)\quad M = f(C, F \mid X_1, X_2, X_3, X_4 \ldots\ldots X_n)$$

Here, only the resources or inputs to the left of the vertical bar are considered variable. Labor required to handle cows under different rations is necessarily varied in an experiment. For experimental purposes, however, labor is assumed to be available in unlimited quantities at no cost. While labor as a variable must be considered in terms of its cost or price in profit decisions on the farm (e.g., more labor may be required to feed a specific grain/hay ratio than to hand-feed grain and self-feed hay), this step is unnecessary in technical experiments involving only estimation of feed substitution rates or feed-milk transformation ratios.

In a generalized production function such as equation 2, variables to the right of the vertical bar are considered fixed. While researchers try to control these variables by selecting cows of similar breed, weight and inherent milk-producing capabilities, controls are never sufficiently rigid to hold their magnitudes constant. However, under the assumption that differences in magnitudes of variables to the right of the bar can be considered as stochastic or random disturbances, the production function may be analyzed in the manner of equation 2. This procedure is followed in the experiment reported, with the exception that most of the estimates include three or four variables.

Geometric Forms

A milk production function involving two variable categories of feed can be represented as a three-dimensional diagram or surface. Milk output per cow in the relevant production period is measured on the vertical axis while each category of feed is measured on the respective horizontal axis. Each point in the feed plane represents a different ration and level of feeding and will correspond to a particular level of milk represented on the milk surface. The particular nature of this surface, including the slope of the inclines over it and the slope of the contours around it, will determine (1) the forage/grain ration which gives lowest cost for any stated level of milk output, (2) the level of feeding which will result in maximum profit per cow over feed costs and (3) the extent to which conventional ENE or TDN evaluations of the energy or heat transformation of feed are appropriate for evaluations concerned with milk transformation of these same feeds.

Unfortunately, little is known about the nature of the milk production surface. Numerous feeding standards suppose that the milk production function is homogeneous of degree 1.0: The surface is implicitly assumed to be linear up to the limits of the cow's milk producing capacity. Some standards, such as the *total digestible nutrient* (TDN) basis for rations consider the milk isoquants and input-output curves to be straight lines since they are not varied to consider the proportion and level of feeding or milk production. Other nutrition recommendations suppose the milk surface to have nonlinear isoquants and inclines. This is because ration recommendations seldom include only one class of feed (i.e., grain or forage), a recommendation which would be the most profitable one if the milk isoquants were linear. Since so little is known about the milk production surface and since its geometric form is extremely important in feed substitution possibilities and ration recommendations, some alternatives and their implications are outlined in the following paragraphs.

Production Surfaces and Milk Isoquant Maps

A production surface resulting from variation in two resources can take numerous forms. One possibility is a linear production function represented by the surface in fig. 1. This surface supposes that the added milk obtained for each added pound of feed from a particular ration is the same (i.e., the milk/feed transformation ratio is a constant) over all levels of feeding. A different ration is represented by each possible diagonal line which can be drawn in the feed plane. Hence, feed input-milk output lines, such as R_1, R_2, R_3 or R_4, are straight lines. A second characteristic of a linear production function is depicted by straight-line product isoquants or milk contours. The product isoquants represent all the possible combinations of feeds which will produce a given level of output. Lines a, b, c and d are such isoquants, representing increasing levels of production as feed intake is increased.

These isoquants are horizontal lines on the production surface, with their vertical distance representing level of milk output. Their counterpart can be reproduced in the feed or input plane to denote the many feed combinations which result in the particular milk output. As is pointed out later, a ration represented by a line such as R_2 or R_3 in fig. 1 is never the least-cost or most profitable ration when the production surface is linear. The best ration for any one milk production level will always be found at the ends of the milk isoquants (i.e., lines a, b, c and d) and will be represented by an extreme ration such as R_1 or R_4.

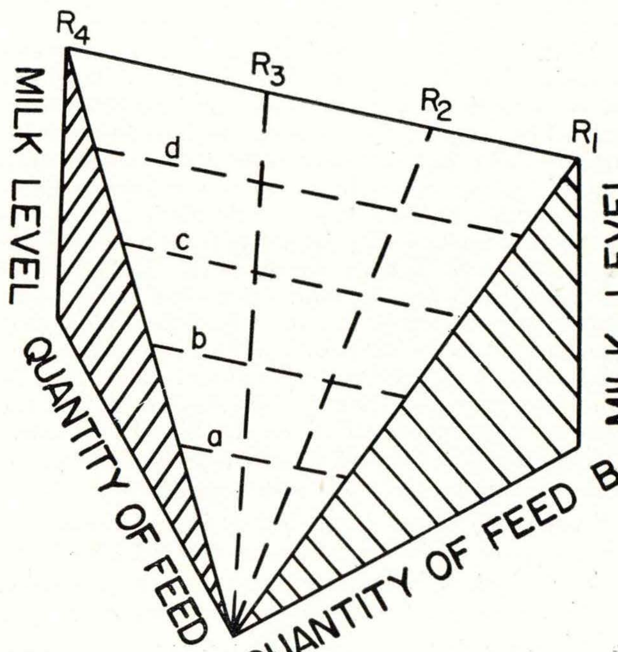

Fig. 1. Production surface for linear production function.

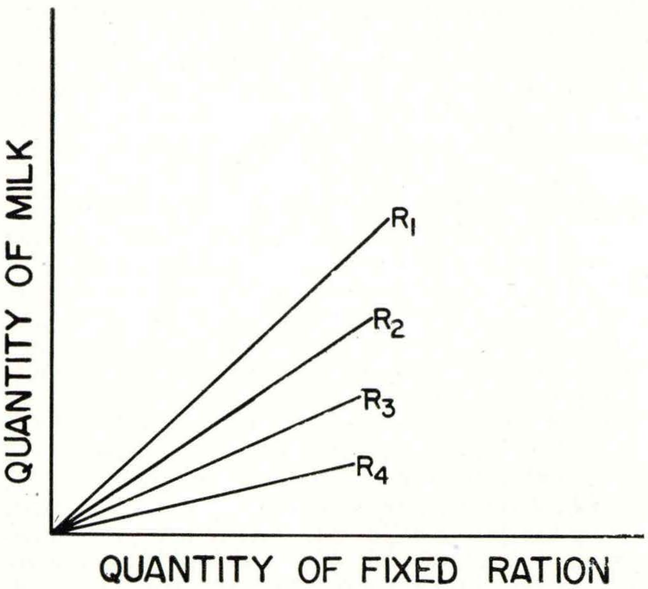

Fig. 2. Feed milk input-output lines from fig. 1.

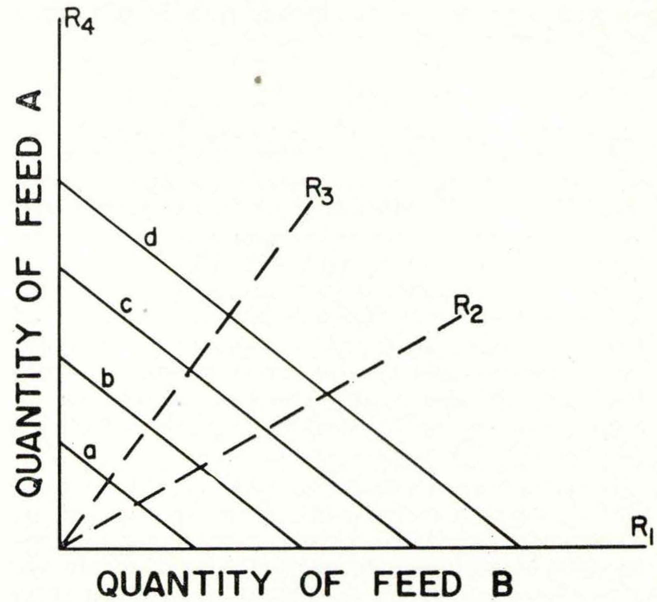

Fig. 3. Milk isoquants for fig. 1.

Two sets of relationships in milk production or dairy cow nutrition are represented by a production surface such as fig. 1. One is the feed/milk transformation line showing level of total milk output as intake of a given ration is increased (e.g., lines R_1, R_2, R_3 and R_4 in fig. 1). In this relationship the ration, or ratio of grain to hay, is a constant but total milk output and total feed input are variable. The second relationship is the grain/hay relationship, where the level of milk output is a constant but the quantities of each feed (the ratio of the two feeds) are variable (e.g., milk isoquants a, b, c and d).

These two sets of relationships can be shown separately by reducing the three-dimensional model of fig. 1 to two-dimensional figures such as figs. 2 and 3. For a production surface homogeneous of degree 1.0, such as shown in fig. 1, the feed input–milk output or transformation lines are linear denoting that a cow should be fed at the limits of her capacity if she is to be fed only grain. The slopes of the input-output lines indicate the rate at which feed is transformed into milk at each level of feeding. Since the slopes of the lines in fig. 2 are constant, the feed/milk transformation ratio also is constant. Similarly, the isoquants form a two-dimensional map such as that shown in fig. 3. The slopes of the isoquants are the substitution rates between the feeds. Since the slopes are constant, substitution rates also are constant for all possible feed combinations. The fixed ration lines represented in figs. 1 and 2 can be reproduced in fig. 3 (lines R_1, R_2, R_3 and R_4). If the function is homogeneous of degree 1.0, the segments of each ration line are of equal length between points of intersection by successive milk isoquants.[2]

While figs. 1, 2 and 3 represent one set of alternatives for the production function with two variable categories of feed, it is unlikely that the milk production function corresponds to this strictly linear form over all possible levels of feeding and all possible feed ratios. If the milk production surface were strictly linear, there could be no limits (1) to milk output per cow or (2) in the feed ratios which could be used to attain any specified level of milk output per cow. While milk isoquants may approach linearity over part of their range, it is unlikely that the isoquants (fig. 3) or input-output lines (fig. 2) are perfectly straight at all extremes.

An alternative production surface is shown in fig. 4. This production function is characterized by (1) non-

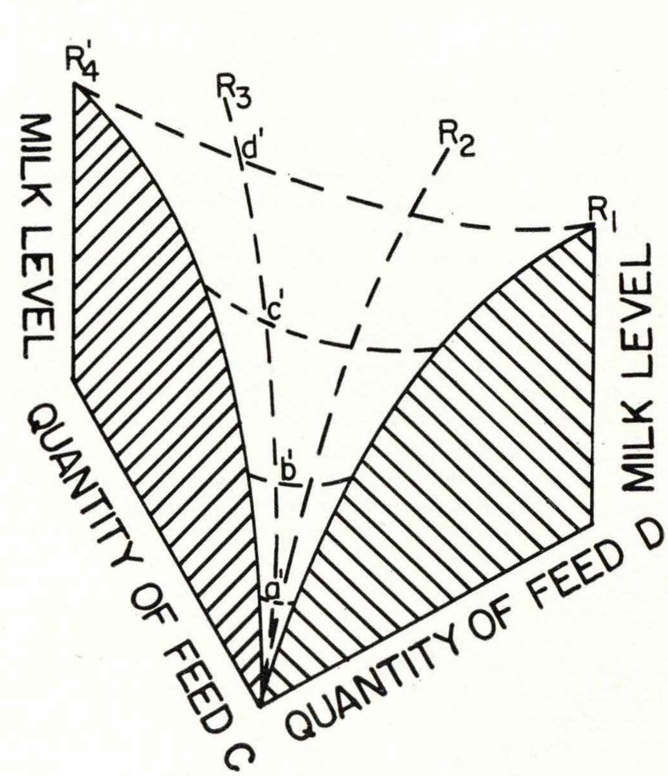

Fig. 4. Production surface with diminishing feed productivity and substitution rates.

[2]This statement applies only where successively higher milk isoquants represent the same increment in output per cow. In other words, "a" in fig. 3 may represent 500 pounds of milk while b, c and d represent 1,000 1,500 and 2,000 pounds, respectively.

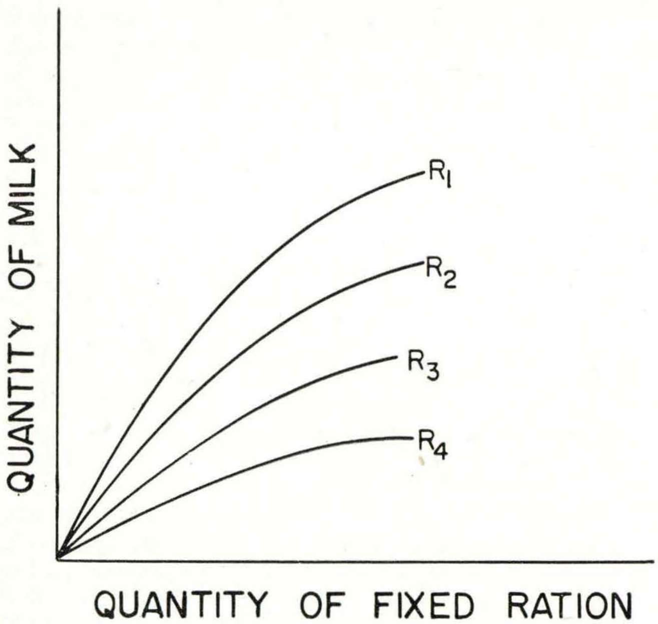

Fig. 5. Feed milk input-output lines from fig. 4.

linear feed input–milk output lines and (2) isoquants which are curved for a particular milk output. These geometric characteristics indicate that the rate at which feed is transformed into milk declines as higher levels of feeding and greater milk outputs are attained. They also indicate that, with milk held constant, the two categories of feed substitute at diminishing rates as more of one and less of the other is fed. In other words, increasing quantities of one feed are required to replace constant quantities of the other.

The basis of these statements also can be explained by use of figs. 5 and 6. In fig. 5, the slope of the feed input–milk output lines decreases with total feed intake denoting that each added quantity of feed results in smaller and smaller increments of milk, with the ration consisting of a constant ratio of the two feeds. Similarly, the isoquants of fig. 6 are nonlinear; their changing slopes indicate that the rate of substitution changes as the ratio of one feed declines relative to the other. Since the slopes of the isoquants decline with a greater input of feed D, the rate of substitution of feed D for feed C declines as the ratio $\frac{\text{feed C}}{\text{feed D}}$ declines along a given isoquant. Conversely, the slope of the isoquant increases as the ratio $\frac{\text{feed C}}{\text{feed D}}$ increases, denoting that the rate of substitution of D for C is increasing. Finally, the fact that the segments of the fixed ration lines (R_1, R_2, etc.) increase in length between points of intersection by milk isoquants (a, b, c and d), representing equal increments in milk level, indicates that diminishing returns exist for any one ration; increasing quantities of the ration are required for equal additions to milk output.

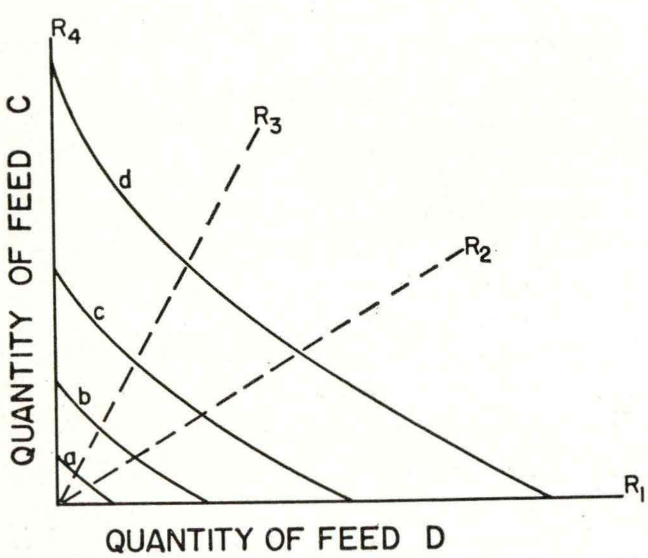

Fig. 6. Milk isoquants for fig. 4.

Again it is unlikely that a surface such as fig. 4 adequately represents feed and production relationships in dairying. To correspond with this model, milk output per cow would not be limited, and an increasingly large number of rations could be used in attaining higher levels of milk production. Also, since the surface extends to either feed axis, one feed could be substituted completely for the other feed in attaining specified levels of milk production. Although the exact nature is yet to be established, a more logical model of the milk production surface is shown in fig. 7: Diminishing returns exist for transforming any one ration into milk since the surface declines in slope at higher levels of output. Also, diminishing substitution ratios are indicated between feeds since the slope of the milk isoquants changes with feed ratios or rations. In contrast to fig. 4, however, there are limits to the level of milk production which can be attained. Maximum milk production per cow is at level M. Only one ration and level of feeding will result in maximum milk production and is denoted at point G in the feed plane.

Milk is forthcoming if a hay ration alone is fed (i.e., if level of feeding is extended along the forage axis OF). However, the maximum level of milk production, FH, under a pure forage ration is lower than the maxi-

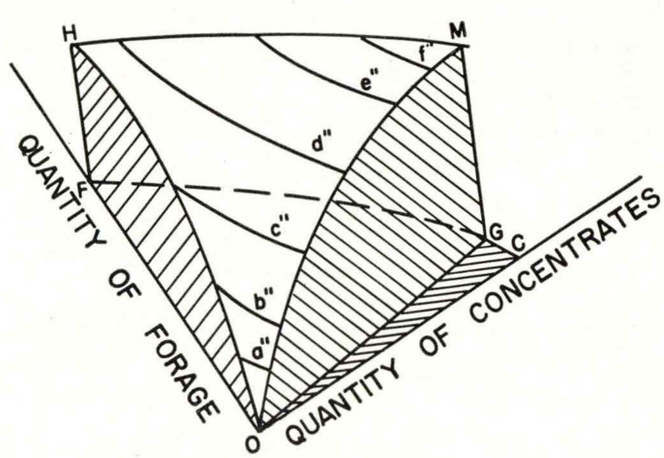

Fig. 7. Production surface with limits of substitution.

mum level GM attainable under the ration following line OG in the feed plane. With the cow fed hay only, OF represents the quantity denoting her stomach capacity. Supposedly, if a cow were fed to the highest milk level possible from a pure hay ration, and if milk level FH were attained, the amount of hay could be reduced along a stomach limit line. By replacing it with successive quantities of grain, milk production could be raised to higher levels by following the line on the surface denoted as HM until the maximum milk level of GM is attained. The *milk level* line denoted as HM parallels the *feed line* FG in the feed plane. Line FG also represents the limit of the cow's stomach capacity and indicates that, to attain higher milk levels along HM, the amount of forage must be decreased as grain intake and milk production are increased. The extreme grain ration denoted as OG does not follow along the grain axis OC, under the postulate that a physiological minimum of forage must be included in the ration if lactation is to extend over a long period. Hence, the area OFG in the feed plane represents the limits of rations with respect to (1) the grain/hay ratios which may be fed and (2) the maximum possible intake of any particular grain/hay ratio or ration. Over the top of the production or milk surface (i.e., the area OMH in fig. 7) are milk contours or isoquants such as a″ through f″, each indicating a particular milk level. If lines corresponding to these milk isoquants are drawn in the *relevant feed plane,* OFG, they indicate all of the possible rations which allow attainment of a particular milk level.

The two-dimensional isoquant map corresponding to the model of fig. 7 is shown in fig. 8. Line H′M′ in fig. 8 is the *stomach limit* line and parallels line FG (HM) in fig. 7; O′M′ is the *physiological limit* line corresponding to OM in fig. 7. The isoquants a″ through f″ are the counterparts of those shown in fig. 7 and, again, indicate all possible rations or feed combinations which allow attainment of the specified milk output. As mentioned previously, the slope of these isoquants indicates the grain/hay substitution ratio. If the isoquants are straight lines over the entire area O′M′H′, the two

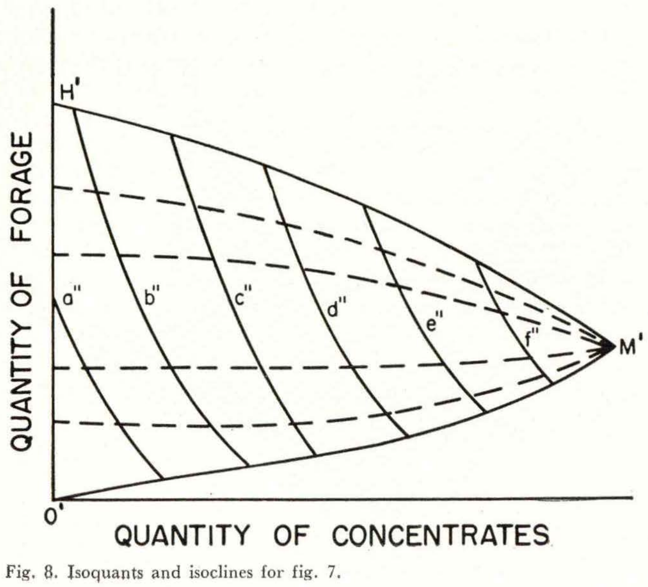

Fig. 8. Isoquants and isoclines for fig. 7.

feeds would substitute at constant rates regardless of the ratio of feeds fed. If they are nonlinear, the substitution rate will change as the ration or feed ratio changes. The slope must decrease along a particular isoquant as the feed ratio approaches the boundary O′M′, if diminishing substitution ratios are to hold true.

An additional relationship shown in fig. 8 was not included in figs. 3 and 6.[3] It is the *isocline.* An isocline connects all points on successive isoquants of the same slope (i.e., all points which have the same substitution ratio). There is a family of isoclines, with one for every possible slope or substitution rate on the milk isoquants, if the isoquants are nonlinear. The dotted lines in fig. 8 provide the concept of a family of isoclines. These converge at the point of maximum milk production, M′. Actually, the stomach limit and the physiological limit lines are the boundaries of the isoclines and represent isoclines with zero substitution rates or possibilities.

An isocline has particular economic meaning; a specific isocline shows, for the appropriate price ratio, the least-cost feed ration for attaining a given milk level. As milk output is pushed to higher levels, it should follow the path of the appropriate isocline (i.e., the one tracing the slope on the isoquants equal to the feed price ratio) if the least-cost ration is to be attained for successive levels of milk production. Of course, the most profitable level of milk production will depend on the price of milk relative to the price of feed. While the isoclines shown in the schematic presentation of fig. 8 are nonlinear, it is possible that they are linear, converging to the point of maximum milk output per cow.

The geometry of this section presents some alternatives in the nature of the milk production surface. Other alternatives also exist.[4] However, the general nature of fig. 7 appears to be the most likely hypothesis of the milk production surface. The nature of the surface needs to be established if problems such as (1) the most profitable ration under various feed and milk price ratios and (2) the possibilities of substituting forage for grain under production control or soil conservation programs, are to be solved. Research studies need to be designed which predict such items as (1) the location of the stomach and physiological limit lines, including indications of whether they are linear, convex or concave, (2) the nature of the milk isoquants, including indications of their degree of linearity and their slopes (or substitution ratios) for different rations and milk levels and (3) the nature of the isoclines, including their curvature. Several complex experiments will likely be needed before these relationships and quantities can be fully established.

Economic Principles for Rations

Prediction of the milk production surface allows the use of economic principles in designating the least-cost feed combination or ration. One problem in nutrition

[3]The input-output lines shown in figs. 2 and 5 paralleling the surface of figs. 1 and 4, respectively, have not been constructed for the surface of fig. 7. The input-output curves for the latter surface would parallel those for fig. 4, with modifications to recognize the stomach limits of FG in fig. 7.

[4]One unlikely extreme would be a "knife's edge" surface such as represented in figs. 3 and 4 in: Heady, Earl O., *et al.* Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Exp. Sta. Res. Bul. 424. (10). Also, see figs. 1 and 5 in this same publication for other alternatives of the production surface.

economics is to determine the ration or feed ratio which gives the lowest cost for a specified level of milk output. The least-cost ration for a given production level is specified when the condition of equation 3 is attained.[5]

$$(3)\quad \frac{\triangle F}{\triangle G} = \frac{P_g}{P_f}$$

Here $\triangle F/\triangle G$ is the substitution ratio of grain for forage, P_g is the price of grain and P_f is the price of hay. Where the mathematical function has been predicted, dF/dG, the derivative of forage with respect to grain takes the place of $\triangle F/\triangle G$ and can be equated with the price ratio to specify the least-cost ration. If dF/dG is a constant, such as is true when the isoquants are straight lines with constant substitution rates, the least-cost ration will fall at the extreme of (1) the stomach limit line (H′M′ in fig. 8) for forage or (2) the physiological limit (O′M′ in fig. 8) for grain. If the substitution ratio, dF/dG, is greater than the price ratio, P_g/P_f, then a ration should be selected along the physiological limit line such as O′M′ in fig. 8, for a particular milk level such as b″. The feed intake or rations should be extended along this limit if higher milk levels are to be attained with lowest feed costs. If dF/dG is less than P_g/P_f, the ration should be selected on the stomach limit line for a particular milk isoquant. It should follow along this line if higher milk levels are to be attained at least-cost.

However, if the isoquants are not linear, the substitution ratio, the derivative or slope of the isoquant, takes on different values for different ratios of the two feeds. It is then possible for the least-cost ration to fall between the limit lines and to change with variations in the relative price of the two feeds.[6] Isoclines such as those shown in fig. 8, will exist for each value of dH/dG denoting, by their point of intersection with each isoquant, the least-cost ration to produce a given milk level.

The optimum level for feeding any one ration is defined by equation 4 where: $\triangle M$ refers to the increment in milk production associated with

$$(4)\quad \frac{\triangle M}{\triangle R} = \frac{P_r}{P_m}$$

$\triangle R$, the increment of the particular ration; $\triangle M/\triangle R$ is the marginal milk product, the amount added to total milk production by a unit increase in R; P_r is the price per unit of the ration and P_m is the price per unit of milk.[7] Once the milk production function or equation is known, dM/dR, the derivative of the appropriate equation can be substituted for $\triangle M/\triangle R$ to measure the marginal productivity of the ration. The quantity dM/dR will be a constant for a linear milk production surface but will vary in magnitude for a nonlinear function. The derivative then is equated to the price ratio to determine the most profitable level of feeding for the particular ration.

Equation 4 specifies the optimum level of feeding if a particular ration is to be fed, but it does not specify which ration is the least-cost or most profitable one. Equation 3 specifies the ration which gives lowest cost, but it does not specify the level of feeding (i.e., the isoquant) which is most profitable. Hence, the two conditions—(1) the optimum ration for any specific level of milk production and (2) the optimum level of feeding and milk production possible from alternative rations—need to be solved simultaneously. This procedure can be accomplished by setting the partial derivatives of milk with respect to each feed to equal the respective feed/milk price ratio as indicated by equations 5 and 6. By solving these equations simultaneously, the quantity of forage and grain to be fed for maximum profit above feed costs

$$(5)\quad \frac{\partial M}{\partial G} = \frac{P_g}{P_m}$$

$$(6)\quad \frac{\partial M}{\partial F} = \frac{P_f}{P_m}$$

per cow can be solved. Once the quantities are determined for G and F, the level of milk production also can be predicted.[8]

BASIC MODELS

The models and principles of this section provide the methodology for the design of the experiment and the empirical analysis which follows. The experiment to be explained is a modest one, although involving considerable outlay of facilities and funds, and applies only to particular circumstances of feeds, breeds and environmental characteristics. Added experiments are needed to provide predictions of the milk production surface under other conditions and to provide more conclusive

[5]When the equality of equation 3 has been attained $(\triangle F)(P_f) = (\triangle G)(P_g)$ and the value of one feed replaced is just equal to the value of the other feed added. If the equality in equation 3 does not hold true, we have a condition such as (a) below where the substitution ratio is greater than the price ratio. The value of feed F replaced is greater than the value of feed G added and

$$(a)\ \frac{\Delta F}{\Delta G} > \frac{P_g}{P_f} \text{ or } (\triangle F)(P_f) > (\Delta G)(P_g)$$

costs can be lowered by making further substitutions of this kind. Condition (b) represents a case, with the marginal rate of substitution smaller than the price ratio, where the opposite holds true.

$$(b)\ \frac{\Delta F}{\Delta G} < \frac{P_g}{P_f} \text{ or } (\Delta F)(P_f) < (\Delta G)(P_g)$$

In these equations, the marginal rate of substitution should have a minus sign before it. However, this technical convention is not used since the results are the same when the substitution rate as a positive quantity is related to a positive price ratio.

[6]It is possible, however, for the curvature of the isoquants to be so little that dF/dG may still be larger or smaller than the price ratio at the limit line, denoting that the latter trace out the alternatives of least-cost rations.

[7]Under conditions of equation 4, it is also true that $(\Delta M)(P_m) = (\Delta R)(P_r)$, and the value of the added milk just equals the value of the added feed. If the marginal product is less than the price ratio, equation (a) holds true and profit can be increased by feeding less since the value of the milk sacrificed is less than the value of the feed subtracted.

$$(a)\ \frac{\Delta M}{\Delta R} < \frac{P_r}{P_m} \text{ or } (\Delta M)(P_m) < (\Delta R)(P_r)$$

If the marginal product is greater than the price ratio, higher feeding levels will add more to the value of milk produced than to the value of the feed added, as indicated in (b):

$$(b)\ \frac{\Delta M}{\Delta R} > \frac{P_r}{P_m} \text{ or } (\Delta M)(P_m) > (\Delta R)(P_r)$$

[8]The principles outlined in preceeding paragraphs refer to maximum profits above feed costs. The solutions might differ somewhat from farmer recommendations or decisions because of the labor costs involved in different methods of feeding. However, these and other costs can be incorporated into the profit equations; a step not taken here.

evidence of the nature of isoquants, isoclines and other relationships even for the particular variables of this study.

Previous Work Relating to Nature of Isoquants and Milk Surface

While no previous experiment has been designed to predict the nature of the milk production surface, the results of other studies do lead to hypotheses about substitution and transformation coefficients. Huffman and Duncan (11) discuss the possible stimulating effect of a small amount of grain when a cow has been fed forage alone. They state that a cow receiving forage alone will not produce as much milk (FCM) as when a small amount of grain is substituted for an equal amount of TDN from hay. These results lead to the supposition that the milk isoquant may curve rather sharply at the forage end and is, therefore, nonlinear.

Jensen *et al.* (12) predict grain input-output curves which are nonlinear. Although the report provides no postulates about milk isoquants, it appears that curved isoquants should be associated with curved input-output curves for any mathematical function used to define the optimum or maximum milk production per cow. Beach (2) indicates that as more grain is fed to a cow, the maintenance requirements (i.e., at the zero milk isoquant) in terms of TDN become less and less. As the ration approaches an all-grain ration, TDN from grain eventually become less efficient in maintaining health and activities of the cow. While the Huffman-Duncan experiment suggests that input-output curves may be nonlinear and that the milk isoquants are curved on the hay extremity, the work of Beach suggests curved isoquants towards the grain extremity. Of course, the isoquants might have greater curvature at the ends but be nearly linear in the middle.

In a study based on a sample of dairy farms, Ashe (1) concludes that the input-output curves follow a near linear relationship up to about 4,000 pounds of grain per cow; between 4,000 and 6,000 pounds of grain, milk increases only slightly; and over 6,000 pounds of grain, milk does not increase at all. Yates and others (24) summarize several research reports implying a diminishing rate of transformation of feed into milk. These studies included both European and American data and an "economic curve" derived from Danish experiments indicating considerable decline in marginal milk yields at high feeding levels.

Recommended daily energy allowances for milk production are summarized by the Committee on Animal Nutrition of the National Research Council (17). Because of lack of appropriate input-output data, its recommendations do not consider the concept of diminishing returns to dairy cow rations. The committee's approximate guide gives 0.32 pound TDN as the requirement for each additional pound of 4-percent, fat-corrected milk (FCM) above maintenance, but it does not indicate any changes in the TDN transformation due to the combination of feeds and/or to the level of feeding. The Morrison (16) recommendations, used extensively in this country, have the same over-all recommendations for TDN, whereas recommendations based upon the net-energy (NE) system estimate 0.30 therms for each additional pound of 4-percent FCM above maintenance requirements. Blaxter (3), in a review of the energy standards for dairy cattle, states that energy standards should express the productivities of feeds as they are. However, he does not indicate that feeding values should vary with the proportions of feed fed.

The dependence of the ruminant upon the catabolic and anabolic activities of the rumen microflora implies that certain combinations of hay and grain may stimulate or depress microbiological activity and, hence, exert an effect on the nature or curvature of the isoquants. This problem is pointed out in studies by Hamilton (9) and Swift *et al* (22) who report a depression of ration digestibility at a hay-to-grain ratio of 1:1. The concepts of the "stomach capacity" and "physiological limit" lines already mentioned have further ramifications in dictating the size and feed capacity of the cow which can be used most economically under varying economic conditions. The problem of genetic ability or "dairy merit," which is defined by the percentage of consumed energy that is converted into milk energy, and its basic interrelationships with the plane of nutrition involved in the lactational level desired, must also be considered in determining the milk production surface.

DESIGN OF EXPERIMENT

The basic experiment for the predictions of this study included 36 cows and is explained in detail elsewhere.[9] This experiment, conducted without the use of pasture, extended over a 17-month period in 1953 and 1954. For certain predictions and estimates, data from an earlier experiment including 15 cows were also used. Both experiments are explained below. The experimental design and over-all study represent an interdisciplinary approach by dairy nutritionists and production economists, with the aid of specialists in statistics. Individuals involved held exploratory seminars to discuss the logical models, based on previous knowledge in nutrition and economics. The design of the experiment and the prediction methods were then selected to conform to these models. It should be pointed out, however, that the researchers involved do not hold that the design used is optimum. The experiment is a relatively small one, but one feasible with the limited resources and facilities available.

Basic 1953-54 Experiment With 36 Cows

The Holstein herd at the Iowa State College Dairy Farm was the source of the 36 cows used from March 1953 to September 1954. From the date of calving, each cow remained under experimental conditions for an initial 14-day adjustment period. A fixed ratio of 7 pounds of hay to 4 pounds of concentrates, initiated during the adjustment period, was maintained throughout the preliminary period. The preliminary period provided the basis for dividing the animals into high-, medium- and low-producing ability groups for their subsequent random allotment to the experimental period. In general, the production ability ranges for the animals in terms

[9]For a detailed explanation of the experiment, see: Bloom, Solomon. Unpublished Ph.D. thesis (4).

TABLE 1. NATURE OF ALLOCATION OF ANIMALS BY LEVEL OF FEEDING AND ABILITY. (FIGURES ARE NUMBERS ASSIGNED TO COWS).

Level of feeding	Ability	Hay-to-concentrate ratio			
		H-75:C-25	H-55:C-45	H-35:C-65	H-15:C-85
High	High	2,553	3,157	2,710	2,982
	Medium	2,392	2,649	3,529	2,976
	Low	3,632	3,263	3,160	3,538
Medium	High	3,266	2,600	2,378	3,142
	Medium	3,469	3,444	2,643	3,493
	Low	3,440	3,597	3,432	3,516
Low	High	3,272	3,291	2,963	3,174
	Medium	3,450	3,483	3,128	2,606
	Low	3,302	3,294	3,439	2,159

of pounds of 4-percent FCM were as follows: (a) "high"—10,500 pounds and over, (b) "medium"—9,000 to 10,500 pounds and (c) "low"—9,000 pounds or less.

The design for the experimental period of 182 days is illustrated in table 1. Four of the hay-to-concentrate ratios were chosen, ranging from a ration in which 75 percent of the energy (ENE) was derived from hay and 25 percent from concentrates, to one in which 15 percent of the energy was derived from hay and 85 percent from concentrates. The four hay-to-concentrate ratios were 75:25, 55:45, 35:65 and 15:85. Each of these hay-to-concentrate ratios was fed at high, medium and low levels. For each of the 12 "hay-to-concentrate ratio-feeding level" treatments, there were three cows—one of high-, one of medium- and one of low-producing ability. At the start of the experimental period, each cow, after its assignment to an ability group, was randomly allotted to one of the 12 "hay-to-concentrate ratio-feeding level" positions.

Auxiliary Experiment

While the main analysis of this study rests on the basic experiment explained above, cows from the following experiment were used for predictions explained later. The auxiliary experiment is that reported by Martin *et al.* (14). Fifteen cows were used from this experiment to evaluate four levels of alfalfa hay feeding. Rates of hay feeding during the experimental period were at 0.5, 1.17, 1.83 and 2.5 pounds per day per 100 pounds of body weight. A 2-week preliminary period was used in which the cows were fed hay at a daily rate of 2 pounds per 100 pounds of body weight and grain to supply 100 percent of Morrison's (16) recommended levels. Cows were on the auxiliary experiment for 16 weeks.

BODY WEIGHT CHANGES

Table 2 summarizes the body weight changes from two aspects: the total changes over the entire 182-day experimental period and the changes from the end of the initial 4 weeks of the experimental period to its conclusion. The results of the weight changes over the two periods are presented in the analyses of variance in tables 3 and 4. When the entire experimental period was considered (table 3), the effects of ration, feeding level and ability upon body weight changes either ap-

TABLE 3. ANALYSIS OF VARIANCE: BODY WEIGHT CHANGES DURING THE ENTIRE EXPERIMENTAL PERIOD.

Source of variation	d.f.	Sum of squares	Mean squares	F	Component of variance	(%)
Ration	3	33,167	11,056	3.30†	$^{2}_{R}$	9
Level	2	30,207	15,104	4.51*	$^{2}_{L}$	15
Ability	2	32,906	16,453	4.91*	$^{2}_{A}$	15
R x L	6	22,949	3,825	1.14	$^{2}_{RxL}$	2
R x A	6	30,826	5,138	1.53	$^{2}_{RxA}$	9
A x L	4	11,792	2,948	0.88	$^{2}_{AxL}$	0
R x L x A	12	40,200	3,350		2	51
Total	35	202,047				

*$P < 0.05$
†Approaches $P < 0.05$

TABLE 4. ANALYSIS OF VARIANCE: BODY WEIGHT CHANGES FROM THE END OF THE INITIAL 4 WEEKS OF THE EXPERIMENTAL PERIOD TO ITS CONCLUSION.

Source of variation	d.f.	Sum of squares	Mean squares	F	Component of variance	(%)
Ration	3	26,268	8,756	2.54	$^{2}_{R}$	6
Level	2	8,951	4,476	1.30	$^{2}_{L}$	0
Ability	2	705	353	0.10	$^{2}_{A}$	0
R x L	6	32,062	5,344	1.55	$^{2}_{RxL}$	14
R x A	6	25,333	4,222	1.23	$^{2}_{RxA}$	5
A x L	4	10,441	2,610	0.76	$^{2}_{AxL}$	0
R x L x A	12	41,301	3,442		2	75
Total	35	145,061				

TABLE 2. BODY WEIGHT CHANGES DURING TWO TIME INTERVALS OF THE EXPERIMENTAL PERIOD.

Level of feeding	Ability	H-75:C-25*			H-55:C-45*			H-35:C-65*			H-15:C-85*		
		Cow	Change (lbs.) 0-26 wks.	4-26 wks.	Cow	Change (lbs.) 0-26 wks.	4-26 wks.	Cow	Change (lbs.) 0-26 wks.	4-26 wks.	Cow	Change (lbs.) 0-26 wks.	4-26 wks.
High	High	2,553	99	200	3,157	-69	-48	2,710	42	79	2,982	-80	-21
	Med.	2,392	59	125	2,649	-33	-46	3,529	92	179	2,976	-82	-38
	Low	3,632	-45	-47	3,263	-24	-7	3,160	110	69	3,538	51	20
Med.	High	3,266	-22	7	2,600	-78	-31	2,378	14	64	3,142	-102	3
	Med.	3,469	40	57	3,444	36	26	2,643	46	22	3,493	-56	10
	Low	3,440	26	39	3,597	-25	-32	3,432	55	29	3,516	58	120
Low	High	3,272	-116	53	3,291	2	40	2,963	-153	-21	3,174	-178	-31
	Med.	3,450	-67	-24	3,483	-61	-5	3,128	23	89	2,606	-164	-91
	Low	3,302	92	55	3,294	-2	19	3,439	-70	-4	2,159	21	36

*Hay-to-concentrate ratio.

proached significance or were significant at the 5-percent level. However, when the last 22 weeks of the experimental period were considered these effects disappeared (table 4). From the first to the second case, the components of variance for ration, feeding level and ability decreased markedly. These results indicate that the body weight changes, after the first 4 to 5 weeks of the experimental period, were largely independent of the ration fed, the level of feeding and the ability of the animals.

MILK PRODUCTION FUNCTIONS FOR 36 COWS IN 26-WEEK BASIC EXPERIMENT IN THE OVER-ALL PERIOD

The production functions estimated below for the 36-cow, basic experiment conducted in 1953-54 over a 26-week experimental period are the algebraic counterparts of the production surfaces discussed in respect to figs. 1 through 8, with the exception that most of them include three variables. The algebraic predictions of the milk production function provide the mathematical basis by which it is possible to derive the isoquants, substitution rates, input-output curves, transformation ratios and isoclines explained earlier.

Two sets of functions have been derived from the basic 36-cow experimental data: (1) those where a single time period is included and (2) those where time is considered as a variable. While predictions for a single and constant time period are included as methodological materials, it is believed that the production functions which include time as a variable provide the most efficient estimates. The time variable allowed some for changes in body weight, as well as the normal trend in milk output over the lactation period. As the data in table 2 indicate, cows attained a near-equilbrium in weight after a month of the experiment.

Initial Equations for 36-Cow Basic Experiment Over 26 Weeks (Time Not a Variable)

Since little is known about the milk production function, three initial types of algebraic equations which do not include time as a variable were fitted to the data. These include (1) a logarithmic equation, (2) a particular form of a quadratic equation and (3) a quadratic equation with square root transformations. The logarithmic equation was selected as one of a general type (including those such as exponential, Mitcherlish, etc.) which does not require specification of a single maxima in milk production but allows asymptotic estimates. It is known that such a function may not conform adequately to a milk production surface since it assumes constant elasticity of production, linear isoclines and increasingly wide ranges of rations for higher levels of milk production.[10] However, it was thought that the function might allow reasonable estimates of substitution ratios and transformation coefficients in the midsection of the milk surface. The other two equations allow specification of one ration consistent with maximum milk production per cow and of isoclines which converge to this point.

[10]Actually, the surface depicted by this equation forms a ridge rather than a peak, with level of production becoming asymptotic to a limit at the ridge peak.

VARIABLES AND REGRESSION EQUATIONS.

Three variables were used in estimating the initial milk production functions. Two of these are the concentrate and hay feeds discussed earlier. The third is cow ability since this variable was found to be highly associated with milk production in the experimental period. Hence, the variables for the functions which follow are those explained below where the milk output and feed input measurements are aggregates extending over the 26-week experimental period of the basic experiment:

M is production in pounds per cow of 4-percent fat-corrected milk in the 26-week experimental period.

H is pounds of alfalfa hay measured in pounds per cow over the 26-week experimental period.

G is concentrate mix, called grain hereafter, measured in pounds per cow over the 26-week experimental period.

A is cow ability measured in pounds of 4-percent fat-corrected milk produced in the 50-day preliminary period when all cows were fed the same ration as detailed earlier.[11]

The 26-week initial functions for the 36-cow basic experiment are shown as equations 6a, 7a and 8a for the logarithmic, quadratic and square root functions, respectively.

$$(6a)\quad M = 15.749H^{0.1213}\,G^{0.2758}\,A^{0.3659}$$

$$(7a)\quad M = 3{,}787.56 - 0.1288H + 0.9842G - 1.0991A + 0.000042H^2 - 0.000064G^2 + 0.000353A^2 + 0.000000032HGA$$

$$(8a)\quad M = 19{,}356.40 + 1.7855H + 1.1258G + 3.2040A - 300.0230\sqrt{H} - 183.0605\sqrt{G} - 226.5986\sqrt{A} + 2.6626\sqrt{HG}$$

The relevant statistics for these three functions are given in table 5. The t values are for the regression coefficients in their respective order within the equations above. From 73 to 78.6 percent of the total variance in milk production is accounted for in the three variables, depending on the function. All of the regression coefficients in the logarithmic function are acceptable at the probability level of 1 percent. However, none of the individual coefficients for the quadratic or square root functions are significant at this probability level, even though a larger portion of the total variance in milk production is accounted for by the variables of the latter two functions. In a pure proba-

[11]The simple correlation coefficient (r) between the ability term used and milk production is 0.64. The comparable statistic using age-corrected fat in previous lactations as the ability term yielded a correlation coefficient of 0.33.

TABLE 5. CORRELATION COEFFICIENTS AND t VALUES FOR EQUATIONS 6a, 7a AND 8a.

		Values of t in order of b's in equation						
Equation	R	b_1	b_2	b_3	b_4	b_5	b_6	b_7
6a	0.8545*	2.84*	5.65*	5.13*	......	......	......	......
7a	0.8832*	0.18	0.94	0.60	0.74	0.54	0.99	0.64
8a	0.8864*	1.21	0.45	0.91	0.84	0.39	0.66	0.73

*$p < 0.01$

bility sense, equation 6a might be accepted for prediction purposes while equations 7a and 8a might be dropped. However, even though their regression coefficients have relatively larger standard errors, the general forms of equations 7a and 8a might provide predictions which conform better to the logic of the milk production surface than does equation 6a, particularly since the latter function: (1) causes milk level to become asymptotic to a limit, rather than to approach a maximum; (2) causes the milk surface to pass up a "ridge" of wide ration latitudes, rather than to narrow to the peak of a single ration for the maximum milk production per cow; (3) does not allow low-level isoquants to be attained with hay or grain alone; and (4) causes linear isoclines which "fan out," rather than converge at the maximum milk production level. As is indicated in table 6, substitution ratios become extremely large or small at extreme rations for the logarithmic function. However, logarithmic equation 6a may provide useful estimates of marginal feed productivity and marginal substitution rates over a narrow range of the surface surrounding the mean level of feeding and the mean ratio of feeds in the experiment.

MARGINAL EQUATIONS FOR LOGARITHMIC FUNCTION.

Equations 6b and 6c, which are derived from 6a, define the marginal or incremental productivity of hay and grain, respectively, when one is variable and one is fixed in quantity. Both indicate diminishing productivity since the exponent on both the H and G variables is less than 1.0 in equation 6a. Also, since the sum of the exponents is less than 1.0, the function indicates a diminishing feed/milk transformation as any fixed ratio of hay and grain is increased in quantity. Equation 6d, also derived from 6a, provides the isoquants paralleling those outlined in the earlier section on logic. Equation 6e defines the slopes of the isoquants and, therefore, indicates the marginal rate of substitution of grain for hay if equation 6a is used for predictions.

TABLE 6. FEED COMBINATIONS AND MARGINAL RATES OF SUBSTITUTION (ΔH/ΔG) PREDICTED FROM LOGARITHMIC EQUATION 6a FOR MILK ISOQUANTS OF 5,300, 6,300 AND 7,300 POUNDS PER COW OVER THE 26-WEEK PERIOD (ABILITY FIXED AT MEAN FOR THE 36 COWS).

5,300 lbs. milk*			6,300 lbs. milk†			7,300 lbs. milk‡		
Lbs. grain§	Lbs. hay§	ΔH**/ΔG	Lbs. grain§	Lbs. hay§	ΔH**/ΔG	Lbs. grain§	Lbs. hay§	ΔH**/ΔG
1,000	5,815	13.22	....	....	....	....	....	....
1,500	2,313	3.51	1,500	9,308††	14.11	....	....	....
2,000	1,204	1.37	2,000	5,001	5.68	2,000	16,843††	19.15
2,500	724††	0.66	2,500	3,008	2.74	2,500	10,152††	9.23
3,000	478††	0.36	3,000	1,991	1.51	3,000	6,709††	5.08
....	....	....	3,500	1,401	0.91	3,500	4,714	3.06
....	....	....	4,000	1,034††	0.59	4,000	3,489	1.98
....	....	....	4,500	792††	0.40	4,500	2,661	1.34
....	....	....	....	....	....	5,000	2,098	0.95

*Columns 1 and 2 show feed combinations which will produce 5,300 pounds of milk; column 3 shows marginal rates of substitution of grain for hay for these combinations.
†Columns 4 and 5 show feed combinations which will produce 6,300 pounds of milk; column 6 shows marginal rates of substitution for these combinations.
‡Columns 7 and 8 show feed combinations which will produce 7,300 pounds of milk; column 9 shows marginal rates of substitution for these combinations.
§Derived from equation 6d.
**Derived from equation 6e.
††Outside of range of observations in experiment.

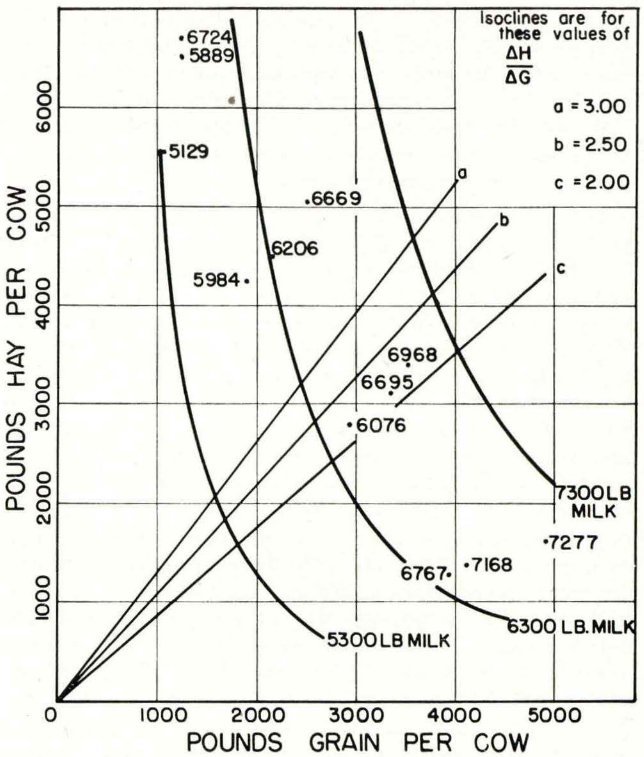

Fig. 9. Milk isoquants for a 26-week period predicted from equation 6a with ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

Obviously, the substitution ratio will be predicted to change as hay/grain ratios change, if function 6a is used for estimation.

$$(6b)\quad \frac{dM}{dH} = 1.906\ H^{-0.8787}\ G^{0.2758}\ A^{0.3659}$$

$$(6c)\quad \frac{dM}{dG} = 4.347\ H^{0.1213}\ G^{-0.7242}\ A^{0.3659}$$

$$(6d)\quad H = \left(\frac{M}{15.75G^{0.2758}\ A^{0.3659}}\right)^{8.2440}$$

$$(6e)\quad \frac{dH}{dG} = (-2.278)\frac{H}{G}$$

Isoquants for three levels of milk production predicted from the logarithmic function (i.e., isoquant equation 6d) are shown in fig. 9. The straight lines denoted as a, b and c are isoclines indicating all feed combinations which give a specified rate of substitution between hay and grain, if predictions are based on equation 6a. Line "a" shows, for example, all quantities of grain and hay where 1 pound of grain substitutes for 3 pounds of hay. The slope of the isoquants does not change greatly above isocline "a" with a $\triangle H/\triangle G$ substitution ratio of 3. The isoquants "bend" rather sharply below line c. These same phenomena are illustrated in table 6, the tabular counterpart of fig. 9, since the $\triangle H/\triangle G$ substitution ratios become very large for rations with a small proportion of grain and very small with a large proportion of grain. Isoquants with sharply changing slopes (rapid changes in substitution rates) at the extremes, such as found in the logarithmic equation, may actually be consistent with physiological processes of milk production. However, isoclines which "fan out" such as those in fig. 10 are inconsistent with a maximum milk output per cow, a condition possible only with converging isoclines.

If the logarithmic functions were accepted as the best predicting equations, the isoclines shown would indicate the least-cost ration for particular price ratios. For example, isocline b shows the points on the milk isoquants where 1 pound of grain substitutes for 2.5 pounds of hay. Hence, if the price of grain divided by the price of milk is 2.5, the points of intersection of the isoclines and isoquants show the least-cost rations for milk production levels of 5,300, 6,300 and 7,300 pounds per cow in the 26-week period. The dots in fig. 10 represent the 12 cows at the medium level of ability for four ration combinations and three feed levels. These dots provide a visual indication of whether the predicted isoquants seem consistent with the data. The average milk production for these 12 cows was 6,463 pounds. (All 36 cows served as the basis for predicting the isoquants in fig. 10.) The marginal rate of substitution of grain for hay at approximately the midpoint (2,800 pounds of hay and 2,600 pounds of grain) of the 6,300-pound milk isoquant is 2.45. This figure indicates that, at the particular feed combination, one more pound of grain would replace 2.45 pounds of hay, with milk output held constant at 6,300 pounds. Conversely, 1 pound of hay would substitute for 0.40 pound of grain if the particular equation were used for the predictions.

MARGINAL EQUATIONS FOR QUADRATIC AND SQUARE ROOT ESTIMATES.

The marginal milk product functions for grain and hay, as single variables, are indicated as equations 7b and 7c, respectively, for quadratic equation 7a; they are indicated as equations 8b and 8c, respectively, for square root function 8a.

$$(7b)\quad \frac{dM}{dH} = -0.1288 + 0.000085H + 0.000000032GA$$

$$(7c)\quad \frac{dM}{dH} = 0.9842 - 0.000129G + 0.000000032HA$$

$$(7d)\quad H = 1{,}524.0 - 0.9556G \pm 11{,}838.0\sqrt{-0.5309 - 0.000187G + 0.000000017G^2 + 0.000169M}$$

$$(7e)\quad \frac{dH}{dG} = \frac{0.9842 - 0.000129G + 0.000000032HA}{-0.1288 + 0.000085H + 0.000000032GA}$$

$$(8b)\quad \frac{dM}{dH} = 1.7855 - 150.0115H^{-0.5} + 1.3313G^{0.5}H^{-0.5}$$

$$(8c)\quad \frac{dM}{dG} = 1.1258 - 91.5302G^{-0.5} + 1.3313H^{0.5}G^{-0.5}$$

$$(8d)\quad H = \left[84.01 - 0.7456\sqrt{G} \pm (0.2800)\sqrt{7.1421M - 290.2395\sqrt{G} - 0.9513G - 24{,}469.10}\right]^2$$

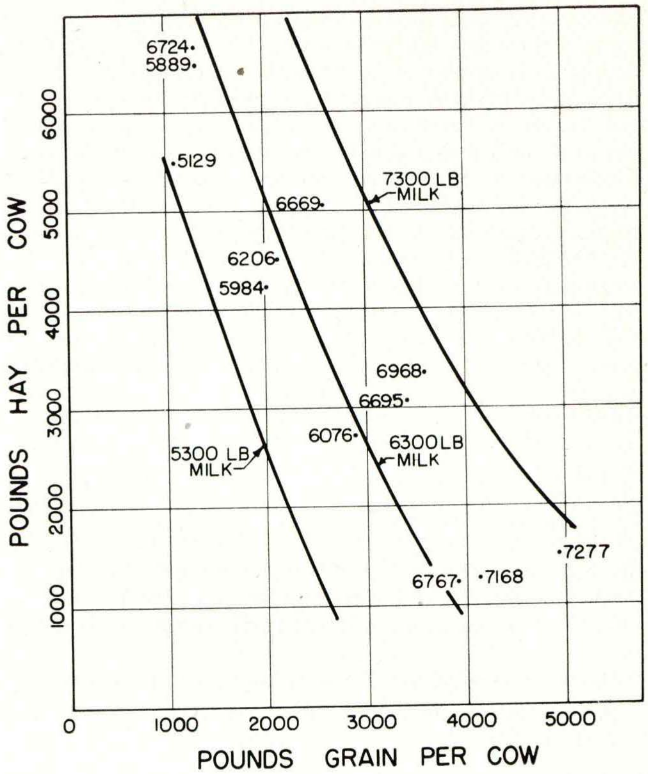

Fig. 10. Milk isoquants for a 26-week period predicted from equation 9a with ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

$$(8e)\quad \frac{dH}{dG} = \frac{1.1258 - 91.5302G^{-0.5} + 1.3313H^{0.5}G^{-0.5}}{1.7855 - 150.0115H^{-0.5} + 1.3313G^{0.5}H^{-0.5}}$$

The milk isoquant functions for the quadratic and square root functions are indicated, respectively, as equations 7d and 8d; the marginal rate of substitution equations are 7e and 8e.

Equations 7a and 8a appear unsatisfactory for prediction purposes since the signs of the H terms in equations 7b and 8b are such that the marginal product of hay increases with higher levels of feeding. In other words, each pound of hay would add more to milk production than the previous pound over all possible feeding levels. If equation 7a is used as the basis for predicting a milk isoquant of 6,300 pounds for the 26-week experimental period, the figures in table 7 result. Similar results are forthcoming for predictions based on equation 8a. In table 7, the marginal rates of

TABLE 7. MILK ISOQUANT OF 6,300 POUNDS AND MARGINAL RATES OF SUBSTITUTION FROM QUADRATIC EQUATION 7a. ABILITY FIXED AT MEAN FOR 36 COWS.

Feed to produce 6,300 lbs. milk*		Marginal rate of substitution: $\triangle H/\triangle G$†
Lbs. grain	Lbs. hay	
1,000	7,711‡	2.45
1,500	6,490	2.43
2,000	5,279	2.41
2,500	4,083	2.37
3,000	2,908	2.32
3,500	1,769	2.23
4,000	688‡	2.09

*Derived from equation 7d.
†Derived from equation 7e.
‡Estimates outside of range of observations.

substitution of grain for hay change slowly between the extremes of the ration and do not differ significantly from the substitution rate of 2.3 from the linear equations presented later.

OTHER FUNCTIONS FOR 36 COWS

Other quadratic or square root equations similar to equations 7a and 8a appear logical in estimating a milk production surface, since (1) the t values are low for equations 7a and 8a and (2) the marginal product equation for hay is increasing. The t values are low in equations 7a and 8a partly because the size of the sample is small relative to the variance of milk production and because a relatively large proportion of the degrees of freedom are exhausted in the many coefficients estimated for the equations. A larger experiment might qualify a quadratic or square root function in estimating the surface. In a simple attempt to increase the number of degrees of freedom for the small sample, equation 9a was estimated with the variables outlined earlier. In this case, A^2 was dropped, with ability entering into the function in linear form only. Only three regression coefficients (in contrast to the seven coefficients of equation 7) were estimated, including one for the term $H - 0.00001H^2$ and one for the term $G - 0.00007G^2$. The coefficients before the H^2 and G^2 were simply estimated from previous nutrition studies and the data of this study.

$$(9a)\quad M = 0.4304(H - 0.00001H^2) + 1.5008(G - 0.00007G^2) + 0.9265A - 665.49$$

The regression coefficients for this "adjusted quadratic" equation can be accepted in a probability sense at the 1-percent level even if two added degrees of freedom are dropped to compensate for direct estimation of the constants for H^2 and G^2 (see table 8).

Equations 9b and 9c have been derived from equation 9a and are, respectively, the milk isoquant and substitution rate equations. The milk isoquants in fig. 10 are based on equation 9b; equation 9c indicates the slopes of the isoquants at particular points in the feed plane.

$$(9b)\quad H = \frac{50{,}000.0 \pm (-116{,}165.61)}{\sqrt{0.2136 + 0.0000258G - 0.0000000018G^2 - 0.0000172M}}$$

$$(9c)\quad \frac{dH}{dG} = \frac{1.5008 - 0.000210G}{0.4304 - 0.0000086H}$$

If used to estimate milk isoquants, equation 9a results in contours which have some curvature, although the change in slope is not great. The slope of the isoquants, and their location in the feed plane, is almost identical with the isoquants presented later in figs. 11 and 12 for equations 11 and 12. Table 9 includes data showing feed combinations, or the milk isoquants derived from equation 9b, and the marginal rates of feed substitution for three milk isoquants derived from equation 9c. The substitution rates do not change as rapidly as do those in table 6 for the logarithmic function.

TABLE 8. CORRELATION COEFFICIENTS AND t VALUES FOR EQUATIONS 9a, 10, 11a, 12a and 13a.

Equation	R	b_1	b_2	b_3	b_4	b_5
	Value of t in order of b's in equation					
9a	0.8755*	3.98*	6.16*	5.28*	......	......
10	0.8770*	0.99	3.57*	4.75*	1.42	0.38
11a	0.8764*	3.85*	3.89*	4.91*	1.46	......
12a	0.8782*	3.76*	0.15	4.94*	1.61	......
13a	0.8672*	3.85*	5.82*	5.27*	......	......

*p < 0.01

TABLE 9. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY PREDICTED FROM MODIFIED QUADRATIC EQUATION 9a. ABILITY FIXED AT MEAN OF 36 COWS FOR 26-WEEK BASIC EXPERIMENT.

Feed to produce 5,300 lbs. milk			Feed to produce 6,300 lbs. milk			Feed to produce 7,300 lbs. milk		
Lbs. grain	Lbs. hay	MRS of grain for hay* ΔH/ΔG	Lbs. grain	Lbs. hay	MRS of grain for hay* ΔH/ΔG	Lbs. grain	Lbs. hay	MRS of grain for hay* ΔH/ΔG
1,000	5,560	3.37	1,000	8,256†	3.59	1,000	11,138†	3.86
1,500	3,969	2.99	1,500	6,566	3.17	1,500	9,328†	3.39
2,000	2,558	2.65	2,000	5,074	2.79	2,000	7,738†	2.97
2,500	1,313	2.33	2,500	3,761	2.45	2,500	6,345	2.60
3,000	221†	2.03	3,000	2,612	2.13	3,000	5,130	2.25
....	....	......	3,500	1,616	1.84	3,500	4,080	1.94
....	....	......	4,000	763†	1.56	4,000	3,182	1.64
....	....	......	....	....	......	4,500	2,428	1.36
....	....	......	....	....	......	5,000	1,813	1.09

*Marginal rate of substitution: Amount of forage replaced by 1 pound of grain for "small change" away from the combinations shown. Figure is a derivative of hay with respect to grain from equation 9c.
†Prediction is outside range of observations.

However, while the isoquants in fig. 10 appear to have but little slope and substitution rates in table 9 do not change as rapidly as those in table 6, a considerable change in substitution rates does occur over the range of observations presented. These data, all derived from equation 9a, have a logical advantage over the estimates for equations 11a and 12a since the former allows diminishing returns for hay as well as grain. However, as differences between equations 10 and 11a show, there is no probability basis for retaining a hay coefficient which results in diminishing returns for this feed category when the 36 cows are used for an aggregate 26-week lactation period.

Equations 10, 11a and 12a represent estimates of the milk production function when particular terms are dropped from the initial quadratic equation 7a and the initial square root equation 8a. In equation 10, the A^2 and interaction terms have been dropped from equation 7a; in equation 11a, H^2 also is dropped. In equation 12a, $\sqrt{H}$, $\sqrt{A}$ and the interaction term have been dropped from equation 8a.

$$(10)\quad M = 0.3010H + 1.5171G + 0.8978A - 0.000106G^2 + 0.000014H^2 - 459.63$$

$$(11a)\quad M = 0.4089H + 1.4423G + 0.9074A - 0.000091G^2 - 573.42$$

$$(12a)\quad M = 0.3977H - 0.1028G + 0.8988A + 102.9030\sqrt{G} - 2{,}335.45$$

The R and t values for these equations were given in table 8. In equation 10, the regression coefficient for H^2 again is non-significant and positive, suggesting the unrealistic condition of increasing marginal productivity of hay. In equation 11a, the regression coefficient for G^2 is significant at a probability level of less than 0.20, and the coefficients for the other terms are significant at a probability level of less than 0.01. An isoquant

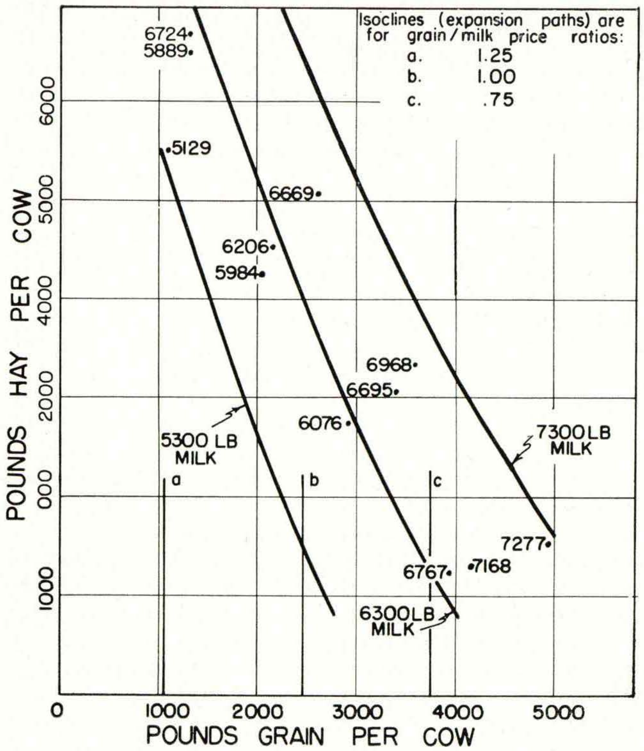

Fig. 11. Milk isoquants for a 26-week period predicted from equation 11a with ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

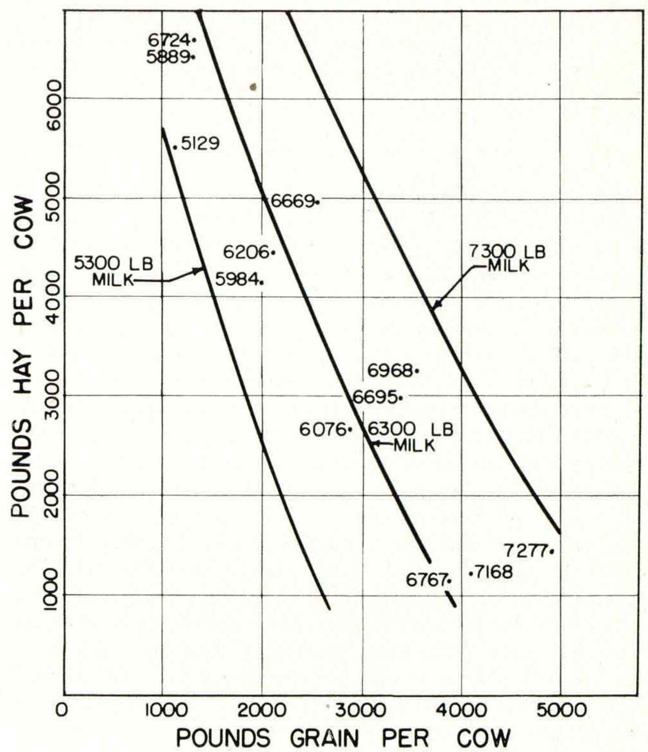

Fig. 12. Milk isoquants for a 26-week period predicted from equation 12a with ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantities from 12 cows at medium level of ability.)

map for quadratic equation 11a is included in fig. 11, while one for square root equation 12a is included in fig. 12. These two sets of isoquant maps are almost identical; milk contours for both equations have only slight curvature. While the location of the milk contours in the feed plane is similar for figs. 10, 11 and 12 (i.e., equations 9a, 11a and 12a, respectively), those in fig. 10 have somewhat greater curvature at the lower end. This difference is due to the fact that the marginal rate of substitution of grain for hay is lower as the proportion of grain increases for isoquants based on equation 9a than for those based on equations 11a and 12a. The difference is apparent in comparisons of table 9, which includes substitution rates based on equation 9a, and table 10, which includes substitution rates based on equations 11a and 12a. For example, with 3,500 pounds of grain used in combination with hay to produce 7,300 pounds of milk, the substitution rate of grain for hay in table 9 (equation 9a) is 1.93; it is 1.97 for equation 11a and 1.93 for equation 12a in table 10. However, with grain at 5,000 pounds the substitution rate for equation 9a in table 9 is only 1.09, while it is 1.30 for equation 11a and 1.57 for equation 12a in table 10.

In table 10, the rates of substitution at the extremes of the isoquants suggest greater difference between the estimates of equations 11 and 12 than visual comparison of figs. 11 and 12 would indicate. However, between

TABLE 10. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY, PREDICTED FROM EQUATIONS 11a AND 12a. ABILITY FIXED AT MEAN OF 36 COWS FOR 26-WEEK BASIC EXPERIMENT.

Lbs. of grain	Pounds of required hay to produce: 5,300 lbs. of milk		6,300 lbs. of milk		7,300 lbs. of milk		Marginal rate of substitution of grain for hay.* All milk levels.	
	Equation 11a	Equation 12a	Equation 11a	Equation 12a	Equation 11a	Equation 12a	Equation 11a	Equation 12a
1,000	5,528	5,642	7,974†	8,157†	10,420†	10,671†	3.08	3.83
1,500	4,043	3,933	6,489	6,447	8,935†	8,961†	2.86	3.08
2,000	2,669	2,512	5,115	5,026	7,561†	7,540†	2.64	2.63
2,500	1,407	1,275	3,853	3,789	6,299	6,304	2.41	2.33
3,000	256†	170†	2,702	2,684	5,148	5,198	2.19	2.10
3,500	....	....	1,663	1,678	4,109	4,192	1.97	1.93
4,000	....	....	735†	750†	3,181	3,264	1.74	1.77
4,500	....	....	....	....	2,365	2,401	1.52	1.67
5,000	....	....	....	....	1,660	1,591	1.30	1.57

*Substitution rates are derivatives from equations 11b and 12b.
†Estimates outside range of observations.

2,000 and 4,000 pounds of grain, the two milk production functions give almost identical slopes as indicated by the similarity of substitution ratios. The predicted amount of hay required with a specified amount of grain also is quite similar for equations 11a and 12a for grain inputs of 2,000 and 4,000 pounds (see table 10). Similarly, the hay quantities in table 9 compare favorably with those of table 10 for grain inputs over the range of 2,000 to 4,000 pounds.

The substitution ratios are the same for a given grain input regardless of the level of milk production in equations 11a and 12a. This condition holds true since equations 11a and 12a have only a linear term for hay. Therefore, the equations derived from them, which define the marginal ratios of substitution of grain for hay, contain no hay term (see equations 11b and 12b, respectively). A given grain input, regardless of the hay input per cow, will have the same substitution rate as higher milk levels are attained. Because of this, the isoclines (lines a, b and c) are vertical and linear, as illustrated in fig. 11, for estimates based on equation 11a with a squared term for G. (The isoclines for fig. 12 also are vertical and linear.)

$$(11b) \quad \frac{dH}{dG} = 3.5270 - 0.000446G$$

$$(12b) \quad \frac{dH}{dG} = 129.3668G^{-0.5} - 0.2584$$

Since the isoclines in fig. 11 are vertical, the optimum ration is found by determining the quantity of grain which results in a marginal productivity of grain ($\triangle M/\triangle G$) equal to the price ratio P_g/P_m. Once the quantity of grain has been determined (i.e., points on the grain axis such as those indicated at the bottom of lines a, b and c), the vertical isocline shows the combinations of grain and hay to be used. Hence, if the grain/milk price ratio were 1.25, line "a" should be followed for any prices of grain and milk giving that ratio. In other words, using equations 11a and 12a as predictors of the milk production surface would lead to this ration recommendation: Feed the cow an amount of grain consistent with the prices of grain and milk and allow her to eat hay to her stomach capacity.

Lines b and c, fig. 11, show estimated optimum amounts of grain for the 26-week period when the grain/milk price ratio is 1.0 and 0.75. In each case, the cow should be given the specified quantity of grain in the period and allowed to eat hay to stomach capacity. This procedure parallels the common feeding practice or nutrition recommendation where the cow is given grain in relation to her milk production and is allowed to consume forage on a free-choice basis. Hence, it can be said that this conventional practice is consistent with greatest profit only if the milk production surface is characterized by equations such as 11a and 12a; namely, the functions must have a linear term only for hay and must give vertical isoclines.

The ration solutions indicated by the isoclines corresponding to the milk and grain prices shown in fig. 11 have been determined by equating the partial derivatives of milk in respect to grain with the appropriate price ratios (equations 5 and 6). Since the derivative for hay is a constant equal to 0.4089, the cow should be fed hay to the stomach limit, after she is fed grain in line with price ratios, as long as the hay/milk price ratio is less than 0.4089. Grain feeding would, however, be adjusted to changes in price ratios (as indicated by lines a, b and c in fig. 11) if profits are to be maximized. With a hay/milk price ratio greater than 0.4089, the cow would be fed hay only at the physiological minimum.

The isoquants for equation 9a (fig. 10) do not have vertical isoclines. However, their slopes are so great that the ration recommendations mentioned above again apply; namely, feed grain in line with the grain/milk price ratio and allow free choice of forage.

LINEAR FUNCTION

A final function fitted to the 26-week data for milk production by the 36 cows of the basic experiment was one with only linear terms for hay, grain and ability (equation 13a). As table 8 indicates, the coefficient for each of the three variables is significant at the 1-percent probability level, a condition which also held true for the logarithmic equation 6a and the modified quadratic equation 9a. In equation 13a, 86.7 percent of the variance in milk production was explained by the linear terms. This proportion is slightly less than the 87.7 percent for modified quadratic equation 9a and slightly more than for logarithmic equation 6a, although the differences are not significant in a probability sense.

$$(13a) \quad M = 0.4154H + 0.9560G + 0.8570A + 50.35$$

$$(13b) \quad \frac{dM}{dH} = 0.4154$$

$$(13c) \quad \frac{dM}{dG} = 0.9560$$

$$(13d) \quad H = 2.4071M - 2.3017G - 2.0628A - 121.19$$

$$(13e) \quad \frac{dH}{dG} = 2.3017$$

If equation 13a were accepted as the estimate of the the production surface, the marginal rate of substitution would be predicted (equation 13e) as 2.3017 regardless of the proportions of hay and grain in the ration. Similarly, 1 pound of hay would be predicted (equation 13b) to yield 0.4154 pound of milk and 1 pound of grain would be predicted (equation 13c) to produce 0.9560 pound of milk, regardless of the level of feeding. Constant substitution and transformation rates such as these are assumed in conventional TDN or ENE evaluations of feeds. The substitution ratio predicted from equation 13a is considerably higher than that assumed by ENE and TDN transformations. One reason why this greater value may be predicted is that it is based on the entire area of the milk isoquant map included in the study (i.e., schematically as an area such as OMH in fig. 7 or O′M′H′ in fig. 8) while TDN and ENE evaluations have probably been made near the peak of the production surface (i. e., point M in fig. 7 or M′ in fig. 8, or an area such as that between isoquant f″ and point M′ in fig. 8). Hence, the 2.3017 figure for the entire surface may be consistent with a smaller substitution value at the convergence of isoclines (the usual point of evaluation), particularly if the production surface is actually nonlinear as illustrated later. Finally, part of the difference in substitution ratios (between the linear estimates of this study as compared to the linear TDN and ENE estimates) may grow out of experimental error in the current data.

The following nutrition recommendations would be followed for the 26-week period if linear equation 13a were used as the predictor of the milk production surface: Hay alone would be fed to stomach capacity if the grain/hay price ratio were greater than 2.3017 and the hay/milk price ratio were less than 0.4154. This is true since the derivative of milk with respect to hay (equation 13b) is 0.4154 while the derivative of hay with respect to grain is 2.3017 (equation 13c). If the grain/hay price ratio were less than 2.3017 and the grain/milk price ratio were less than 0.9560, grain alone would be fed to stomach capacity. With a grain/hay price ratio greater than 2.3017 and a hay/milk price ratio greater than 0.4154, milk production would not be profitable from the standpoint of feed costs alone, other costs disregarded. Hence, a linear estimate of the production function would always call for extremes in rations: Cows should be fed only grain or hay (and never a combination of grain and hay, except in the unique case where the feed substitution ratio equals the feed price ratio) to stomach capacity, if the feed/milk price ratio is less than the milk/feed derivative; they should not be kept in production if the feed/milk price ratio is greater than the milk/feed derivative.

Since the substitution ratios in table 11 are constant at 2.3017 (i. e., the value of the derivative in equation 13e) all isoclines have this same value and can be vertical, horizontal or positively sloped. There is, in fact, no single line representing an isocline value; rather every point in the feed plane of fig. 13 has the hay/grain substitution value of 2.3017.

TDN and ENE Transformations in Relation to Linear Functions

Procedures which use TDN or ENE transformations in evaluating feeds assume a linear production function with straight-line milk isoquants and, hence, constant (1) hay/grain substitution coefficients and (2) milk/feed grain transformation coefficients. This is true since 1 pound of feed is given the same TDN or ENE value regardless of the ratio of feeds used or the level of feeding. If diminishing rates of substitution were assumed (such as illustrated by the curved milk isoquants in figs. 9, 10, 11 and 12), the TDN or ENE transformation coefficients would need to be changed as the ration changes in proportions of feeds and level of feeding. Using the linear relationships from equation 13a, a pound of hay is predicted to produce 0.4154 pound of milk and a pound of grain to produce 0.9560 pound of milk, regardless of the ration fed or the level of feeding. A pound of grain is predicted to have a feeding value 2.3017 greater than a pound of hay, regardless of the ration fed. If, however, equation 9a is used for predictions, a pound of hay is predicted to produce 0.4089 pound of milk, and a pound of grain is predicted to

TABLE 11. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY PREDICTED FROM LINEAR EQUATION 13a. ABILITY FIXED AT MEAN OF 36 COWS FOR 26-WEEK BASIC EXPERIMENT.

Feed to produce 5,300 lbs. milk*			Feed to produce 6,300 lbs. milk*			Feed to produce 7,300 lbs. milk*		
Lbs. grain	Lbs. hay	Substitution rate ΔH/ΔG	Lbs. grain	Lbs. hay	Substitution rate ΔH/ΔG	Lbs. grain	Lbs. hay	Substitution rate ΔH/ΔG
1,000	5,194	2.3017	1,000	7,602†	2.3017	1,000	10,010†	2.3017
1,500	4,043	2.3017	1,500	6,451	2.3017	1,500	8,859†	2.3017
2,000	2,892	2.3017	2,000	5,300	2.3017	2,000	7,708†	2.3017
2,500	1,741	2.3017	2,500	4,149	2.3017	2,500	6,557	2.3017
3,000	590†	2.3017	3,000	2,998	2.3017	3,000	5,406	2.3017
....	....	....	3,500	1,847	2.3017	3,500	4,255	2.3017
....	....	....	4,000	697†	2.3017	4,000	3,105	2.3017
....	....	....	....	....	....	4,500	1,954	2.3017
....	....	....	....	....	....	5,000†	803	2.3017

*Isoquant predicted from equation 13d, and substitution rates predicted from equation 13e.
†Outside range of observation.

produce 0.9534 pound of milk only at the mean level of feeding used in the experiment. At this feed combination and feeding level, a pound of grain has 0.9534/0.4089 or 2.3316 times as much feed value, in relation to milk production, as does hay. But in contrast to the linear function, this feed value relationship holds true only for the particular feed combination and level. Turning back to table 9, we note that for a 6,300-pound milk isoquant, grain has a value as great as 3.17 times that of hay when the feed combination includes 1,500 pounds of grain and 6,566 pounds of hay over a 26-week period; it has a value of only 1.84 times the value of hay

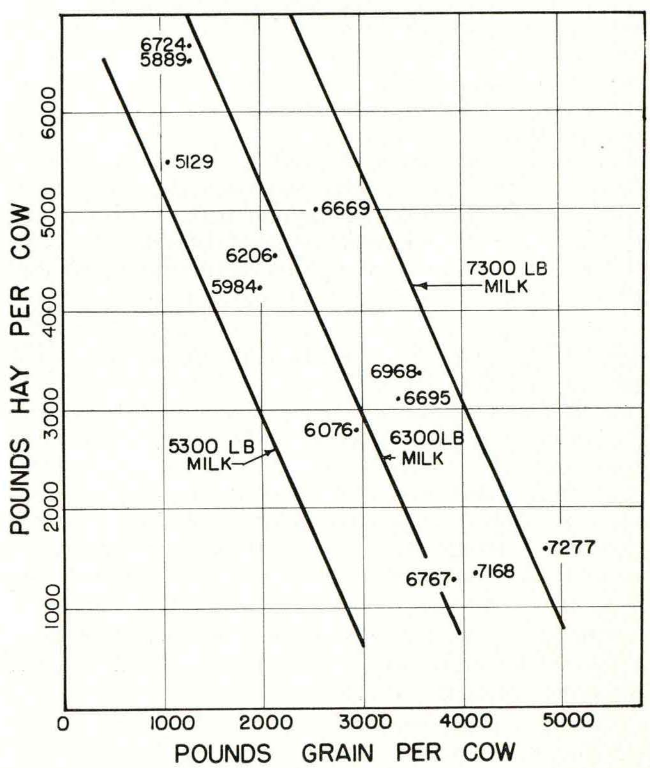

Fig. 13. Milk isoquants for a 26-week period predicted from linear equation 13a with ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

when the ration includes 3,500 pounds of grain and 1,616 pounds of hay. With feeding at a higher level of milk production, as denoted by the 7,300-pound isoquant, a pound of grain has a feeding value 1.94 times that of hay when the ration includes 3,500 pounds of grain and 4,080 pounds of hay in the 26 weeks.

The ratios of the marginal productivities or the substitution ratios provide a basis for relative evaluation of feeds. If it can be proven that the milk production function is linear, constants such as those assumed in the traditional TDN and ENE evaluation of feeds are appropriate. However, if the function is proven to be nonlinear, constant transformations are not the appropriate basis for ration evaluation. Under curved isoquants (such as in figs. 9, 10, 11 and 12), grain must be given a continuously lower value relative to hay for rations which move along a milk isoquant nearer to the grain axis; hay must be given a lower value relative to grain for rations representing movements towards the hay axis. Actually each isocline (i. e., such as those outlined for figs. 3, 6 and 8) traces the only path of rations over which each feed can appropriately be given a constant feeding value; the constant value which is appropriate will differ for each isocline, except for a milk production function which is linear with straight-line isoquants.

Most nutritionists accept the notion that grain and hay do not substitute at constant rates over the entire milk surface. However, wide use of TDN and ENE transformations has been continued because of lack of other data to indicate the rate at which substitution ratios may change. Also, some believe that the milk isoquant may be "near linear" in the middle, with curvature especially near the ridge isocline. Because of these indications from previous research, grain alone is never recommended and hay alone is seldom recommended.

Equation 13a with linear coefficients can be used to test the feeds and results of this study against TDN and ENE transformation ratios which implicitly assume linearity. The constant substitution rate of 2.3017 (i. e., the relative feed value of grain and hay) predicted from linear equation 13a compares with the constants of 1.7884 predicted from the ENE evaluation and 1.350 for the TDN evaluation, using Morrison's standards, of feeds in the experimental ration. The ENE ratio of 1.7884 is calculated as 0.7404 therms per pound of grain divided by 0.4140 therms per pound of hay. The TDN ratio of 1.3550 is calculated as 0.6863 pound of TDN per pound of grain divided by 0.5065 pound of TDN per pound of hay.

The question posed is: "Does the ratio predicted from equation 13a, indicating the relative feeding value of grain and hay in producing a given amount of milk, correspond to the conventional ratios used for indicating relative values of grain and hay in producing energy or heat?" Using the linear equation for the 36 cows, we examine whether the ratio of marginal productivities of grain and hay (i.e., the substitution ratio of the feeds in producing a given amount of milk) is similar to the ENE and TDN ratio, when sampling or experimental error is considered for the regression coefficients.

Table 12 includes the regression coefficients for grain and hay from the linear equation and their 95-percent fiducial limits. The computed values of regression coefficients which are necessary to give ENE and TDN ratios of feeds in the study, based on Morrison, have then been entered on lines 4 and 5, respectively. These computed values fall within the 95-percent confidence limits for the ENE evaluation but not for the TDN evaluation. Similarly, the t values testing the differences between the regression coefficients and coefficient values necessary to give the TDN ratio are at a higher probability level than those for the ENE evaluation. Hence, for the feeds, cows and data of this study, it appears that the TDN ratio expressing the value of grain and hay in producing a given amount of energy or number of therms is not appropriate for comparison of these feeds in producing a given amount of milk.[12]

TABLE 12. REGRESSION COEFFICIENTS FROM EQUATION 13a, 95-PERCENT CONFIDENCE LIMITS AND MAGNITUDES TO GIVE ENE AND TDN RATIOS.

Item	Hay	Grain
1. Upper confidence limit	0.64	1.29
2. Regression coefficient	0.42	0.96
3. Lower confidence limit	0.20	0.62
4. Magnitude of regression coefficient to give 1.7884 ENE ratio	0.53*	0.74†
5. Magnitude of regression coefficient to give 1.3530 TDN ratio	0.71*	0.56†
6. Value of t in testing regression coefficient (2) against coefficient needed (4) to give ENE ratio	0.58	1.01‡
7. Value of t in testing regression coefficient (2) against coefficient needed (5) to give TDN ratio	1.40§	1.86††

*If regression coefficient for grain of 0.9560 is accepted as a parameter.
†If regression coefficient for hay of 0.4154 is accepted as a parameter.
‡$p < 0.30$
§$p < 0.20$
††$p < 0.10$

Additional research is needed to study further the appropriateness of the ENE energy evaluation as compared with a milk production evaluation of feeds. A larger study with lower standard errors also might prove differences to be significant for ENE evaluations. While the comparisons above rest on the linear regression equation, it is unlikely that the milk production function is of this empirical nature. The above analysis was made to examine differences or similarity of conclusions if the purely linear relationships (a condition assumed by ENE and TDN evaluations) were accepted. The functions derived with variable substitution rates in other sections of this report are likely more appropriate for a fundamental analysis of feed values. Inclusion of time as a variable in later sections of this manuscript undoubtedly improves the estimates of feed substitution rates.

PRODUCTION FUNCTIONS INCLUDING AUXILIARY EXPERIMENT FOR 51 COWS IN AN OVER-ALL PERIOD

Production functions paralleling those for the 36-cow, 26-week basic experiment were derived when data from this experiment were pooled with the 15-cow auxiliary experiment explained previously. The auxiliary experi-

[12]The test was made with the regression coefficients, but, since in the linear equation, these form the substitution coefficient for producing a given amount of milk, the statement holds true for a linear production function.

ment included only 16 weeks and the data for the 36-cow, basic experiment was transformed similarly: Milk output and feed input include data for the first 16 weeks of the experimental period only for the 36-cow basic experiment. Except that measurements refer to 16 weeks and 51 cows, the variables in the function explained below are the same as those outlined earlier for the basic experiment. Time is not included as a variable.

Quadratic-Type Functions

Since functions with squared terms and square root terms gave results which did not differ significantly for the 36-cow experiment, only the former type of equation has been used for the 51-cow data over the 16-week period.[13] The equations are listed below in the following sequence: (a) refers to the basic production function of a particular algebraic form; (b) is the isoquant equation derived from equation a with ability set at the mean; (c) is the marginal rate of substitution equation derived from equation b.[14]

$$(14a)\quad M = 0.3449H + 2.2711G + 0.8745A - 0.000264G^2 + 0.000072H^2 - 2{,}264.61$$

$$(14b)\quad H = -2398.0 \pm (6954.0)\sqrt{0.1610 - 0.000653G + 0.000000076G^2 + 0.000288M}$$

$$(14c)\quad \frac{dH}{dG} = \frac{2.2711 - 0.000527G}{0.3449 + 0.000144H}$$

$$(15a)\quad M = 0.8076\,(H - 0.00005H^2) + 1.9895\,(G - 0.0001G^2) + 0.9220A - 2{,}472.35$$

$$(15b)\quad H = 10{,}000 \pm (-12{,}383.0)\sqrt{0.6130 + 0.000321G - 0.000000032G^2 - 0.000162M}$$

$$(15c)\quad \frac{dH}{dG} = \frac{1.9895 - 0.000398G}{0.8076 - 0.000081H}$$

$$(16a)\quad M = 0.6650H + 2.0436G + 0.8860A - 0.000199G^2 - 2{,}404.71$$

$$(16b)\quad H = -3.0731G + 0.000299G^2 + 1.5038M + 393.6412$$

$$(16c)\quad \frac{dH}{dG} = 3.0731 - 0.000599G$$

Related statistics for the production functions are given in table 13. All functions account for a greater proportion of the variance in milk production than the parallel functions for 36 cows presented earlier. For example, equation 15a accounts for 86.7 percent of the variance in milk production while equation 11a accounts for only 75.7 percent. The t values for regression coefficients are all significant at a probability level of 0.05 or lower, except those for H and H^2 in equation 14a. However, this latter equation again appears unrealistic because the coefficient for H^2 is positive, indicating an increasing return to hay as more of this feed is consumed. In equation 15a, the coefficients for H^2 and G^2 were estimated prior to prediction of the regression equation. (Coefficients were predicted only for the terms $H-0.00005H^2$, $G-0.0001G^2$ and A.) Accepting this form for the production function would allow diminishing returns to both feeds and diminishing substitution rates between them. The milk isoquants for equation 15a do not have a sharp curvature. The same statement holds true for equation 16a which has a power term only for grain (see isoquants in fig. 14 for equation 16a). Again, the isoclines for equation 16a are vertical with the implications for hay feeding mentioned in respect to equation 11a and fig. 11. Too, the substitution ratios are always the same for a given amount of grain, regardless of the amount of hay or the level of milk production,

TABLE 13. VALUES OF R AND t FOR 51-COW DATA POOLED FOR BASIC AND AUXILIARY EXPERIMENTS FOR 16-WEEK PERIOD (t VALUES REFER TO REGRESSION COEFFICIENTS IN ORDER PRESENTED IN EQUATIONS 14a, 15a AND 16a).

Equation	R	Value of t for: b_1	b_2	b_3	b_4	b_5
14a	0.9338*	1.25†	5.92*	7.10*	2.67*	1.25†
15a	0.9275*	6.35*	8.78*	7.66*	......	......
16a	0.9314*	6.43*	6.01*	7.17*	2.28**	......
17a	0.9233*	6.23*	8.40*	6.64*	......	......

*p<0.01 **p<0.05 †p<0.30

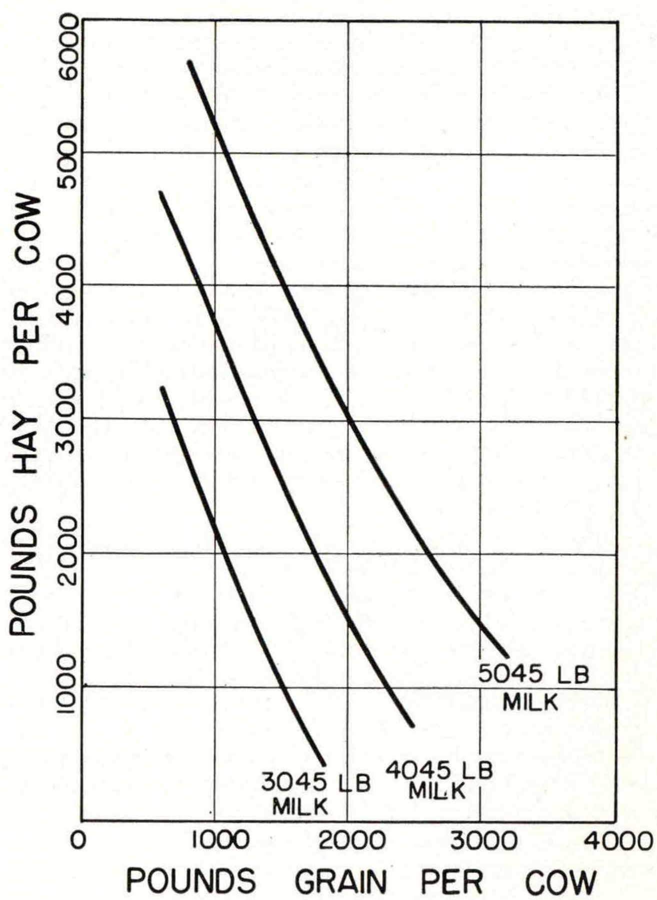

Fig. 14. Milk isoquants for a 16-week period predicted from equation 16a with ability fixed at mean for 51 cows.

[13] One exception was prediction of a production function with a square root term for G. The function, $M = 0.6601H + 0.0589G + 0.8763A + 101.9959\sqrt{G} - 3{,}776.83$, has an R value of 0.9308 and t's for regression coefficients of 6.35, 0.17, 7.08 and 3.92, respectively. The relation of this function to equation 16a in the text is similar to the relation of equation 12a to equation 11a for 36 cows.

[14] In each (b) equation, ability has been set at the mean preliminary period milk production for 51 cows. The 2-week preliminary period data for the 15 auxiliary cows were extrapolated to get an estimate comparable to the basic experiment.

TABLE 14. FEED COMBINATIONS IN PRODUCING THREE MILK LEVELS PREDICTED FROM EQUATIONS 15a AND 16a (FOR 16-WEEK PERIOD WITH ABILITY FIXED AT MEAN OF 51 COWS).

Feed combinations for 3,045 lbs. milk				Feed combinations for 4,045 lbs. milk				Feed combinations for 5,045 lbs. milk				Marginal rate of substitution of grain for hay ($\Delta H/\Delta G$) for 4,045 lbs. milk	
Equation 15a		Equation 16a		Equation 15a		Equation 16a		Equation 15a		Equation 16a			
Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Equation 15a	Equation 16a
600	3,190	600	3,236	600	5,351*	600	4,740*	....	....	....	....	4.66	2.71
1,000	2,067	1,000	2,199	1,000	3,822	1,000	3,702	1,000	6,339*	1,000	5,206*	3.19	2.47
1,400	1,173	1,400	1,256	1,400	2,709	1,400	2,760	1,400	4,677*	1,400	4,264	2.43	2.23
1,800	443*	1,800	410*	1,800	1,841	1,800	1,914	1,800	3,535	1,800	3,418	1.93	2.00
....	....	....	....	2,200	1,146	2,200	1,165	2,200	2,677	2,200	2,669	1.56	1.76
....	....	....	....	2,600	584	2,600	510*	2,600	2,007	2,600	2,014	1.26	1.52
....	....	....	....	....	....	....	....	3,000	1,481	3,000	1,455	......	......
....	....	....	....	....	....	....	....	3,200	1,264	3,200	1,212	......	......

*Estimates outside range of observation.

because the equation of substitution rates (16c) has only a linear grain term and does not include a term for hay. Table 14 allows comparisons of rations for three levels of milk production predicted by equations 15a and 16a. For both functions, the relative feeding value of hay and grain would change with the proportions fed for a given milk isoquant. Since constants for G^2 and H^2 have been "forced" into equation 15a, it is likely that the substitution ratios for equation 16a in table 14 are most appropriate for the 51 cows.

Linear Function

Equation 17a is a linear function fitted to the 16-week period for 51 cows. For this equation, an additional pound of hay would always produce 0.6717 pound of milk, regardless of the grain/hay ratio or the level of

(17a) $M=0.6717H+1.3527G+0.8500A-1{,}841.40$

(17b) $H=1.4888M-2.0138G-1.2655A+2{,}741.40$

(17c) $\frac{dH}{dG} = 2.0137$

feeding; an additional pound of grain would always produce 1.3527 pounds of milk. Grain would replace 2.0137 pounds of hay in producing a given amount of milk, regardless of the ration. Hay fed to the stomach limit line (or all hay) would give the lowest cost ration for any grain/hay price ratio of greater than 2.0137; grain fed to the physiological limit (or all grain) would give the lowest cost ration for a grain/hay price ratio less than 2.0137. The ratio, 2.0137, expressing the relative feeding value of grain and hay is less than the 2.3017 ratio computed from equation 13a for the 36-cow basic experiment.

A comparison of (1) the substitution ratio of grain and hay in producing a given amount of milk for the 51-cow linear function with (2) the ENE and TDN ratios for producing a given amount of energy, had the same results as the same test for the 36-cow linear function: The substitution ratio could be said to differ significantly from the TDN ratio but not the ENE ratio, supposing that a linear production function could be accepted in predicting the milk production surface.

PRODUCTION FUNCTIONS WITH TIME AS VARIABLE FOR 36-COW BASIC EXPERIMENT

Since time, or point of time in the lactation period, is an important variable affecting milk production, regression equations have been computed with time included and are based on the variables following. Too, as mentioned earlier, changes in cow weights approached an equilibrium after the outset of the experiment and allowed better prediction after this change had been taken into account.

M is production of 4-percent, fat-corrected milk in each 4-week period, measured in pounds.

G is quantity of grain mix (see earlier explanation), measured in pounds, consumed in each 4-week period.

H is quantity of hay, measured in pounds, consumed in each 4-week period.

A is ability measured as pounds of milk produced in the preliminary period.

T is time in 4-week periods. Six such periods were included and were measured as the cardinal numbers, 1 through 6. The last 2 weeks of the 26-week experimental period were not included.

Four production functions have been derived with variables defined as above. The first of these, equation 18a, is the logarithmic form explained earlier. The next three equations, 19a, 20a and 21a, are of a general quadratic form with variations in the term included. Basic statistics for these functions are given in table 15. Here, as in previous estimates, equations which are acceptable in a probability sense are less appropriate on logical grounds; while equations which have logical basis are acceptable only at higher probability levels.

(18a) $M=4.1937H^{0.1506}\ G^{0.3082}\ A^{0.3716}\ T^{-0.1973}$

(19a) $M=1.6302H+3.1309G+0.1497A+14.2243T$
$-0.000388H^2-0.001192G^2+4.3792T^2-$
$0.001056HG-0.1570GT-0.0865HT-731.76$

(20a) $M=0.5513H+1.3285G+0.1488A-110.2640T-$
$0.000081H^2-0.000360G^2+6.0320T^2+151.36$

(21a) $M=0.6601H+1.4276G+0.1553A-$
$0.000054H^2T-0.000152G^2T-2.0752T^2-157.24$

While all of the regression coefficients of equation 18a

TABLE 15. CORRELATION COEFFICIENTS AND t VALUES FOR EQUATIONS 18a, 19a, 20a AND 21a (36 COWS WITH TIME AS VARIABLE).

Equation	R	Values of t in order of b's in equation									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
18a	0.8654	6.69*	12.04*	9.27*	15.61*	......	......	......	......	......	......
19a	0.9016	1.94‡	2.60*	9.86*	0.34	1.14	1.67§	1.62§	1.10	3.49*	2.82*
20a	0.8953	3.77*	6.58*	9.60*	5.74*	0.75	1.56§	2.24†	......	......	......
21a	0.8960	8.70*	13.72*	10.22*	4.54*	5.92*	1.65§	......	......	......	......

*$p<0.01$ †$p<0.05$ ‡$p<0.10$ §$p<0.20$

are acceptable at a probability level of less than 1 percent, this equation explains a somewhat smaller proportion of the variation in milk production than do the other three equations. The percentage of variation in milk production explained by the regression equations is 74.9, 81.3, 80.2 and 80.3, respectively, for functions 18a, 19a, 20a and 21a. Equation 19a appears to have the greatest logical basis since (1) it allows definition of a single ration with a single milk level maximum per cow and (2) it allows a more complete specification of interaction between variables than do equations 20a and 21a. Also, as indicated later, it results in an isocline map which seems more consistent with nutrition logic. In equation 19a, the crossproduct terms reduce the sum of squares of deviation from regression by an amount acceptable in probability terms, as compared with equation 20a (see appendix table A-6). Equation 19a gives estimates quite similar to equation 21a, which has t values acceptable at lower levels of probability. Because of these several reasons, the writers believe that equation 19a is as appropriate, or more appropriate, than any of the equations in estimating the milk production surface, feed substitution rates and profit maximizing rations. It may, however, tend to overestimate substitution ratios near the end of the lactation period. It is likely that, in a larger experiment with a smaller proportion of the degrees of freedom used in estimating regression coefficients, values would fall at probability levels as low as those for equations 18a and 21a.

In evaluating the t values for the 216 observations (36 cows × 6 months) in table 15, it must be remembered that the six observations for each cow are not independent. If, however, the number of degrees of freedom is considered to be as low as 25 (36 minus 11) for equation 19a, rather than 205 (216 minus 11), the probability statements indicated in the footnotes of table 15 still hold true. To obtain complete independence with 205 degrees of freedom for equation 19a, for example, it would be necessary to have 216 cows, with each one used for a single observation.

Functions By Months

In addition to estimating production functions with time (months) as a variable, functions of the form of equation 19a were estimated for each month separately. In these six separate equations, variables included feed input and milk output measured over the particular month. (Time was not included as a variable, and there were only 36 observations for each month.) The regression coefficients for these monthly equations are given in table 16. Here equation 22a is for the first month in the experimental period, equation 23a is for the second month, etc. Table 17 includes the R and t values for the six monthly functions. Because of the large proportion of degrees of freedom used in the regression estimates and the relative within month variability of milk production, the t values correspond to quite high probability levels. Selected estimates based on these functions are provided in Appendix E.

Predictions From Functions With Time (Month of Lactation) as a Variable

In this section results are shown using equations 19a and 21a, with time as a variable, in predicting feed combinations possible in attaining specified milk levels (isoquants), and in estimating the rate of substitution between hay and grain in the various months. Equation 19b is the milk isoquant, based on production function equation 19a, where T has been set at 1 and A has been set at 2,492 (the mean of the 36 cows in the experimental period). While equation 19b provides estimates for the first month ($T=1$), similar estimates can be provided other months by assigning a particular value to T. Equation 19c provides estimates of substitution rates for equation 19b. Equations 21b and 21c are, respectively, the isoquant and substitution functions for func-

TABLE 16. REGRESSION COEFFICIENTS FOR QUADRATIC EQUATIONS ESTIMATED FOR EACH MONTH OF EXPERIMENTAL PERIOD.

Month (equation)	Constant	Regression coefficient for:					
		H	G	A	H^2	G^2	HG
1; 22a	– 506.0	0.7495	1.7177	0.3075	–0.000065	–0.000545	–0.000392
2; 23a	– 764.0	1.1904	2.6425	0.2014	–0.000228	–0.001096	–0.000682
3; 24a	1,851.0	–2.5569	–3.2638	0.1257	0.001103	0.002296	0.003868
4; 25a	–1,140.0	2.9770	4.5732	0.1085	–0.001188	–0.002368	–0.003026
5; 26a	– 993.0	1.9343	4.1901	0.1038	–0.000577	–0.002542	–0.002402
6; 27a	–2,313.0	4.4090	7.8840	0.0591	–0.001644	–0.005090	–0.005749

TABLE 17. CORRELATION COEFFICIENTS AND t VALUES FOR EQUATIONS 22a, 23a, 24a, 25a, 26a AND 27a (PRESENTED IN TABLE 16).

Equation	R	Value of t in order of b's in equation					
		b_1	b_2	b_3	b_4	b_5	b_6
22a	0.9614	0.72	1.28	11.37	0.14	0.68	0.34
23a	0.9464	0.66	1.10	7.64	0.30	0.78	0.33
24a	0.8396	0.89	0.76	3.49	0.94	0.88	1.14
25a	0.7935	1.30	1.30	3.04	1.26	1.06	1.07
26a	0.7379	0.76	1.06	2.57	0.56	0.98	0.72
27a	0.6720	1.64	1.92	1.32	1.40	1.80	1.60

tion 21a, with the values for T and A mentioned above.

$$(19b)\quad H = 1{,}989.36 - 1.3608G \pm (-1{,}288.66)\sqrt{1.8553 + 0.001355G - 0.00000073G^2 - 0.001552M}$$

$$(19c)\quad \frac{dH}{dG} = \frac{2.9740 - 0.002384G - 0.001056H}{1.5437 - 0.000776H - 0.001056G}$$

$$(21b)\quad H = 6{,}106.28 \pm (-9{,}250.69)\sqrt{0.4850 + 0.000309G - 0.000000033G^2 - 0.000216M}$$

$$(21c)\quad \frac{dH}{dG} = \frac{1.4276 - 0.000304G}{0.6601 - 0.000108H}$$

Functions with time as a variable also allow estimates of the rate of change in milk production as the lactation period progresses. Equations 19d and 21d, based respectively on equations 19a and 21a, are marginal time-yield equations. They indicate the decline in milk production associated with each unit progress of time beyond the beginning of the experimental period and are used for estimates in a following section.

$$(19d)\quad \frac{dM}{dT} = 14.22 + 8.7584T - 0.1570G - 0.0865H$$

$$(21d)\quad \frac{dM}{dT} = -\ 0.000054H^2 - 0.000152G^2 - 4.1504T$$

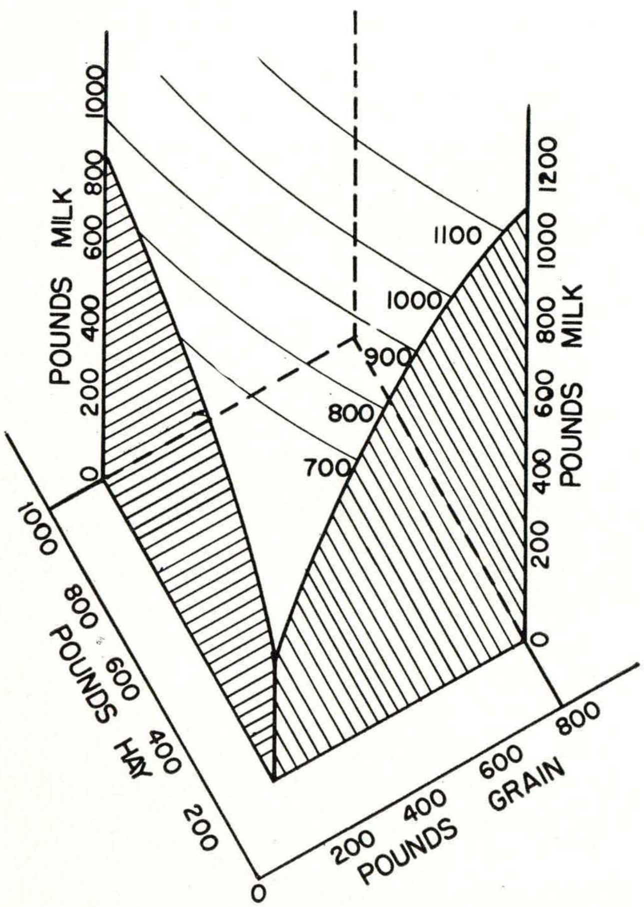

Fig. 15. Milk production surface and milk isoquants estimated from equation 19a for mean month of experiment. Ability at mean for 36 cows.

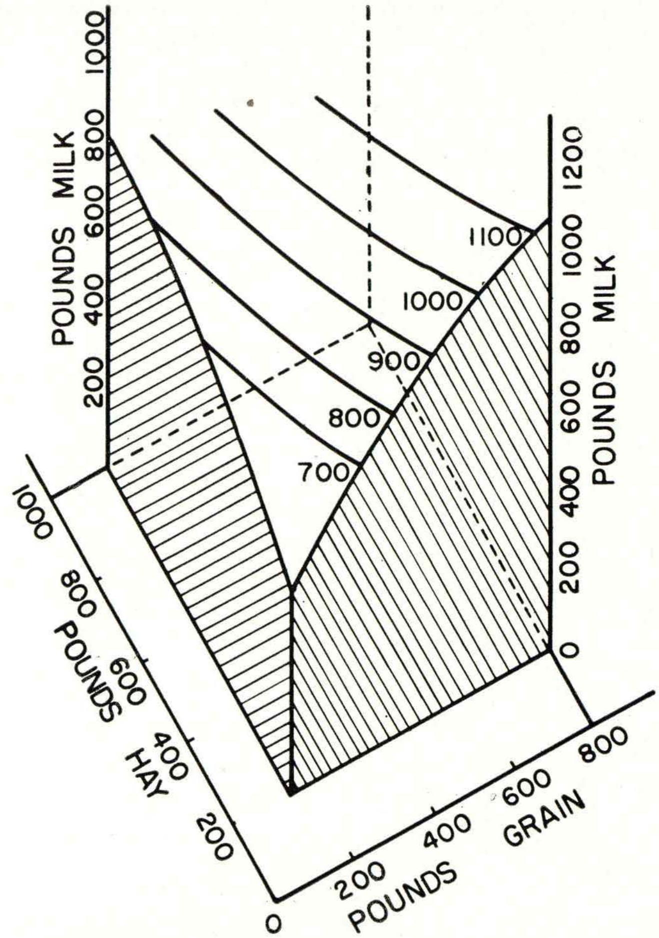

Fig. 16. Milk production surface and milk isoquants from equation 21a for mean month of experiment. Ability at mean for 36 cows.

The milk production surfaces derived from equations 19a and 21a are presented in figs. 15 and 16, respectively, for the "mean" month of the experiment (i.e., T= 3.5).[15] The surfaces are quite similar with respect to the milk contours. Feed quantities for producing stated milk production levels and marginal rates of substitution derived from equation 19a are presented in tables 18, 19 and 20, respectively, for the first, the "mean" and the sixth month of the experimental period. Parallel quantities are provided in tables 21, 22 and 23 for the first, the mean and the sixth month predictions based on equation 21a. As data of the tables indicate, increasing inputs of hay are predicted for higher milk levels, for a given grain level, because of diminishing productivity of feed. Also, the marginal rate of substitution of grain for hay increases (1) as the ration includes a greater proportion of hay for given milk level and (2) as higher levels of milk are attained with greater hay inputs, grain remaining constant. While not directly apparent from parallel tables, the marginal rates of substitution between grain and hay for the two equations, for a given combination of the two feeds and time and ability fixed at the same level, are similar in early parts of the period. However, the rate of substitution tends to widen between the two functions as time increases. Substitution ratios

[15]For the "mean" month, T has been set at 3.5, to include the last half of the third month and the first half of the fourth month.

TABLE 18. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS IN 28 DAYS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY BASED ON EQUATION 19a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. FIRST MONTH OF EXPERIMENTAL PERIOD (T = 1).

Level of grain (pounds)	Pounds hay required to maintain milk output of:*					Marginal rates of substitution ($\Delta H/\Delta G$): pounds hay replaced by 1 additional pound of grain along indicated milk isoquant.†				
	1,000 lbs.	1,100 lbs.	1,200 lbs.	1,300 lbs.	1,400 lbs.	1,000 lbs.	1,100 lbs.	1,200 lbs.	1,300 lbs.	1,400 lbs.
150	883	1,040	....	....	....	2.41	2.63	......	......	......
200	766	913	1,093	....	....	2.29	2.46	2.77	......	......
250	654	793	960	....	....	2.18	2.32	2.55	......	......
300	547	680	837	1,037	....	2.09	2.20	2.38	2.76	......
350	445	573	722	906	....	2.01	2.10	2.24	2.51	......
400	346	470	613	785	1,024	1.94	2.01	2.13	2.33	2.87
450	250	371	509	673	889	1.87	1.93	2.02	2.18	2.54
500	158	276	410	567	768	1.81	1.86	1.93	2.06	2.31
550	....	185	315	467	657	......	1.79	1.85	1.95	2.14
600	....	....	225	372	554	......	......	1.78	1.85	2.00
650	....	....	....	282	457	......	......	......	1.76	1.87
700	....	....	....	196	366	......	......	......	1.08	1.76
750	....	....	....	....	280	......	......	......	......	1.67
800	....	....	....	....	199	......	......	......	......	1.57

*Predicted from equation 19b.
†Predicted from equation 19c.

TABLE 19. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS IN 28 DAYS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY BASED ON EQUATION 19a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. MEAN MONTH OF EXPERIMENTAL PERIOD (T = 3.5).

Level of grain (pounds)	Pounds hay required to maintain milk output of:					Marginal rates of substitution ($\Delta H/\Delta G$): pounds hay replaced by 1 additional pound of grain along indicated milk isoquants.				
	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.	1,190 lbs.	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.	1,190 lbs.
150	815	1,036	....	....	....	2.54	3.09	......	......	......
200	692	891	....	....	....	2.37	2.74	......	......	......
250	577	760	1,032	....	....	2.24	2.50	3.41	......	......
300	468	640	877	....	....	2.12	2.32	2.85	......	......
350	365	528	744	....	....	2.02	2.17	2.53	......	......
400	266	423	623	973	....	1.93	2.05	2.31	4.00	......
450	172	323	513	806	....	1.85	1.94	2.13	2.90	......
500	....	229	410	674	....	......	1.85	1.99	2.45	......
550	....	139	314	559	....	......	1.76	1.87	2.17	......
600	....	....	223	455	....	......	......	1.76	1.97	......
650	....	....	138	361	....	......	......	1.66	1.80	......
700	....	....	....	275	716	......	......	......	1.66	4.79
750	....	....	....	194	575	......	......	......	1.53	2.09
800	....	....	....	122	486	......	......	......	1.41	1.53

TABLE 20. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY BASED ON EQUATION 19a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. SIXTH MONTH OF EXPERIMENTAL PERIOD (T = 6).

Level of grain (pounds)	Pounds hay required to maintain milk output of:				Marginal rates of substitution ($\Delta H/\Delta G$): pounds hay replaced by 1 additional pound of grain along indicated milk isoquant.			
	600 lbs.	700 lbs.	800 lbs.	900 lbs.	600 lbs.	700 lbs.	800 lbs.	900 lbs.
150	564	801	......	......	2.40	2.98	......	......
200	448	661	......	......	2.24	2.62	......	......
250	339	537	867	......	2.11	2.38	3.88	......
300	236	422	701	......	2.06	2.20	2.93	......
350	......	316	566	......	......	2.06	2.50	......
400	......	216	448	......	......	1.93	2.23	......
450	......	......	341	......	......	......	2.04	......
500	......	......	244	732	......	......	1.88	14.76
550	......	......	......	530	......	......	......	2.67
600	......	......	......	414	......	......	......	2.06
650	......	......	......	320	......	......	......	1.71
700	......	......	......	242	......	......	......	1.44
750	......	......	......	176	......	......	......	1.18
800	......	......	......	......	......	......	......	......

TABLE 21. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS IN 28 DAYS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY, BASED ON EQUATION 21a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. FIRST MONTH OF EXPERIMENTAL PERIOD (T =1).

Level of grain (pounds)	Pounds hay required to maintain milk output of:					Marginal rates of substitution (ΔH/ΔG): pounds hay replaced by 1 additional pound of grain along indicated milk isoquant.				
	1,000 lbs.	1,100 lbs.	1,200 lbs.	1,300 lbs.	1,400 lbs.	1,000 lbs.	1,100 lbs.	1,200 lbs.	1,300 lbs.	1,400 lbs.
150	920	....	....	....	....	2.46	......	......	......	......
200	799	976	....	....	....	2.38	2.46	......	......	......
250	682	855	1,034	....	....	2.31	2.38	2.47	......	......
300	568	738	913	1,095	....	2.23	2.30	2.38	2.47	......
350	459	625	796	973	....	2.17	2.23	2.30	2.38	......
400	352	515	683	857	1,036	2.10	2.16	2.23	2.30	2.38
450	249	409	574	743	919	2.04	2.10	2.16	2.23	2.30
500	148	306	467	634	806	1.98	2.04	2.09	2.16	2.23
550	....	205	364	528	696	......	1.98	2.03	2.09	2.16
600	....	108	264	425	590	......	1.92	1.97	2.03	2.09
650	....	....	167	325	487	......	......	1.92	1.97	2.03
700	....	....	....	228	388	......	......	......	1.91	1.97
750	....	....	....	134	291	......	......	......	......	1.91
800	....	....	....	....	197	......	......	......	......	1.85

TABLE 22. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS IN 28 DAYS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY, BASED ON EQUATION 21a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. MEAN MONTH OF EXPERIMENTAL PERIOD (T = 3.5).

Level of grain (pounds)	Pounds hay required to maintain milk output of:					Marginal rates of substitution (ΔH/ΔG): pounds hay replaced by 1 additional pound of grain along indicated milk isoquant.				
	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.	1,200 lbs.	800 lbs.	900 lbs.	1,000 lbs.	1,100 lbs.	1,200 lbs.
150	763	1,084	....	....	....	3.41	5.07	......	......	......
200	608	870	....	....	....	2.82	3.67	......	......	......
250	477	706	1,003	....	....	2.42	2.96	4.14	......	......
300	363	570	822	....	....	2.12	2.49	3.18	......	......
350	263	454	678	965	....	1.88	2.16	2.62	3.58	......
400	174	352	557	806	....	1.69	1.90	2.23	2.82	......
450	....	262	453	677	963	......	1.69	1.94	2.35	3.21
500	....	182	362	568	820	......	1.52	1.71	2.01	2.56
550	....	....	281	474	702	......	......	1.52	1.75	2.14
600	....	....	208	391	603	......	......	1.36	1.54	1.83
650	....	....	144	318	522	......	......	1.13	1.36	1.59
700	....	....	....	254	443	......	......	......	1.21	1.48
750	....	....	....	196	377	......	......	......	1.07	1.21
800	....	....	....	145	320	......	......	......	0.95	1.07

TABLE 23. FEED COMBINATIONS IN PRODUCING SPECIFIED MILK LEVELS IN 28 DAYS AND MARGINAL RATES OF SUBSTITUTION OF GRAIN FOR HAY, BASED ON EQUATION 21a. ABILITY FIXED AT MEAN OF COWS IN EXPERIMENT. SIXTH MONTH OF EXPERIMENTAL PERIOD (T = 6).

Level of grain (pounds)	Pounds hay required to maintain milk output of:					Marginal rates of substitution (ΔH/ΔG): pounds hay replaced by 1 additional pound of grain along indicated milk isoquant.				
	600 lbs.	700 lbs.	800 lbs.	900 lbs.	1,000 lbs.	600 lbs.	700 lbs.	800 lbs.	900 lbs.	1,000 lbs.
150	507	....	....	....	....	3.49	......	......	......	......
200	360	666	....	....	....	2.49	4.66	......	......	......
250	250	488	....	....	....	1.95	2.83	......	......	......
300	162	367	678	....	....	1.59	2.09	4.01	......	......
350	....	274	523	....	....	......	1.64	2.46	......	......
400	....	201	418	792	....	......	1.32	1.79	4.78	......
450	....	141	339	628	....	......	1.07	1.38	2.40	......
500	....	....	278	529	....	......	......	1.08	1.63	......
550	....	....	230	459	962	......	......	0.83	1.17	11.81
600	....	....	194	409	769	......	......	0.63	0.85	2.08
650	....	....	167	373	691	......	......	0.44	0.58	1.14
700	....	....	....	350	646	......	......	......	0.35	0.63
750	....	....	....	337	625	......	......	......	0.14	0.24
800	....	....	....	335	621	......	......	......	......	......

for equation 19a differ more from customary evaluation standards than do those for equation 21a.

Figures 17, 18 and 19 include milk isoquants for the first, the "mean" and the sixth month of the experimental period, based on equations 19a and 21a. The fact that the slopes of the isoquants are similar emphasizes the point mentioned above—namely, that estimates of the feed substitution rates based on equations 19a and 21a do not differ significantly. However, one condition is apparent from the isoquant figures. The milk contours in figs. 17, 18 and 19 take on greater curvature for higher milk levels, for both equations 19a and 21a, indicating that as feeding levels become greater (1) relatively small variations in feed combinations tend to cause larger variations in the substitution rates or (2) smaller ranges of feed combinations will allow a specified level of milk production. Only one ration or ratio of grain and hay would allow maximum production per cow (i.e., the point of isocline convergence). Similarly as time progresses, from fig. 17 to fig. 18 to fig. 19, the milk isoquants tend to take on greater curvature, indicating a more rapid change in substitution rates as feed proportions are varied in attaining a particular milk level. The tendency of the milk isoquants to increase in curvature with progress of the lactation period also is emphasized in fig. 20. In this figure, 1,000-pound milk isoquants based on the two functions are presented for each of the 6 months of the experimental period. The isoquants for the two functions tend to "spread apart" with greater amounts of time because the coefficients for time, or the product of time and feeds, is relatively greater in equation 21a than in equation 19a.

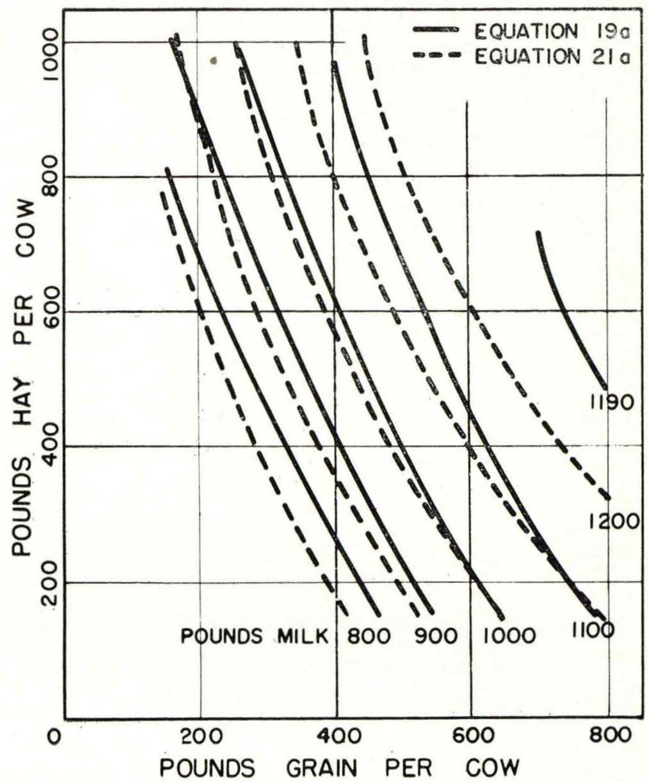

Fig. 18. Milk isoquants for mean month of experiment, based on equations 19a and 21a. Ability fixed at mean for 36 cows.

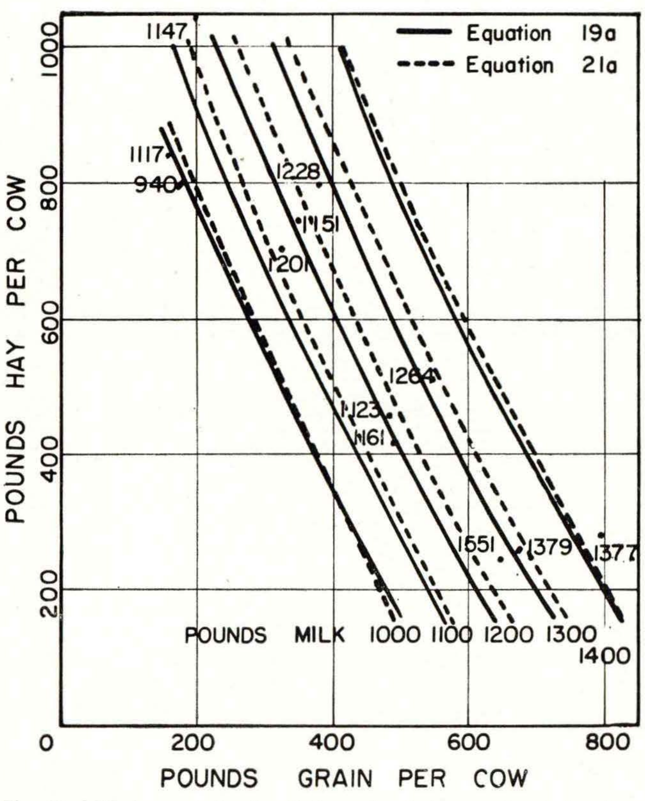

Fig. 17. Milk isoquants for the first month of the experiment, based on equations 19a and 21a. Ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

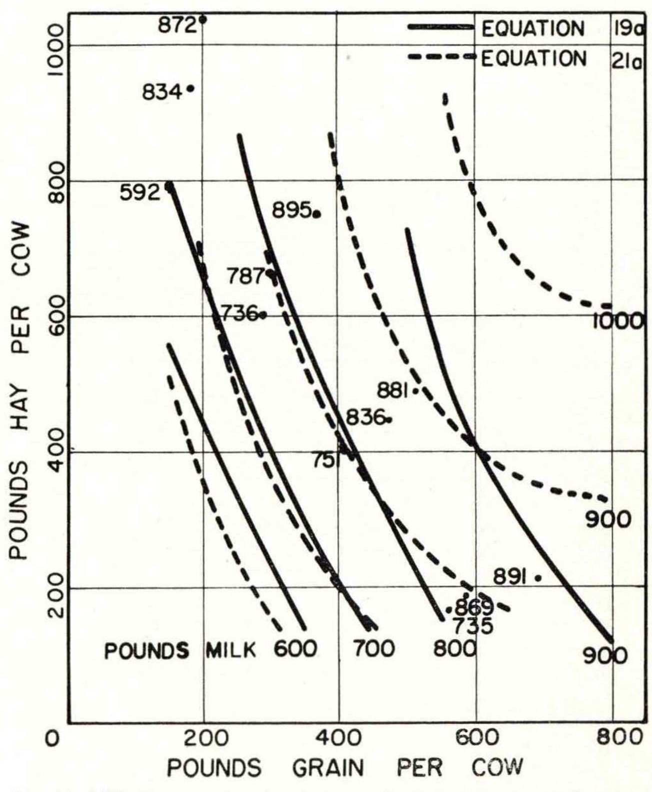

Fig. 19. Milk isoquants for the sixth month of the experiment, based on equations 19a and 21a. Ability fixed at mean for 36 cows. (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

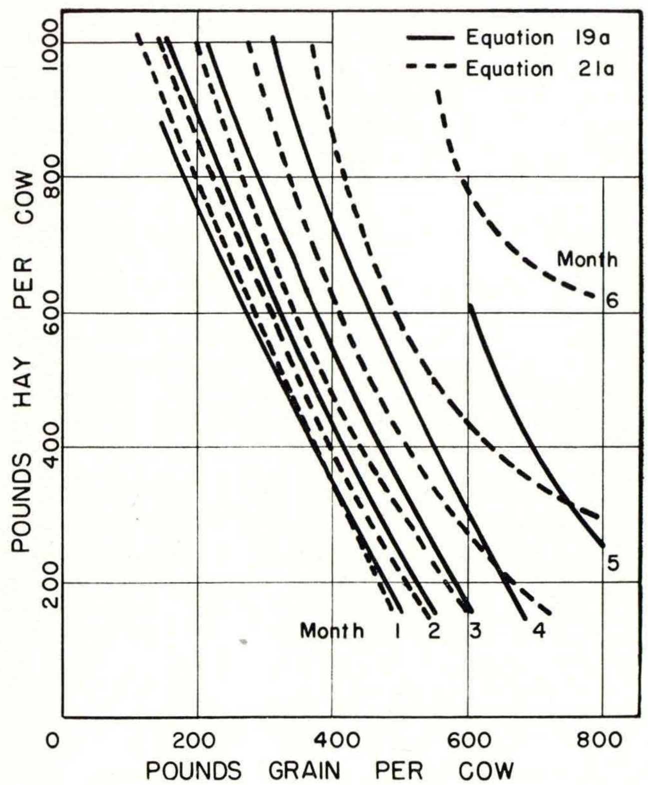

Fig. 20. Feed combinations to produce 1,000 pounds milk in each month of the experiment. Ability set at mean for 36 cows. (All isoquants represent a 1,000-pound milk level. Equation 19a does not permit a 1,000-pound estimate in the sixth month within the grain input range shown.)

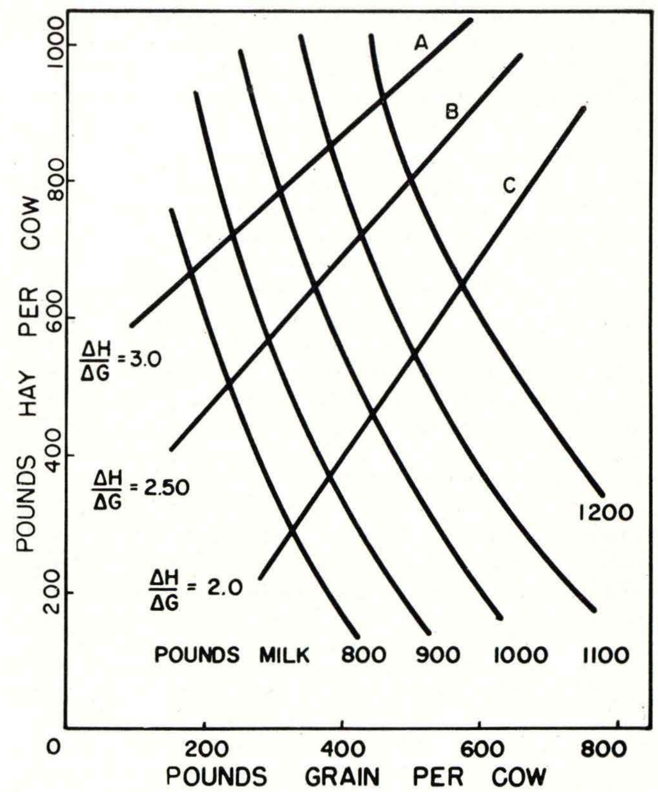

Fig. 22. Isoquants and isoclines for mean month of experimental period and for given substitution rates, estimated from equation 21a. Ability at mean for 36 cows.

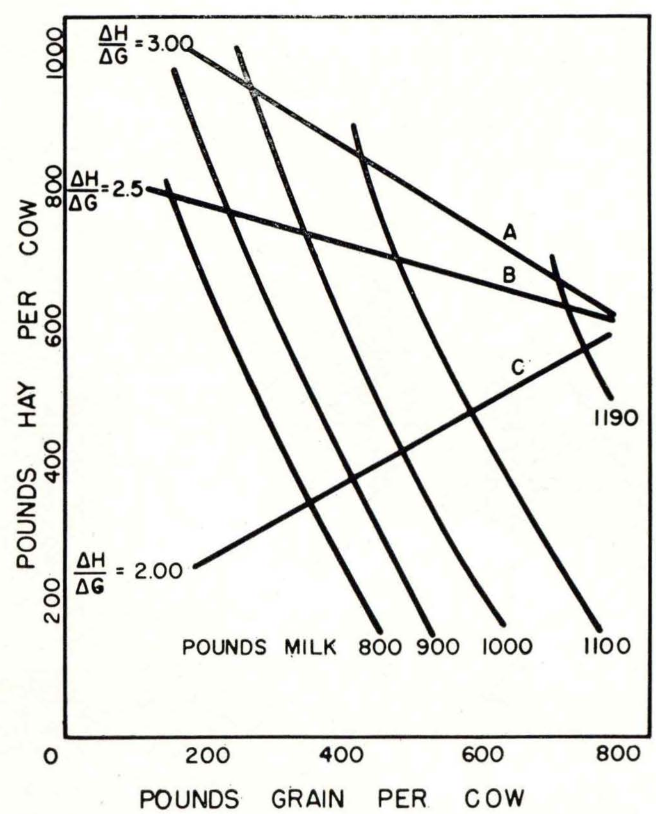

Fig. 21. Isoquants and isoclines for mean month of experiment and for given substitution rates, estimated from equation 19a. Ability at mean for 36 cows.

Isoclines Predicted From Functions With Time Variable

Use of the functions with time as a variable would indicate that standard ratios, such as ENE or TDN constants, are not appropriate for evaluating feeds in relation to milk production. The basis for this statement is the changing nature of substitution rates in tables 18 through 23. (These substitution rates have been determined relative to a given milk level.) Since the hay/grain substitution ratio declines with an increased proportion of grain and increases with a decreased proportion of grain in the ration, the coefficient for evaluating feeds should change accordingly if evaluation is relative to milk production. The marginal substitution rates (i.e., equations such as 19c and 21c) are the basis for coefficients which can serve as the basis for feed evaluation when the entire milk production surface is considered in the evaluation.

The isoclines explained earlier also provide a basis for feed evaluation and recommendations: Feeds have a constant value relative to each other only along an isocline. Isoclines, along with milk isoquants, are presented in fig. 21 for milk function 19a and in fig. 22 for function 21a. These isoclines again trace out the path of feed combinations which result in a given substitution rate between hay and grain as milk is taken to higher levels (i.e., is denoted by isoquants higher in the feed plane). In other words, 1 pound of grain substitutes for 3.00 pounds of hay for the feed combinations traced out by isocline A. If the grain/hay price ratio were also 3, isocline A traces the least-cost combination of feeds for the various milk levels. Similarly, isocline C traces out the least-cost ration when the grain/hay price ratio is 2 if equation 19a is used for predicting the milk production surface. Similarly, isoclines A, B and C in fig. 22 indicate the paths of least-cost rations if equation 21a is used for the estimates.

While the regression coefficients for equation 21a have lower standard errors than do the coefficients for equation 19a, the slopes of the isoclines in fig. 21 appear more in line with physiological characteristics of milk production. They converge more rapidly, suggesting a maximum possible level of milk production. The point at which the isoclines converge defines the single ration consistent with maximum milk production per cow. Convergence and definition of a maximum milk level is not so apparent in fig. 22 for equation 21a. However, the slopes of the isoclines in fig. 22 do approach those in fig. 11, which are more nearly the historic basis for recommendations on rations. However, the historic basis is appropriate only if the marginal rate of substitution equation includes only one feed variable with a linear coefficient, an unlikely situation unless it can be proven that the other feed has constant marginal productivity.

Surfaces With Time and Ability Varied

The isoquant and isocline maps in figs. 21 and 22 are for the mean month (T=3.5) of the experimental period and the mean level of ability of the 36 cows in the experiment. The basic production functions can be used to predict feed combinations, substitution rates and milk levels for other points of time in the lactation period and/or other ability levels. Figure 23 includes milk production surfaces, estimated from equation 19a when ability is fixed at the mean of the 36 cows, for the first, the mean and the sixth month of the experimental period. Parallel estimates for equation 21a are provided in fig. 24. Because of the time coefficients in the milk production functions, the milk level declines at a decreasing rate with time (see Appendix D). Also, the slope of the surface declines, indicating a lower marginal productivity of either feed as the lactation period progresses.

Figure 25 provides production surfaces estimated from equation 19a for cow ability at three different levels with time fixed at the mean month. The slopes of these surfaces are identical; the only difference between them is the height of milk level at which the surface slopes begin. In other words, the entire surface is moved upward as the level of ability increases, because ability is included as a linear term only in the equation used. However, the rates of hay/grain substitution differ between ability levels for a given milk level. This is true because a milk contour, such as 1,000 pounds, falls lower on the sloping portion of the surface as the level of ability increases. As is illustrated in figs. 17 and 19, the curvature of the isoquants (i.e., the marginal rates of substitution) changes for milk contours spaced further over the feed plane. (While surfaces from equation 21a have not been provided for different ability levels, they have the same basic differences illustrated in fig. 25.)

SPECIFICATIONS OF ECONOMIC OPTIMA IN RATIONS

Prediction of the milk production function or surface allows specification of the ration which will maximize returns above feed costs. The basic conditions for profit maximization were indicated in the section on logic; equation 3 indicates the condition for determining the least-cost ration when milk is held at a particular level (e.g., the one of the many feed combinations indicated by the 1,000-pound milk contour in fig. 23 to give this

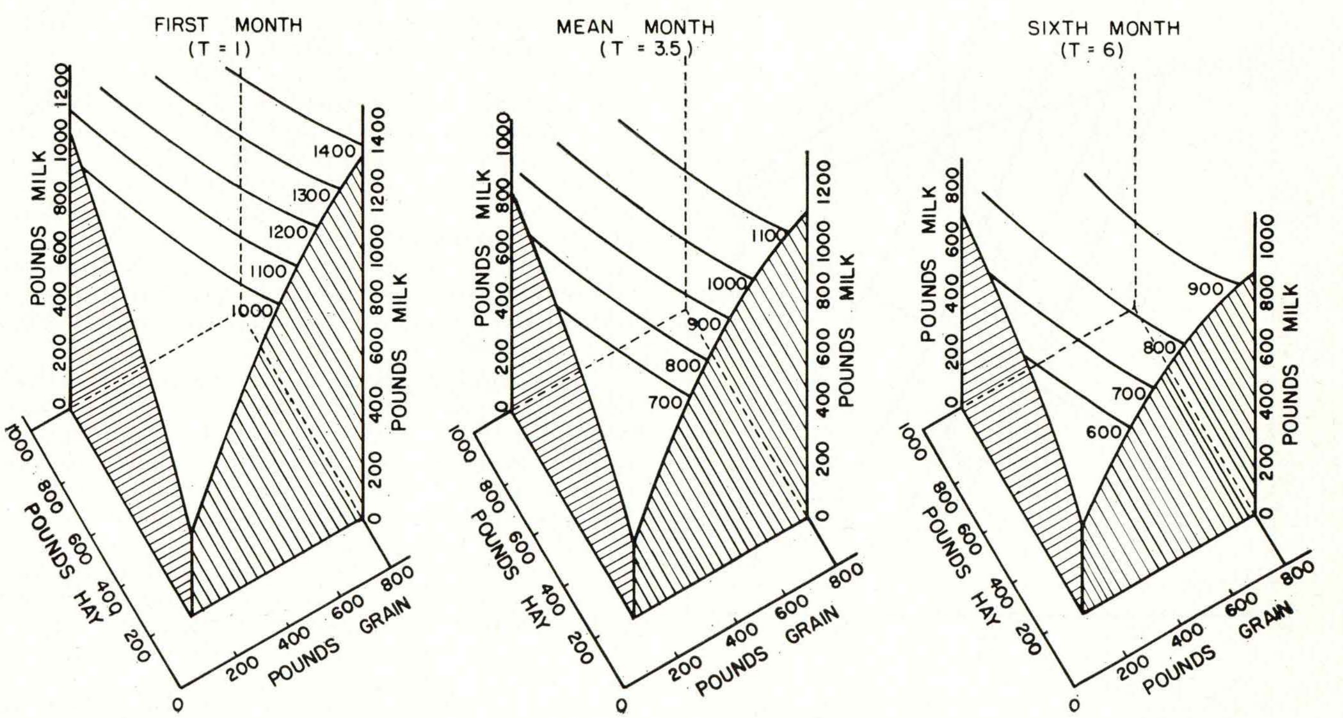

Fig. 23. Milk production surfaces estimated from equation 19a for first, mean and sixth month (28 days) of experiment, ability at mean.

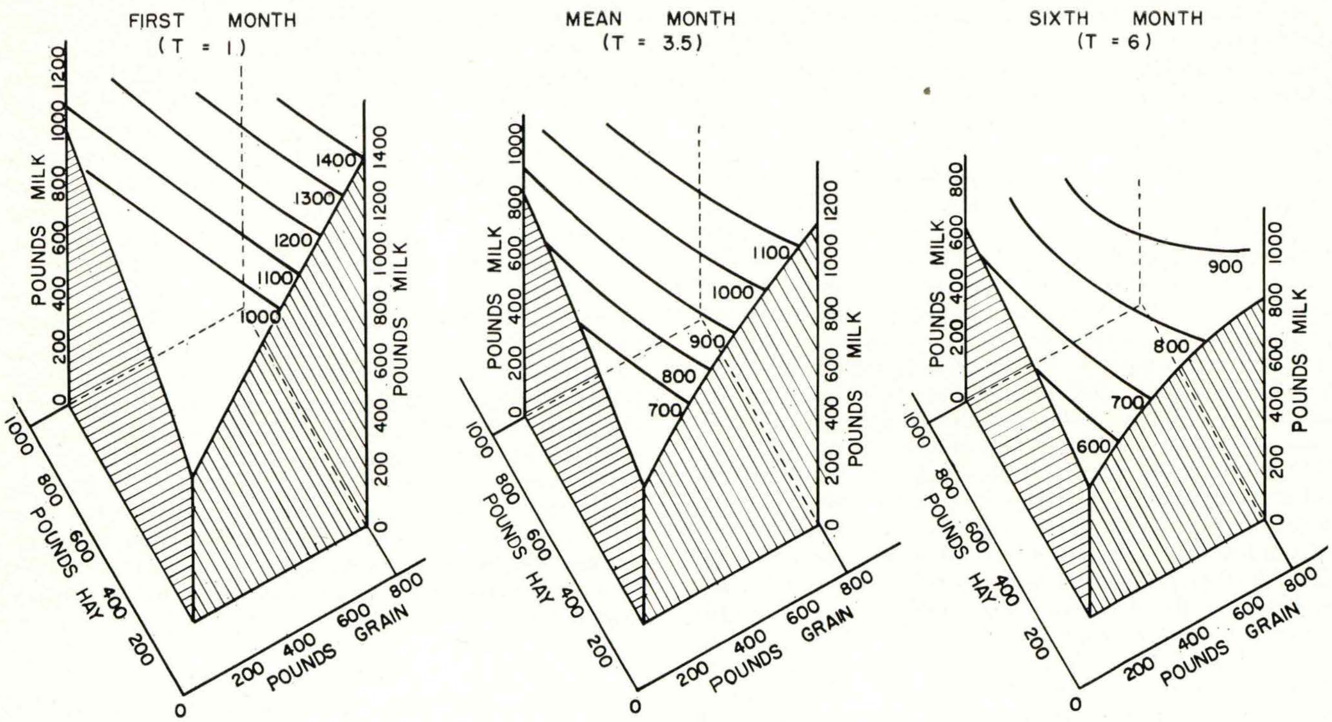

Fig. 24. Milk production surfaces estimated from equation 21a for first, mean and sixth month (28 days) of experiment, ability at mean.

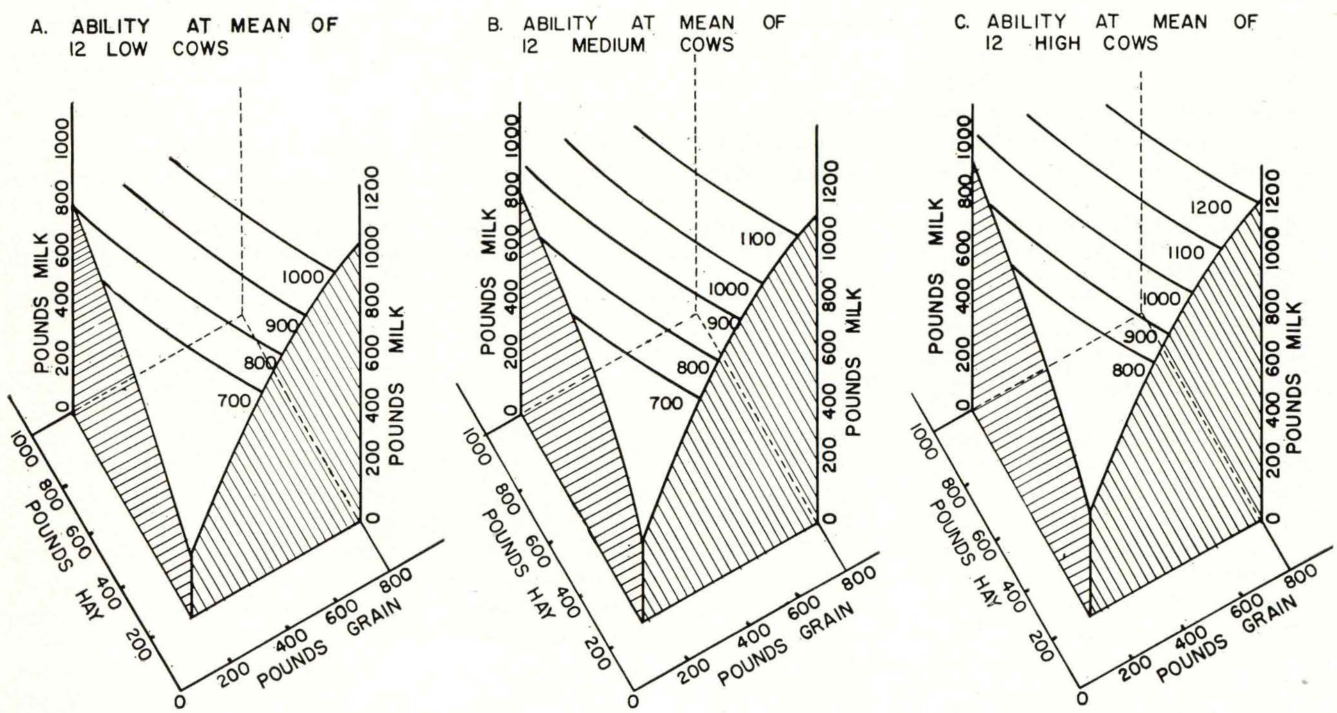

Fig. 25. Milk production surfaces estimated from equation 19a for "high," "medium" and "low" ability cows, and for mean month of experiment.

output at lowest cost); equation 4 indicates the condition for determining the optimum level of feed (i.e., the optimum milk level) for a given ration; equations 5 and 6 provide the conditions for simultaneously defining both the optimum ration and the optimum level of milk production. Using production function 19a for predictions, equation 19c can now be substituted for the $\triangle H/\triangle G$ of equation 3 and set to equal any grain/hay price ratio with time and ability at previously stated levels, as indicated in equation 28. (This equation is a particular use of equation 19c.)

$$(28) \quad \frac{2.9740 - 0.002384G - 0.001056H}{1.5437 - 0.000776H - 0.001056G} = \frac{P_g}{P_h}$$

In this case, with equation 28 serving as the basis for predictions, ability is set at the mean of cows in the experiment, and time is set at the first month of the experimental period. By reference to table 18, it may be seen that for any grain/hay price ratio equal to 2.01, the least-cost combination to produce 1,000 pounds of milk in 28 days with a cow of mean ability is 350 pounds of grain and 445 pounds of hay. This is true for any level of grain and hay prices (per pound) which yields a ratio of 2.01 and makes no reference to the most profitable level of production. Similarly, with grain at 1.93 cents and hay at 0.8 cent per pound (or any other pair of prices yielding a ratio of 2.41) the least-cost feed combination to produce 1,000 pounds of milk under previously stated conditions of time and ability is 150 pounds of grain and 883 pounds of hay in the 28 days (5.4 pounds of grain and 31.5 pounds of hay per day). The substitution rates in tables such as 18, 19 and 20 allow prediction of least-cost rations for particular milk levels in any month of the experiment (months 3 to 8 of the lactation). For example, when the price of grain is 2.45 cents per pound and the price of hay is 1 cent per pound, the grain/hay price ratio is 2.45. In table 19, when the marginal rate of substitution of grain for hay, $\triangle H/\triangle G$, is 2.45, the minimum cost feed combination to produce 1,100 pounds of milk in the mean month of the experiment with a cow of mean ability is 500 pounds of grain and 674 pounds of hay.

Other price ratios can be figured similarly with interpolations made between feed combinations. Increases in costs or sacrifices in profit are not great for small deviations away from the feed combination where the substitution rate is equal to the price ratio. In the case above, for example, the cost of the optimum ration for 1,100 pounds of milk is $18.99. If a ration of 600 pounds of grain and 455 pounds of hay were used, the feed cost would be $19.25; for 800 pounds of grain and 122 pounds of hay, the feed cost would be $20.82. These differences in feed costs are perhaps not great enough to offset the added labor costs for feeding particular hay rations. When labor costs are figured, the least-cost milk production may be obtained when the optimum level of grain feeding is determined in relation to the grain/milk price ratio, with self-feeding of hay to stomach capacity as was explained in respect to fig. 11.

The gain from feeding one ration rather than others along a milk contour increases with greater curvature of the isoquant. Since the curvature of the milk isoquants for equations such as 19a or 21a tends to increase with level of milk production, gains in feeding the unique optimum ration are greater as the level of milk production increases (i.e., the grain/milk price ratio decreases) or as time in the lactation period increases. While the gains from feeding unique rations are relatively low from the predictions of this study, the final advantages of particular rations for given milk levels can be determined only as the nature of the milk surface and its isoquant family are better established.

Simultaneous Specification of Ration and Milk Level

Equation 19a now can be used to derive the partial derivatives outlined in equations 5 and 6. In equation 29 below (based on equation 19a), the partial derivative for grain is equated to the grain/milk price ratio when

$$(29) \quad \frac{\partial M}{\partial G} = 2.9740 - 0.002384G - 0.001056H = \frac{\$3.00}{\$4.00}$$

TABLE 24. ESTIMATED OPTIMUM FEED QUANTITIES AND MILK PRODUCTION IN THE FIRST MONTH, MEAN MONTH AND SIXTH MONTH OF EXPERIMENTAL PERIOD, FOR VARIOUS PRICE RATIOS. ESTIMATES FROM EQUATION 19a WITH ABILITY AT MEAN.*

Feed prices		Price ratio:† grain/hay	Hay, grain and milk quantities (pounds) with milk prices per cwt. at:								
			$3.00			$4.00			$5.00		
Grain per cwt.	Hay per ton		Hay	Grain	Milk	Hay	Grain	Milk	Hay	Grain	Milk
						Month 1					
$2	$15	2.67	882	577	1,501	846	663	1,544	843	715	1,563
3	15	4.00	1,361	225	1,328	1,204	400	1,446	1,110	504	1,500
3	25	2.40	820	465	1,387	799	579	1,479	786	648	1,522
3	35	1.71	275	707	1,357	390	761	1,462	458	793	1,511
4	35	2.29	762	350	1,224	756	493	1,388	752	579	1,464
						Mean month					
2	15	2.67	745	473	1,100	710	560	1,143	686	612	1,163
3	15	4.00	1,223	121	926	1,066	296	1,045	972	400	1,100
3	25	2.40	682	361	986	662	476	1,080	649	544	1,121
3	35	1.71	........‡	503	....	252	657	1,062	322	690	1,111
4	35	2.29	624	247	824	618	390	988	613	476	1,064
						Month 6					
2	15	2.67	608	369	825	570	456	867	548	508	887
3	15	4.00	1,085	18	651	928	193	770	834	297	824
3	25	2.40	545	257	710	524	372	803	511	440	845
3	35	1.71	........‡	499	....	........‡	553	....	184	586	835
4	35	2.29	486	144	549	480	286	712	475	372	787

*Figures show the most profitable ration and milk level for each combination of hay, grain and milk prices. For example, with hay at $25.00, grain at $3.00 and milk at $3.00, the most profitable ration includes 465 pounds of grain, 820 pounds of hay and produces 1,387 pounds of milk in the first month.
†Price per pound of grain divided by price per pound of hay.
‡Physiological minimum hay quantity.

TABLE 25. ESTIMATED OPTIMUM FEED QUANTITIES AND MILK PRODUCTION IN THE FIRST MONTH, MEAN MONTH AND SIXTH MONTH OF EXPERIMENTAL PERIOD, FOR VARIOUS PRICE RATIOS. ESTIMATES FROM EQUATION 21a WITH ABILITY AT MEAN.

Feed prices		Price ratio:* grain/ hay	Hay, grain and milk quantities (pounds) with milk prices per cwt. at:†								
			$3.00			$4.00			$5.00		
Grain per cwt.	Hay per ton		Hay	Grain	Milk	Hay	Grain	Milk	Hay	Grain	Milk
							Month 1				
$2	$15	2.67	3,797	2,492	....	4,353	3,051	....	4,723	3,380	....
3	15	4.00	3,797	1,407	....	4,353	2,229	....	4,723	2,722	....
3	25	2.40	2,223	1,407	....	3,242	2,229	....	3,797	2,722	....
3	35	1.71	742	1,407	....	2,038	2,229	....	2,871	2,722	....
4	35	2.29	742	321	....	2,038	1,407	....	2,871	2,064	....
							Mean month				
2	15	2.67	1,085	712	1,444	1,244	872	1,573	1,349	966	1,633
3	15	4.00	1,085	402	1,186	1,244	637	1,427	1,349	778	1,539
3	25	2.40	635	402	1,035	926	637	1,347	1,085	778	1,486
3	35	1.71	212	402	823	582	637	1,218	820	778	1,407
4	35	2.29	212	92	462	582	402	1,012	820	590	1,275
							Month 6				
2	15	2.67	633	415	878	726	508	953	787	563	988
3	15	4.00	633	234	727	726	372	868	787	454	934
3	25	2.40	370	234	639	540	372	822	633	454	903
3	35	1.71	124	234	516	340	372	747	478	454	857
4	35	2.29	124	54	306	340	234	626	478	344	780

*Price per pound of grain divided by price per pound of hay.
†Pounds milk not computed for feed quantities in first month because of obvious inability of animal to consume such quantities.

$$(30)\quad \frac{\partial M}{\partial H} = 1.5437 - 0.000776H - 0.001056G = \frac{\$1.25}{\$4.00}$$

milk is 4 cents per pound and grain is 3 cents per pound, in the first month of the experiment for a cow of mean ability; in equation 30 (also derived from 19a), the partial derivative for hay is equated to the hay/milk price ratio when hay is 1.25 cents per pound ($25 per ton). By simultaneous solution of equations 29 and 30, it is determined that the ration which will maximize return above feed costs should include 799 pounds of hay and 579 pounds of grain fed over 28 days (28 pounds of hay and 21 pounds of grain per day) to produce 1,479 pounds of milk and a return of $31.80 above feed costs. If a radically different ration such as 275 pounds hay and 707 pounds grain (see table 24, month 1, line 4) had been fed in the first month under these price relationships, the return above feed costs is estimated at $29.63. With feeds remaining at the above prices and the milk price rising $5 per hundred pounds, the optimum ration would include 786 pounds of hay and 648 pounds of grain, with milk production at 1,522 pounds. If the milk price were $3 per hundred pounds, the optimum ration would include 820 pounds of hay and 465 pounds of grain, with milk production at 1,387 pounds.

If the milk price remains at $4 per hundred pounds and hay increases to 1.75 cents per pound ($35 a ton), an increase in grain price to 4 cents per pound has this effect on the optimum ration: The feed combination should include 756 pounds of hay and 493 pounds of grain to produce 1,388 pounds of milk in the first month from a cow of medium ability. With these same prices in the sixth month, the ration would include only 480 pounds of hay and 286 pounds of grain, producing an estimated 712 pounds of milk.

Feed and milk quantities in table 24 represent optimum rations and milk production levels derived from equation 19a for certain feed and milk price situations. Estimates in table 25 are based on equation 21a. However, since the production surface for equation 21a is nearly linear with time fixed at the first month (see fig. 24), the feed quantities suggested for price variations in this month generally exceed the cow's stomach capacity. The estimates from equation 21a for the mean and sixth months are more similar to the estimates from equation 19a in table 24. The wide differences for the first month, and the fact that equation 21a provides estimates far outside the range of stomach capacity, again suggests that equation 19a is preferable to equation 21a, even though the latter has lower standard errors than the former.

It is of interest to compare the optimum rations indicated in table 24 with those based on Jensen et al. (12). Estimates based on the Jensen study indicate that for prices of $25 per ton for hay and $3 and $4 per hundred pounds respectively, for grain and milk, the optimum ration would include about 350 pounds of grain; leaving capacity for 800-900 pounds of hay taken free choice (see Appendix C). This estimate is for a cow producing 40 pounds of milk per day (1,120 pounds in 28 days). Estimates for the same prices, based on equation 19a of this study (table 24), indicate 476 pounds of grain and 662 pounds of hay to produce 1,080 pounds of milk, for a cow of mean ability in the mean month of the experimental period. With prices of $35 per ton for hay, $4 for grain and

$5 for milk, the estimates based on the Jensen study include about 330 pounds of grain and a residual of 800-900 pounds of hay; the estimates of the current study include 476 pounds of grain and 613 pounds of hay to produce 1,064 pounds of milk in 28 days.

Predictions of the daily rate of grain and hay feeding to give maximum return above feed costs in the first, mean and sixth months of the experimental period can be made by dividing the quantities of table 24 by 28 days. These estimates are for a cow of mean ability and are based on equation 19a. Estimates for other ability levels can be predicted by adding to or subtracting from these quantities an amount based on the coefficient for A in equation 19a.

APPENDIX A

ANALYSIS OF VARIANCE OF REGRESSION

The analysis of variance shown in tables A-1, A-2 and A-3 has been made to further indicate the appropriateness of certain of the equations used in the text. Quadratic, square root and linear functions are presented, some of which have been rejected on logical grounds while others have seemed appropriate estimates of the milk production surface. All are acceptable at the 99-percent probability level.

TABLE A-1. ANALYSIS OF VARIANCE OF FUNCTIONS FOR THE 26-WEEK PERIOD.

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Quadratic crossproduct equation 7a.				
Total	35	29,919,103		
Due to regression	7	23,337,648	3,333,950	14.18
Deviations from regression	28	6,581,455	235,052	
Square root crossproduct equation 8a.				
Total	35	29,919,103		
Due to regression	7	23,510,700	3,358,671	14.67
Deviations from regression	28	6,408,403	228,872	
Quadratic equation 10a.				
Total	35	29,919,103		
Due to regression	5	23,013,235	4,602,647	19.99
Deviations from regression	30	6,905,868	230,196	
Quadratic equation 11a.				
Total	35	29,919,103		
Due to regression	4	22,980,025	5,745,006	25.67
Deviations from regression	31	6,939,078	223,841	
Square root equation 12a.				
Total	35	29,919,103		
Due to regression	4	23,075,647	5,768,912	26.13
Deviations from regression	31	6,843,456	220,757	
Linear equation 13a.				
Total	35	29,919,103		
Due to regression	3	22,505,359	7,501,786	32.38
Deviations from regression	32	7,413,744	231,680	

TABLE A-2. ANALYSIS OF VARIANCE OF FUNCTIONS FOR THE 16-WEEK PERIOD WITH 51 COWS.

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Quadratic equation 14a.				
Total	50	42,189,052		
Due to regression	5	36,785,563	7,357,113	61.27
Deviations from regression	45	5,403,489	120,078	
Quadratic equation 15a.				
Total	50	42,189,052		
Due to regression	5	36,295,537	7,259,107	55.42
Deviations from regression	45	5,893,515	130,967	
Quadratic equation 16a.				
Total	50	42,189,052		
Due to regression	4	36,597,821	9,149,455	75.27
Deviations from regression	46	5,591,231	121,549	
Linear equation 17a.				
Total	50	42,189,052		
Due to regression	3	35,966,884	11,988,961	90.56
Deviations from regression	47	6,222,168	132,387	

TABLE A-3. ANALYSIS OF VARIANCE OF FUNCTIONS FOR 28-DAY PERIODS WITH TIME AS A VARIABLE.

Source of variation	Degrees of freedom	Sum of squares	Mean square	F
Quadratic equation 19a.				
Total	215	10,002,585		
Due to regression	10	8,130,771	813,077	89.05
Deviations from regression	205	1,871,814	9,131	
Quadratic equation 20a.				
Total	215	10,002,585		
Due to regression	7	8,017,012	1,145,287	119.98
Deviations from regression	208	1,985,573	9,546	
Quadratic equation 21a.				
Total	215	10,002,585		
Due to regression	6	8,031,055	1,338,509	141.90
Deviations from regression	209	1,971,530	9,433	

APPENDIX B

ANALYSIS OF VARIANCE, ADDED TERMS

Analysis of variance is presented in tables A-4 and A-5 to test the reduction in the error sum of squares due to successive addition of certain terms to the simple linear equation for the over-all period. None of the squared terms, square root terms or crossproducts added significantly to the explanation of variation in milk production.

Analysis of variance is presented in table A-6 for monthly functions, testing the addition of crossproduct terms in explanation of variation in milk production. F is acceptable at the 99-percent confidence level with 3 and 205 degrees of freedom.

TABLE A-4. SUM OF SQUARES AND VALUES OF F FOR ADDED REGRESSION TERMS.

Equation and independent variables	Degrees of freedom	Sum of squares of deviations from regression	F
13a (H, G, A)	32	7,413,744	
11a (H, G, A, G^2)	31	6,939,078	2.12
Reduction due to added term	1	474,666	
11a (H, G, A, G^2)	31	6,939,078	
10a (H, G, A, G^2, H^2)	30	6,905,868	0.14
Reduction due to added term	1	33,210	
10a (H, G, A, G^2, H^2)	30	6,905,868	
7a (H, G, A, G^2, H^2, A^2, HGA)	29	6,581,455	0.69
Reduction due to added terms	2	324,413	

TABLE A-5. SUM OF SQUARES AND VALUE OF F FOR ADDED REGRESSION TERMS.

Equation and independent variables	Degrees of freedom	Sum of squares of deviations from regression	F
13a (H, G, A)	32	7,413,744	
12a (H, G, A, $\sqrt{G}$)	31	6,843,456	2.58
Reduction due to added term	1	570,288	
12a (H, G, A, $\sqrt{G}$)	31	6,843,456	
8a (H, G, A, $\sqrt{G}$, $\sqrt{H}$, $\sqrt{A}$, $\sqrt{HG}$)	28	6,408,403	0.63
Reduction due to added terms	3	435,053	

TABLE A-6. SUMS OF SQUARES AND VALUE OF F FOR ADDED REGRESSION TERMS.

Equation and independent variables	Degrees of freedom	Sum of squares of deviations from regressions	F
20a (H, G, A, T, H^2, G^2, T^2)	208	1,985,573	
19a (H, G, A, T, H^2, G^2, T^2, HG, HT, GT)	205	1,871,814	4.15
Reduction due to added terms	3	113,759	

APPENDIX C

STOMACH CAPACITY

Dairy animal stomach capacity depends partly upon the size of the cow and, hence, upon the breed of cow. For the large dairy breeds, estimates may be made from data of this and other studies. While animals in this study were not fed rations of hay or grain alone, other studies have included such rations. The United States Department of Agriculture conducted hay feeding trials in which animals were full-fed alfalfa for 365 days (8). Consumption averaged 14,352 pounds hay per cow or 39.3 pounds per day. Maximum consumption was 47.1 pounds per day, while the minimum taken *ad. lib.* was 30.4 pounds. Other agricultural experiment stations reporting such experiments include Kansas (21), Nevada (18), Oregon (20) and California (23). Missouri also has reported feeding only a concentrate mixture for 15 entire lactations without materially lowering production (15). However, other trials suggest a minimum hay requirement of 5 to 6 pounds per day (13 and 19).

Table A-7 is derived from data of this study and others mentioned. It suggests maximum daily, 28-day and 182-day intakes of several ration combinations and is presented only as a guide for use with large breed dairy animals. Figure A-1 shows the estimates of table A-7 for a 28-day period, with line ab defining the estimated maximum feed intake of various ration combinations.

TABLE A-7. ESTIMATED MAXIMUM GRAIN AND HAY CONSUMPTION BY LARGE BREED DAIRY COWS IN POUNDS PER DAY, PER 28 DAYS AND PER 182 DAYS.

Pounds per day		Pounds per 28 days		Pounds per 182 days	
Grain	Hay	Grain	Hay	Grain	Hay
0	40	0	1,120	0	7,280
5	37	140	1,036	910	6,734
10	32	280	896	1,820	5,824
15	27	420	756	2,730	4,914
20	21	560	588	3,640	3,822
25	13	700	364	4,550	2,366
29	5	812	140	5,278	910

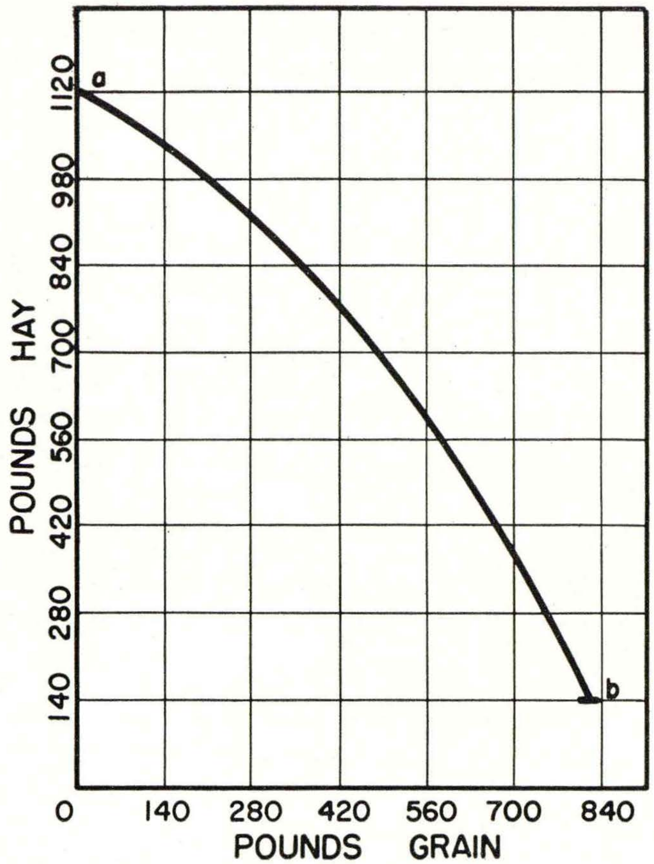

Fig. A-1. Estimates of stomach capacity of large breed dairy cows in 28-day period.

APPENDIX D

PERSISTENCY IN MILK PRODUCTION

Estimates of the ability of dairy cows to maintain production over time, or of the decline in milk production as lactation proceeds, are implicit in equations 18a, 19a, 20a and 21a. Similar estimates also have been made by dairy scientists (6). It has been suggested that the rate of decline per month averages about 6 percent of the previous month's production (5), indicating a declining absolute decrease as estimated here. However, the estimates in table A-8 do not represent a constant proportion of previous output, but rather a declining proportion. For example, from an initial production of 1,200 pounds of milk per month—approximately the mean for the first month of the experiment—equation A-2 estimates declines of 7.8, 7.6, 7.3, 7.0, 6.6 and 5.9 percent per month through months 3 to 8 of the lactation, rather than a fixed percentage of decline. It should be recognized that previous estimates of decline in production over the lactation period usually have included "stomach capacity" feeding of the cow. In this study, at least two-thirds of the cows were fed at a lower level. Hence, the difference in rations may also have caused the rate of decline to differ from the studies cited. Other studies suggest highly erratic persistency patterns with low interyear correlation (7).

TABLE A-8. ESTIMATED MONTHLY DECLINE IN MILK PRODUCTION FOR MONTHS 3 TO 8 OF THE LACTATION, FOR COWS FED AT THE MONTHLY MEAN OF GRAIN AND HAY INPUT. ABILITY (EQUATION A-1) IS SET AT THE AVERAGE OF ALL COWS IN THE EXPERIMENT.

Month of lactation	Decline in milk production per month (28 days)	
	Equation A-1	Equation A-2
3	253.0	93.0
4	110.0	84.0
5	68.0	75.0
6	48.0	66.0
7	37.0	58.0
8	30.0	49.0

The marginal change in milk production over the 6 months of the experimental period (months 3-8 in the lactation) is estimated from equation 18a by:

$$\text{(A-1)}\quad \frac{\partial M}{\partial T} = \frac{-0.1976(4.1937G^{0.3082}\ H^{0.1506}\ A^{0.3716})}{T^{1.1976}}$$

From equation 19a:

$$\text{(A-2)}\quad \frac{\partial M}{\partial T} = 14.224 + 8.758T - 0.157G - 0.0865H$$

Equations 18a and A-1 imply that the decline in milk production over time is associated with cow ability. As seen in equation A-1, the higher the ability index (as estimated in this study by production in the preliminary period), the greater would be the decline in production per time period. There seems to be no *a priori* basis for inferring such a relationship. However, the relatively low correlation coefficient (r) between the ability term and milk production in the sixth month indicates that such a relationship is not grossly misleading. This coefficient for the sixth month was 0.3623, while for the first month it was 0.8239[16]. This suggests that the high producing cows in early months were not easily distinguishable in later months and may have declined further than had cows of lower ability. Mainly, however, this difference results because high capacity cows could draw on body weight during early phases of the experiment. By the later phases, they had used up the weight reserve through which their greater production potential could be expressed. Consequently, all cows were on a more nearly equal basis in respect to their ability to produce milk.

As noted previously, equation A-1 estimates a greater absolute decline for a cow producing at a high rate early in the lactation, since she would have a high "ability" index. However, the array of mean milk production by months (1,210, 1,094, 1,028, 943, 883, 791) for the 36 animals and the differences between months (116, 66, 85, 60, 92), while erratic, suggest that equation A-2 gives the better estimate of persistency.

[16] Acceptable at 0.05 and 0.01 probability levels, respectively.

APPENDIX E

PREDICTIONS FROM MONTHLY FUNCTIONS

Functions for the 6 individual months of the experimental period were included in table 16 of the text. These monthly functions have not been used to predict optimum rations because (1) they did not give consistent estimates of marginal rates of substitution, and (2) equation 19a appears to estimate milk production surfaces for the several months conforming more closely to production logic. Regarding the first point for example, predicted marginal rates of substitution are nearly constant in equation 22a (month 1), while equation 23a indicates a wide range of substitution rates in the second month.

Estimates of milk isoquants for the production function of the first and sixth month (table 16) are presented in Appendix figs. A-2 and A-3. These isoquants are for observed ranges of milk production and are nearly linear for the 2 months shown.

APPENDIX F

INPUT-OUTPUT RELATIONSHIPS

Most nutrition studies have dealt only with input-output relationships, rather than with the isoquant and isocline concepts used here. Since input-output quantities have had widespread use, such estimates are provided here for comparison with earlier work. Predictions from equations 19a and 21a include total milk output and marginal productivity of (1) varied amounts of grain with hay fixed, (2) varied amounts of hay with grain fixed and (3) varied amounts of each of the four ration combinations of this study.

Varied Proportions of Hay and Grain

Tables A-9 and A-10 show estimates of milk production and of the marginal productivity of feeds when grain is varied while hay is fixed at approximately the mean and extremes observed. Tables A-11 and A-12 give similar information for varied hay inputs with grain at the mean and extreme levels. Only equations 19a and 21a are used in these estimates. Figures A-4 and A-5 are drawn from tables A-9 through A-12.

Fixed Proportions of Hay and Grain

Input-output relationships also may be derived from equations presented earlier for fixed ration combinations. The use of ENE coefficients to convert the hay and grain ration to a common input is illustrated in table A-13, showing feed combinations used in deriving table A-14. In table A-14, estimates of milk production as a function of varied inputs (therms ENE) are shown for the four fixed ration combinations of the experiment. Figures A-6 and A-7 show input-output curves from the two equations used as estimators in table A-14. It is seen that the use of constant ENE coefficients to convert the rations to therms, regardless of the composition of the ration, leads to the apparent conclusion that a therm ENE provided by a ration made up largely

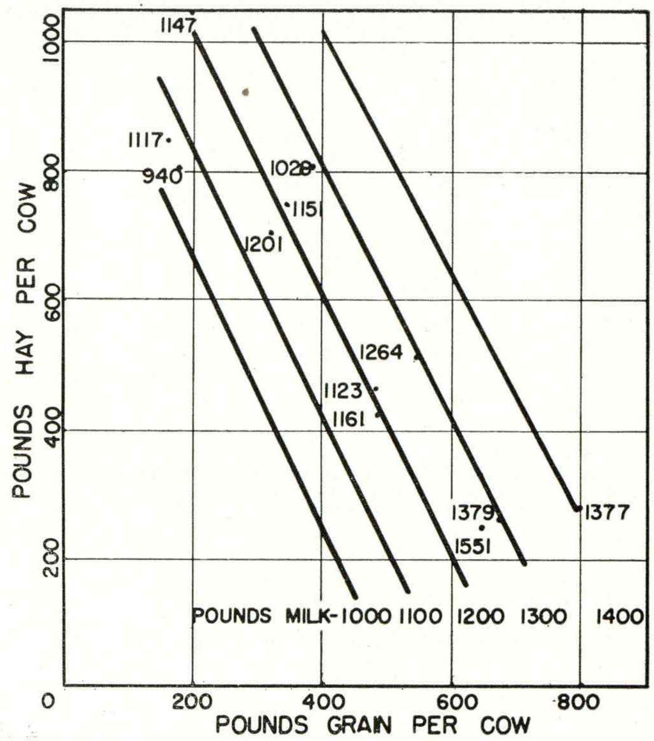

Fig. A-2. Milk isoquants for first 28-day period, predicted from equation 22a. Ability fixed at mean for 36 cows. (Dots show feed combinations which produced stated milk yields in first month from medium ability cows.)

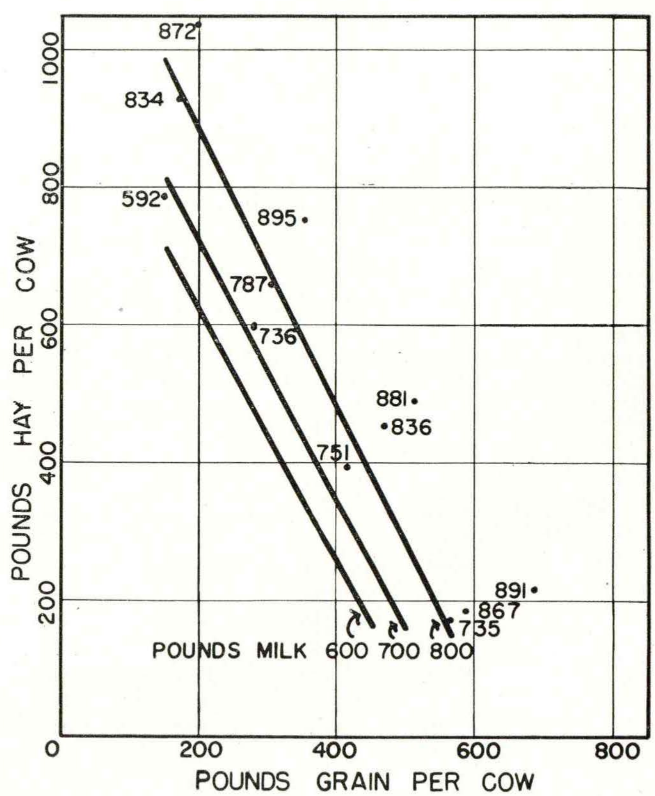

Fig. A-3. Milk isoquants for sixth 28-day period, predicted from equation 27a. Ability fixed at mean for 36 cows. (Dots show feed combinations which produced stated milk yields in the sixth month from medium ability cows.)

TABLE A-9. MILK PRODUCTION IN A 28-DAY PERIOD, AND MARGINAL PRODUCTIVITY OF FEEDS, GRAIN VARIED WITH HAY FIXED AT THREE LEVELS. ESTIMATED FROM EQUATION 19a, ABILITY SET AT MEAN OF 36 COWS, MEAN MONTH OF EXPERIMENTAL PERIOD.

Level of grain feeding (pounds per 28 days)	Level of hay feeding (per 28 days)								
	196 pounds			588 pounds			980 pounds		
	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay
150	319	2.02	1.02	658	1.60	0.71	878	1.19	0.41
200	417	1.90	0.96	736	1.48	0.66	935	1.07	0.36
300	595	1.66	0.86	872	1.24	0.55	1,030	0.83	0.25
400	749	1.42	0.75	985	1.01	0.45	1,110	0.59	0.14
500	879	1.18	0.65	1,073	0.77	0.34	....	....	....
600	986	0.94	0.54	1,109	0.53	0.24	....	....	....
700	1,068	0.71	0.44	....	....	....	....	....	....
800	1,127	0.47	0.33	....	....	....	....	....	....

*"Marginal product," refers to the estimated addition to milk production (pounds) if a pound of grain (or hay) were added to stated ration. Thus, with conditions as given, it is estimated that if 151 pounds grain were fed in the 28 days, along with 196 pounds hay, 321 pounds milk would be produced. The same explanation applies to the marginal product of hay.

TABLE A-10. MILK PRODUCTION IN A 28-DAY PERIOD, AND MARGINAL PRODUCTIVITY OF FEEDS, GRAIN VARIED WITH HAY FIXED AT THREE LEVELS. ESTIMATED FROM EQUATION 21a, ABILITY SET AT MEAN OF 36 COWS, MEAN MONTH OF EXPERIMENTAL PERIOD.

Level of grain feeding (pounds per 28 days)	Level of hay feeding (per 28 days)								
	196 pounds			588 pounds			980 pounds		
	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay
150	529	1.27	0.59	729	1.27	0.44	872	1.27	0.29
200	591	1.22	0.59	791	1.22	0.44	934	1.22	0.29
300	707	1.10	0.59	907	1.10	0.44	1,050	1.10	0.29
400	813	1.00	0.59	1,013	1.00	0.44	1,156	1.00	0.29
500	908	0.90	0.59	1,108	0.90	0.44	....	....	....
600	992	0.79	0.59	1,192	0.79	....	....	....	....
700	1,066	0.68	0.59	....	....	....	....	....	....
800	1,129	0.58	0.59	....	....	....	....	....	....

* See footnote, table A-9 for general explanation of term "marginal product." In equation 21a, derivatives are such that the marginal product of one input is not dependent upon the level of the other.

TABLE A-11. MILK PRODUCTION IN A 28-DAY PERIOD, AND MARGINAL PRODUCTIVITY OF FEEDS, HAY VARIED WITH GRAIN FIXED AT THREE LEVELS. ESTIMATED FROM EQUATION 19a, ABILITY SET AT MEAN OF 36 COWS, MEAN MONTH OF EXPERIMENTAL PERIOD.

Level of hay feeding (pounds per 28 days)	Level of grain feeding (per 28 days)								
	140 pounds			420 pounds			700 pounds		
	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay	Milk output (lbs.)	Marginal product:* Grain	Marginal product:* Hay
200	303	2.04	1.02	780	1.37	0.73	1,070	0.70	0.43
300	402	1.93	0.95	849	1.26	0.65	1,109	0.60	0.36
400	493	1.83	0.87	910	1.16	0.57	1,141	0.49	0.28
500	576	1.72	0.79	964	1.05	0.50	....	....	....
600	651	1.61	0.71	1,009	0.95	0.42	....	....	....
700	718	1.51	0.64	1,047	0.84	0.34	....	....	....
800	778	1.40	0.56	....	....	....	....	....	....
900	830	1.30	0.48	....	....	....	....	....	....
1,000	874	1.19	0.40	....	....	....	....	....	....

*See footnote table A-9.

TABLE A-12. MILK PRODUCTION IN A 28-DAY PERIOD, AND MARGINAL PRODUCTIVITY OF FEEDS, HAY VARIED WITH GRAIN FIXED AT THREE LEVELS. ESTIMATED FROM EQUATION 21a, ABILITY SET AT MEAN OF 36 COWS, MEAN MONTH OF EXPERIMENTAL PERIOD.

Level of hay feeding (pounds per 28 days)	Level of grain feeding (per 28 days)								
	140 pounds			420 pounds			700 pounds		
	Milk output (lbs.)	Marginal product:* Grain	Hay	Milk output (lbs.)	Marginal product:* Grain	Hay	Milk output (lbs.)	Marginal product:* Grain	Hay
200	518	1.28	0.58	834	0.98	0.58	1,067	0.68	0.58
300	575	1.28	0.55	891	0.98	0.55	1,124	0.68	0.55
400	628	1.28	0.51	944	0.98	0.51	1,177	0.68	0.51
500	677	1.28	0.47	993	0.98	0.47	....	....	....
600	722	1.28	0.43	1,038	0.98	0.43	....	....	....
700	763	1.28	0.40	1,079	0.98	0.40	....	....	....
800	801	1.28	0.36	....	....	....	....	....	....
900	835	1.28	0.32	....	....	....	....	....	....
1,000	865	1.28	0.28	....	....	....	....	....	....

*See footnotes tables A-9 and A-10.

TABLE A-13. POUNDS GRAIN AND HAY REQUIRED TO PROVIDE 28-DAY ENERGY QUANTITIES WITHIN THE RANGE OF THE STUDY, SUBJECT TO RESTRICTIONS UPON PROPORTION OF ENERGY PROVIDED BY EACH INPUT.*

Range of input quantities (therms ENE)	Percent of ENE provided by hay, and hay/grain combinations providing stated ENE quantity							
	75 percent		55 percent		35 percent		15 percent	
	Hay	Grain	Hay	Grain	Hay	Grain	Hay	Grain
425	770	144	565	258	359	373	......	......
475	860	160	631	289	401	417	172	545
525	951	177	698	319	444	461	190	603
575	1,041	194	764	349	486	505	208	660
625	....	......	830	380	528	549	226	718
675	....	......	......	......	......	......	244	775

*Based upon Morrison's ENE standards, the hay fed averaged 0.414 therm and the grain 0.7404 therm ENE per pound.

TABLE A-14. MILK PRODUCTION IN 28-DAY PERIOD AND MARGINAL PRODUCTIVITIES PER THERM ENE ALONG FOUR FIXED RATION LINES, ABILITY SET AT MEAN FOR 36 COWS, MEAN MONTH OF EXPERIMENTAL PERIOD.

Input (therms ENE)	Source of energy							
	75% ENE from hay		55% ENE from hay		35% ENE from hay		15% ENE from hay	
	Pounds milk	Marginal product* of a therm ENE	Pounds milk	Marginal product* of a therm ENE	Pounds milk	Marginal product* of a therm ENE	Pounds milk	Marginal product* of a therm ENE
				Equation 9a				
425	766	1.53	804	1.65	827	1.70	....	......
475	837	1.33	881	1.45	908	1.51	916	1.49
525	898	1.12	950	1.25	978	1.31	985	1.28
575	949	0.92	1,007	1.06	1,039	1.11	1,044	1.07
625	......	......	1,055	0.86	1,089	0.91	1,092	0.87
675	......	......	....	......	....	......	1,131	0.66
				Equation 21a				
425	795	1.10	849	1.29	875	1.35	....	......
475	847	1.04	913	1.24	941	1.29	932	1.19
525	897	0.94	974	1.18	1,005	1.24	990	1.12
575	943	0.90	1,031	1.13	1,065	1.18	1,044	1.04
625	......	......	1,087	1.08	1,123	1.13	1,094	0.97
675	......	......	....	......	....	......	1,141	0.90

*Marginal product of a therm ENE refers to the estimated addition to milk production (pounds) if an additional therm ENE were provided. If 426 therms were provided, 0.75 of the added therm being fed as hay, estimated milk production would be 767.5 pounds. Maintenance requirements are not subtracted.

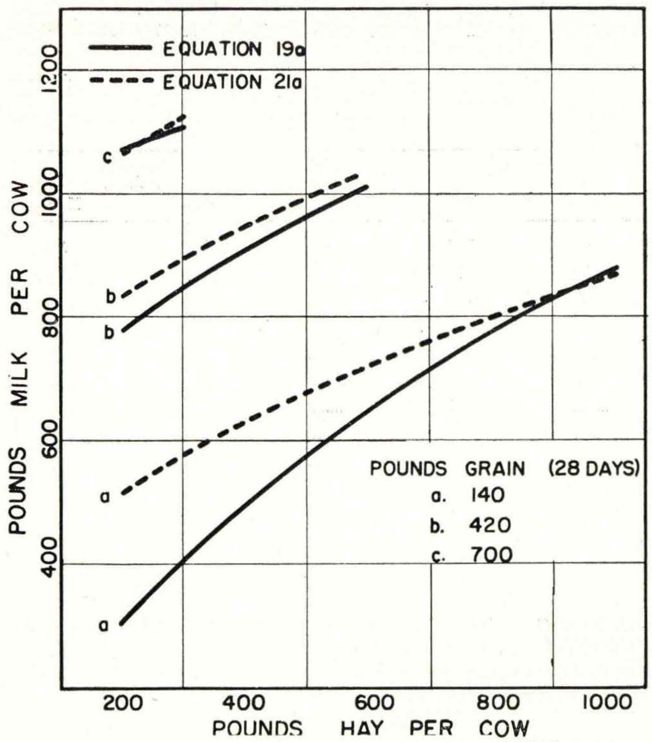

Fig. A-4. Input-output relationships for a 28-day period, with grain fixed at three levels, hay varied. Estimated from equations 19a and 21a for mean month of experimental period. Ability set at mean for 36 cows.

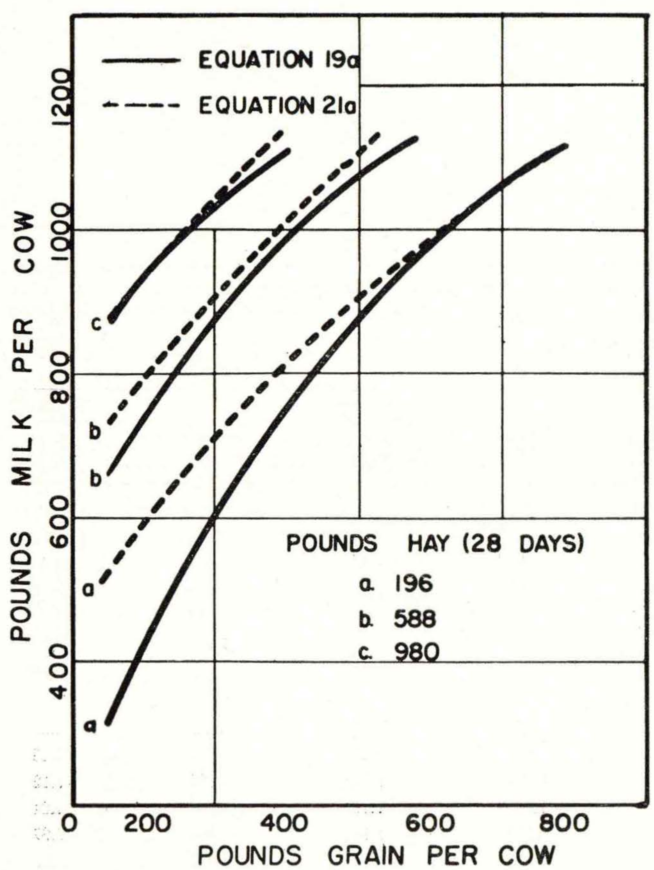

Fig. A-5 Input-output relationships for a 28-day period with hay fixed at three levels, grain varied. Estimated from equations 19a and 21a for mean month of experimental period. Ability set at mean for 36 cows.

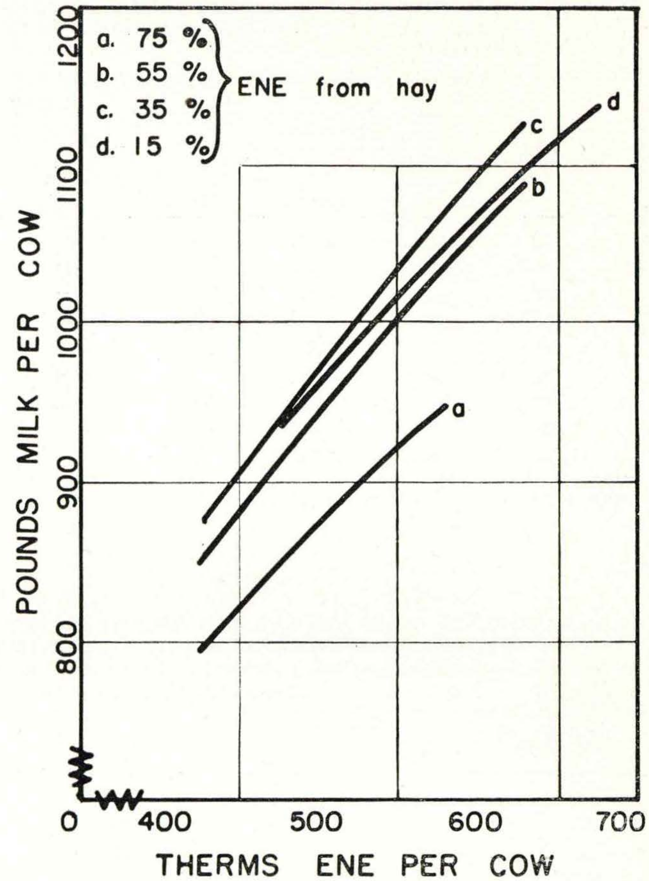

Fig. A-6. Input-output relationships for the four ration combinations of the experiment fed in fixed proportions. Estimated from equation 19a for the mean month of the experiment.

of hay will not produce as much milk as will a therm ENE made up primarily of grain. For example, with an input of 525 therms, estimated milk production is 898, 950, 978 and 985 pounds (equation 19a) for the four ration combinations.

This does not contradict the previous conclusion from data of this study that the ENE system of feed evaluation is more suitable for dairy animals than is the TDN system (4). It simply says that the ENE system does not express a unit from one feed as equivalent to a unit from another feed across the possible range of grain-hay proportions in the ration. Nor can any system possibly do so with constant coefficients over the whole range, if the marginal productivity of grain and hay vary with changing quantities of each. Only a concept such as marginal productivities or marginal rates of substitution, varying with the composition of the ration and the level of feeding, can accurately portray the value of feeds in milk production if the milk production surface is decidely curved.

Inter-month comparisons of input-output relationships are shown in fig. A-8 estimated from equation 19a for the two extreme ration combinations. It shows for both rations a sharp drop in marginal productivity (as seen in the slope of the lines) and in level of output as the lactation progresses.

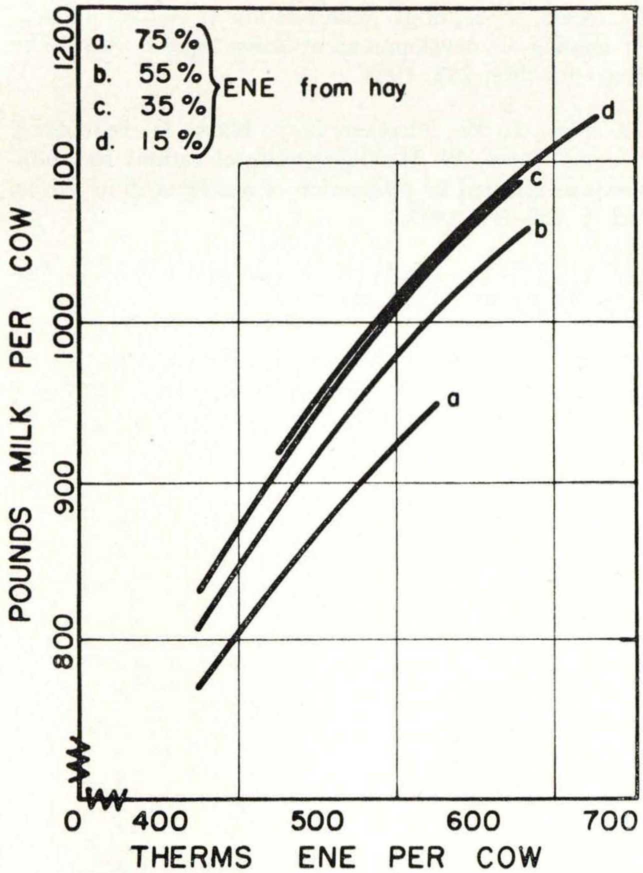

Fig. A-7. Input-output relationships for the four ration combinations of the experiment fed in fixed proportions. Estimated from equation 21a for the mean month of the experiment.

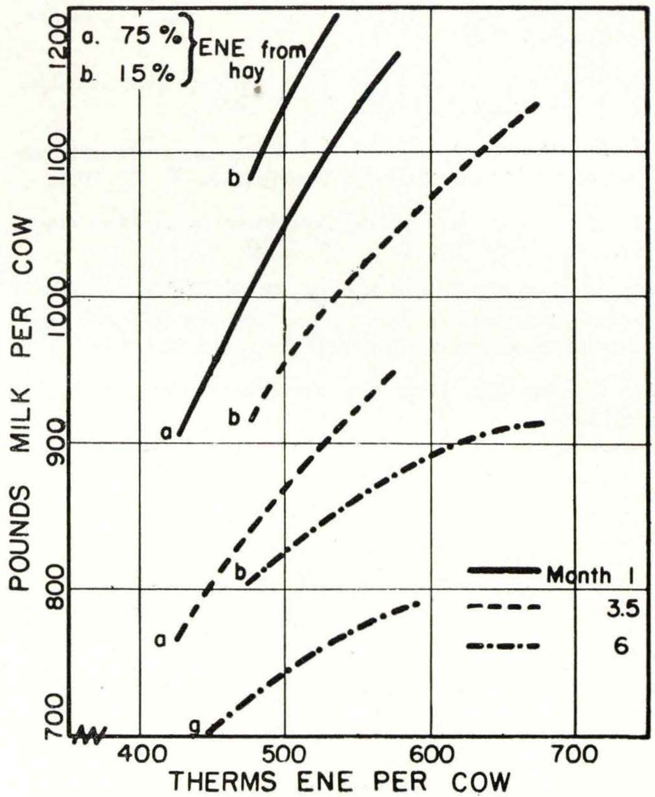

Fig. A-8. Inter-month comparisons of input-output relationships estimated from equation 19a. Ability set at mean.

LITERATURE CITED

1. Ashe, A. J. Response of milk production to increased grain feeding. Farm Econ. No. 174. Cornell Univ. 1950.

2. Beach, C. L. The facility of digestion of foods as a factor in feeding. Conn. Agr. Exp. Sta. Bul. 43. 1906.

3. Blaxter, K. L. Energy feeding standards for dairy cattle. Nutr. Abst. and Rev. 20:1-18. 1950.

4. Bloom, Solomon. Effects of various dietary hay-concentrate ratios on nutrient utilization and production responses of dairy cows. Unpublished Ph.D. thesis. Iowa State College Library, Ames, Iowa. 1955.

5. Brody, S., Ragsdale, A. C. and Turner, C. W. The rate of decline in milk secretion with the advance of the period of lactation. Jour. Gen. Physiol. 5:441-444. 1923.

6. Espe, D. and Smith, V. R. Secretion of milk. The Iowa State College Press, Ames, Iowa. Fourth ed. 1952.

7. Gaines, W. C. Persistency of lactation in dairy cows. Ill. Agr. Exp. Sta. Bul. 288. 1927.

8. Groves, R. R., et al. Feeding dairy cows on alfalfa hay alone. USDA Tech. Bul. 610. 1938.

9. Hamilton, T. S. The effect of added glucose upon the digestibility of protein and of fiber in rations for sheep. Jour. Nutr. 23:101-110. 1942.

10. Heady, Earl O., et al. Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Exp. Sta. Res. Bul. 424. 1955.

11. Huffman, C. F. and Duncan, C. W. The nutritive value of alfalfa hay. Jour. Dairy Sci. 32:465-471. 1949.

12. Jensen, E. et al. Input-output relationships in milk production. USDA Tech. Bul. 815. 1942.

13. Loosli, J. K., Lucas, H. L. and Maynard, L. A. The effect of roughage intake upon the fat content of milk. Jour. Dairy Sci. 28:147-153. 1945.

14. Martin, T. G., Stoddard, C. E. and Allen, R. S. Effects of varied rates of hay feeding on body weight and production of lactating dairy cows. Jour. Dairy Sci. 37:1233-1240. 1954.

15. Missouri Agr. Exp. Sta. Bul. 444. Agricultural investigations. 1942.

16. Morrison, F. G. Feeds and feeding. 21st ed. The Morrison Pub. Co., Ithaca, N. Y. 1948.

17. National Research Council. Recommended nutrient allowances for dairy cattle. Washington, D. C. 1950.

18. Headley, F. B. Feeding experiment with dairy cows. Nev. Agr. Exp. Sta. Bul. 119. 1930.

19. Monroe, C. F. and Krauss, W. E. Relationship between fat content of dairy grain mixtures and milk and butterfat production. Ohio Agr. Exp. Sta. Bul. 644. 1943.

20. Oregon Agr. Exp. Sta. Biennial Report, p. 41-43. 1924-26.

21. Reed, O. E., et al. The relation of feeding and age of calving to development of dairy heifers. Kan. Agr. Exp. Sta. Bul. 233. 1924.

22. Swift, R. W., Thacker, E. J., Black, A., Bratzler, J. W. and Jones, W. H. Digestibility of rations for ruminants as affected by proportion of nutrients. Jour. Anim. Sci. 6:432-444. 1947.

23. Wall, F. W. Alfalfa as the sole feed for dairy cows. Jour. Dairy Sci. 1:447-461. 1918.

24. Yates, F., Boyd, D. and Pettit, G. Influence of changes in level of feeding on milk production. Jour. Agr. Sci. 32:428-456. 1942.