REPORT

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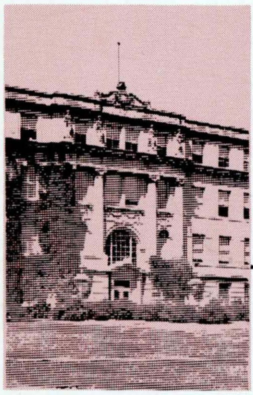
# DISCONTINUITIES IN A RELATIVISTIC GAS

by G. A. Nariboli



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# DISCONTINUITIES IN A RELATIVISTIC GAS

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Price: $2.00

April, 1967
ENGINEERING RESEARCH INSTITUTE
IOWA STATE UNIVERSITY AMES

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# DISCONTINUITIES IN A RELATIVISTIC GAS

by

G. A. Nariboli

1. Introduction: The study of Singular Surfaces in General Relativity has received much attention recently[1,2,3,4]. In any field theory wave propagation is an important phenomenon. Let us make precise the meaning of the term 'wave', which we employ herewith. We assume that there exists a surface ($\Sigma$), called as the singular surface, across which some at least of the field variables or their derivatives are discontinuous; we further take that the wave-front ($\Sigma$) is not stationary, but that it propagates. The strength of the discontinuity, defined in a suitable way, varies as the surface moves. A feature of a nonlinear field theory is that such a discontinuity may grow indefinitely.

Such a study in General Relativity is complicated by two facts: one has to distinguish between intrinsic discontinuities, which cannot be transformed away; the other point, related to the above one, is the question of admissible and non-admissible coordinates. Here the basic metric, which gives the gravitational potentials, may itself suffer a discontinuity. Study of these points can be found in the literature quoted and elsewhere too.

In order to bring out the main idea of the paper, we limit ourselves to Special Relativity. The basic aim of this paper is to present the Ray Theory. This result is important in the integration of the equation which governs the growth of the discontinuity. For the non-relativistic case[5], we have shown the power of this theory in the study of the growth equation. Non-linearity[6], non-homogeneity[7] and anisotropy[8] can all be discussed in a straight-forward manner. The purpose of the present discussion is to generalise these results to relativistic phenomena and illustrate their applicability by means of a simple model. So we limit our model to Special Relativity. When the necessary continuity requirements on the gravitational potentials are satisfied, the results are valid for General Relativity also (with the necessary modifications, when the model adopted for the material-energy tensor changes).

2. Basic Equations: We study here the perfect relativistic gas on the assumption of isentropic changes. Then the vanishing of the divergence of this tensor provides all the equations of our study. We adopt the simplest of the models for this tensor[4,9], which goes over to the known model for the non-relativistic gas, in absence of gravity.

We adopt the metric of special relativity, which gives the separation as

$$ds^2 = h_{AB}dx^A dx^B \qquad (A,B = 0,1,2,3) \tag{2.1}$$

with non-zero components of $(h_{AB})$ given by

$$h_{00} = c^2,\ h_{11} = h_{22} = h_{33} = -1,$$

$$h^{00} = \frac{1}{c^2},\ h^{11} = h^{22} = h^{33} = -1\ . \tag{2.1,a}$$

Here $x^0 = t$, is the time, $(x^i)$ $(i,j = 1,2,3)$ are the spatial co-ordinates and $(c)$ is the velocity of light in vacuo. Capital Latin indices run over $(0,1,2,3)$; small case Latin letters run over $(1,2,3)$ as indices. The latter denote Cartesian tensors and as such, there is no difference between superfixes and suffixes; still we retain them as such, since we identify them with former ones. The capital Latin indices are tensor indices that have to be distinguished as covariant and contravariant. However, since $(h_{AB})$ are constants, covariant differentiation is identical with partial differentiation. With requirements on their continuity satisfied, the final results remain valid, even when they are functions of $(x^A)$.

We describe $(\Sigma)$, the singular surface by means of the parametric form

$$x = x\,(u^{\alpha}),\ (\alpha,\beta = 0,1,2), \tag{2.2}$$

with separation on it given as

$$d\sigma^2 = a_{\alpha\beta}du^{\alpha}du^{\beta}, \tag{2.3}$$

where

$$a_{\alpha\beta} = h_{AB}x^A_{\alpha}x^B_{\beta}\ ,\ x^A_{\alpha} = \frac{\partial x^A}{\partial u^{\alpha}}\ . \tag{2.3,a}$$

We note that the Greek indices are strict tensor indices. We further note a few geometric results[4].

$$x^A_{\alpha,\beta} = b_{\alpha\beta}N^A,\ N^A{}_{,\alpha} = b_{\alpha}{}^{\beta}x^A{}_{\beta}, \tag{2.4}$$

$$2\Omega = a^{\alpha\beta}b_{\alpha\beta} = N^A{}_{,A}\ .$$

The first of the above give the formulea of Gauss; the second set gives those of Weingarten[10]; the last one is an additional result; not given in (4), which we find useful[11]. We note that the sign convention is so chosen that an expanding spherical wave-front will have positive value of $(\Omega)$. Also it is necessary to remember the space we are discussing is flat. This is necessary for (2.4). These will be used only in the study of the growth equation, where necessary modifications are needed, when the gravitational effects are considered.

In the above formulea, $(N_A)$ is the unit space-like normal to $(\Sigma)$. So if

$$f(x^A) = 0 \tag{2.5}$$

is the equation for $\Sigma$, then we set

$$p_A = f_{,A}, \; p_A p^A = -p^2, \; p_A = pN_A. \tag{2.6}$$

We then have

$$x^A_{\alpha} N_A = 0, \; N^A N_A = -1. \tag{2.7}$$

Let (G) be the velocity with which $\Sigma(t)$, considered as a moving surface in the $(x^i)$ space, moves normal to itself. Let $(n^i)$ be the unit normal to it in $(x^i)$ space. Then we have[4]

$$N^A = \left(\frac{\delta G}{c^2}, \delta n_i\right), \; N_A = (\delta G, -\delta n_i), \; \delta^{-2} = 1 - \frac{G^2}{c^2}. \tag{2.8}$$

These hold, of course, for the metric used.

Let $(W^A)$ be the unit time-like vector of velocity. Then for the metric adopted, we have, with $(u^i)$ as the velocity in $(x^i)$ space,

$$W^A = \left(\frac{\gamma}{c}, \frac{\gamma u^i}{c}\right), \; W_A = \left(\gamma c, -\frac{\gamma u_i}{c}\right), \; W^A W_A = 1, \; \gamma^{-2} = 1 - \frac{u^2}{c^2}, \; u^2 = u_i u^i. \tag{2.9}$$

We take the material-energy tensor as[4]

$$T_{AB} = \rho \, W_A W_B - \frac{1}{c^2} \, p h_{AB}, \tag{2.10}$$

with $p = p(\rho)$, $a^2 = \frac{dp}{d\rho}$.

Here $(\rho)$ is the material density and $(p)$ is the pressure, assumed to be a function of density only. Different models[9] for the material energy tensor are obtained by different interpretations of $(\rho)$.

We finally note the jump conditions, that if $(Z)$ is any field variable and, with square brackets denoting jumps,

$$[Z] = 0, \; [Z,_A]N^A = -\lambda, \; [Z,_{AB}]N^A N^B = \bar{\lambda} \tag{2.11}$$

then we have

$$[Z,_A] = \lambda N_A, \; [Z,_{AB}] = \bar{\lambda} N_A N_B + a^{\alpha\beta}\lambda,_\alpha (N_A x_{A,\beta} + N_B x_{A,\beta}) - \lambda b^{\alpha\beta} x_{A,\alpha} x_{A,\beta}. \tag{2.12}$$

Here we have used the notation

$$x_{A,\alpha} = h_{AB} x^B{}_\alpha. \tag{2.12}$$

3. <u>Velocity of Propagation and Ray Theory</u>: Equating the divergence of the material energy tensor (2.10) to zero, we get

$$\rho,_B W_A W^B + \rho W_{A,B} W^B + \rho W_A W^B,_B - \frac{a^2}{c^2} \rho,_A = 0 \tag{3.1}$$

We also differentiate the condition which states that $(W_A)$ is a unit vector. This gives

$$W^A W_{A,B} = 0. \tag{3.1,a}$$

Let the discontinuities in density and velocity be denoted by

$$[\rho,_A]N^A = -\zeta, \; [W^A,_B]N^B = -\lambda^A. \tag{3.2}$$

The junction conditions (2.11) lead to

$$\zeta(LW_A - \frac{a^2}{c^2} N_A) + \rho L \; \lambda_A + \rho W_A \; \lambda^B N_B = 0, \tag{3.3}$$

$$W^A \; \lambda_A = 0, \tag{3.3,a}$$

where $L = W_A N^A$.

We note that $(\rho)$, $(W_A)$, when they appear after taking jumps, denote values ahead of the front.

Multiply (3.3) by $(N^A)$ and $(W^A)$, sum over the repeated index and use (3.3,a), to obtain

$$\zeta(L^2 + \frac{a^2}{c^2}) + 2\rho L \; \lambda^B N_B = 0, \tag{3.4,a}$$

$$\zeta L(1 - \frac{a^2}{c^2}) + \rho \; \lambda^B N_B = 0. \tag{3.4,b}$$

Eliminating $(\rho \; \lambda^A N_A)$ between these, we get, for $\zeta \neq 0$,

$$L^2 + \frac{a^2}{c^2} = 2L^2(1 - \frac{a^2}{c^2}), \tag{3.5}$$

$$\frac{L^2}{1 + L^2} = \frac{L^2}{\ell^2} = \frac{a^2/c^2}{1 - (a^2/c^2)} \; . \tag{3.5,a}$$

Using the value of $(\lambda^B N_B)$ from (3.4,b) (or (3.4,a)) in (3.3), we rewrite it as

$$\frac{a^2}{c^2} \zeta (LW_A - N_A) + \rho L \lambda_A = 0. \tag{3.6}$$

This shows that $(\lambda_A)$ is parallel to the vector $(M_A)$ defined by

$$\ell M_A = N_A - LW_A, \quad M_A M^A = -1, \quad M_A N^A = -\ell, \quad M_A W^A = 0. \tag{3.7}$$

We can then set the discontinuities as

$$\lambda_A = M_A \Psi, \quad \zeta = \frac{\rho(1 + 2L^2)}{L\ell} \Psi, \tag{3.8}$$

where $\Psi$ may be called the strength of the discontinuity.

In non-relativistic gaseous medium, if the wavefront is moving into a medium at rest, we know that the discontinuity vector for velocity is parallel to the normal. This is no more so here.

In relativity, the velocity of propagation (v), or the frequency (n = c/v), is defined by[12]

$$\frac{v^2}{c^2} = \frac{L^2}{\ell^2}. \tag{3.9}$$

Here we obtain by the use of (3.5,a)

$$\frac{v}{c} = \frac{L}{\ell} = \frac{(a/c)}{\sqrt{1 - (a/c)^2}}. \tag{3.10}$$

Using the limit that (c) is much larger than all the velocities, we obtain the limit as

$$G - u_n = a, \quad u_n = u_i n_i. \tag{3.11}$$

Noting that, for the metric of special relativity,

$$L = \frac{\gamma\delta}{c} (G - u_n), \tag{3.12}$$

we can see that the result coincides with that obtained by Thomas[4].

To formulate the ray theory[12], we rewrite (3.9) as

$$2H \equiv h^{AB} p_A p_B + (n^2 - 1)\,(p_A W^A)^2 = 0, \tag{3.13}$$

where $(n = c/v)$ is given from the medium equation (3.10).

The equation (3.13) is a first order non-linear partial differential equation for the wave-front (f). Its solution is equivalent to solving the following system of ordinary differential equations, given by

$$\frac{dx^A}{dw} = \frac{\partial H}{\partial p_A}\,, \quad \frac{dp_A}{dw} = -\frac{\partial H}{\partial x^A}\,. \tag{3.14}$$

Here $(x^A)$ and $(p^A)$ are to be regarded as <u>independent</u> variables. The medium is said to be dispersive if the frequency (n) depends on (L). In our discussion, (n) clearly does not depend on (L). The variable (w) that occurs in (3.14) may be taken as any curve parameter. The first system of (3.14) defines curves, known as rays and the second describes the variation of the gradient-vector $(p_A)$ along the rays. We normalise to unity the ray vector, by changing the curve parameter as $p\ell dw = L ds$. The system (3.14) can then be written as

$$V^A = \frac{dx^A}{ds} = \frac{1}{\ell}\,(LN^A + W^A), \tag{3.15,a}$$

$$\frac{dN_A}{ds} = \frac{L}{\ell}\Big\{(1 + 2L^2)\big((\log a)_{,A} + N^B (\log a)_{,B} N_A\big) - (N_c W^c{}_{,A} + N_A N^B N_c W^c{}_{,B})\Big\}\,. \tag{3.15,b}$$

We note that $(V^A)$ is a time-like vector; it is called the Ray-velocity. The parameter (s), now denotes the arc-length along the curve. Again the rays are no more parallel to the normal trajectories, even when the medium ahead is at rest. This brings out the basic role played by the rays in the study of wave-propagation. Further the rays have no normal component; the ray velocity has only the tangential component. The variation of any quantity (F) along the rays, is given by

$$\frac{dF}{ds} = V^A F_{,A} = V^\alpha F_{,\alpha} = \frac{1}{\ell} W^\alpha F_{,\alpha}. \qquad (3.16)$$

These results are valid under wider conditions than those of special relativity. With relevant modifications, their extension to the study of gravitational waves is obvious. To come to the particular study we are making, we take the state ahead as a constant state. Then we obtain

$$\frac{dN_A}{ds} = 0. \qquad (3.17)$$

Thus the normal vector is unchanged in direction as we move along the rays. Since it is a unit vector, it remains a constant vector as one proceeds along the rays.

Let $(\Delta S)$ be the 3-volume of the <u>normal</u> cross-section of a tube, ds an element of separation along world lines forming the tube and $(V_A)$ a unit vector tangent to these world lines. Then using the theorem of divergence, one can prove[13]

$$\frac{d(\log E)}{ds} = V^A{}_{,A}. \qquad (3.18)$$

where (E) is the ratio of $(\Delta S)$ to its initial value.

This result helps us to express a number of terms, linear in the strength of the discontinuity, as a ray-derivative. But for a constant state ahead, a related result suffices; it is[11]

$$2\Omega = N^{A}{}_{,A} = \delta n^{i}{}_{,i} = 2\underset{o}{\Omega}\delta. \tag{3.19}$$

Here $(\underset{o}{\Omega})$ is the mean curvature of the wavefront, when it is regarded as a two-dimensional surface moving in the $(x^{i})$ space. This result is physically obvious; the multiplier $(\delta)$ accounts for the Lorentz-contraction.

We close this section after a final remark. From (3.5), we see that for propagation, the following inequality must hold

$$c^{2} > 2a^{2}. \tag{3.20}$$

The numerical factor (2) changes with the form of the material-energy tensor considered.

4. Growth of the Wave: We now proceed to study the variation of the strength $(\Psi)$ of the discontinuity, as the wave-front moves. In the present section, we limit ourselves to the case when the state ahead is constant. Let the discontinuities in the second derivatives of density and velocity be denoted by

$$[\rho_{,AB}]N^{A}N^{B} = \bar{\zeta}, \quad [W^{A}{}_{,BC}]N^{B}N^{C} = \bar{\lambda}^{A}. \qquad ((4.1)$$

We differentiate (3.1) and (3.1,a) with respect to $(x^{C})$, multiply by $(N^{C})$ and take the jumps. We obtain three types of terms: the first type consists of the barred quantities only; the second group, denoted by $(-P_{A})$, consists of terms where the jumps are differentiated; the last set, denoted by $(Q_{A})$, is made up of products of jumps and is quadratic in jumps. Without going into details, we obtain

$$\bar{\zeta}(LW_A - \frac{a^2}{c^2} N_A) + \rho L\bar{\lambda}_A + \rho W_A \bar{\lambda}^B N_B = Q_A - P_A \tag{4.2,a}$$

$$\bar{\lambda}_A W^A = \Psi^2. \tag{4.2,b}$$

Note the comparison with (3.3); this comparison is quite general[5]. We multiply (4.2,a) by $(N^A)$ and $(W^A)$ and use (4.2,b), as needed; we get

$$\bar{\zeta}L(1 - \frac{a^2}{c^2}) + \rho\bar{\lambda}^B N_B = (Q_A - P_A)W^A - \rho L\Psi^2 \tag{4.3,a}$$

$$\zeta(L^2 + \frac{a^2}{c^2}) + 2\rho L\bar{\lambda}^B N_B = (Q_A - P_A)N^A. \tag{4.3,b}$$

Eliminating $(\bar{\lambda}^B N_B)$ between these, we obtain

$$Q - P + 2\rho L^2\Psi^2 = \bar{\zeta}\left\{\ell^2 \frac{a^2}{c^2} - L^2(1 - \frac{a^2}{c^2})\right\}, \tag{4.4}$$

with

$$Q = Q_A N^A - 2LQ_A W^A, \quad P = P_A N^A - 2LP_A W^A.$$

In view of the value of $(L)$, the multiplier of $(\bar{\zeta})$ vanishes. Thus the growth equation is

$$Q - P + 2\rho L^2\Psi^2 = 0. \tag{4.5}$$

Evaluating the expressions, for a constant state ahead, we obtain

$$\frac{d\Psi}{ds} + \frac{\Omega\ell}{\ell}\Psi - \Psi^2\left\{1 + \frac{\rho aa'}{(cL)^2}\,\frac{(1 + 2L^2)^3}{\ell^2}\right\} = 0. \tag{4.6}$$

This can be converted into one with respect to ordinary time by noting that

$$dt = V^0 ds = \frac{\ell}{c}\, ds. \tag{4.7}$$

Comparing the discontinuity in the density for the nonrelativistic case, we also need set

$$\Psi = -\frac{\phi}{c}\,. \tag{4.8}$$

Then this reduces to

$$\frac{d\phi}{dt} + \frac{\Omega(cL)}{\ell^2}\,\phi + \frac{\phi^2}{\ell^2}\left\{1 + \frac{\rho a a'}{(cL)^2}\;\frac{(1+2L^2)^3}{\ell^2}\right\} = 0 \tag{4.9}$$

Discussion of the integration is now straight-forward, if we use the known results and the relation (3.19). Since, such discussions are found enough in the literature, we omit the details.

5. Conclusion: Well-known results of the Ray-theory are given in an organized form; it is hoped that these provide a powerful tool in the study of waves in general relativity. The technique is illustrated by means of wave-propagation in a relativistic gas, with flat space and constant state ahead.

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