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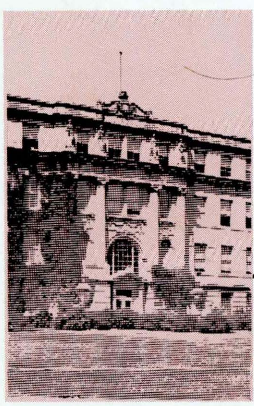
by G. A. Nariboli



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REPORT 57

SOME ASPECTS OF WAVE-PROPAGATION

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SOME ASPECTS OF WAVE-PROPAGATION

by

G. A. Nariboli

1. Introduction:

An important feature of the field theory is the finiteness of the velocity with which a disturbance is propagated through a medium. Continuum mechanics is a field theory, or a phenomenological theory. Any mathematical model, constructed to describe a continuum is, in general, non-linear, non-homogeneous and anisotropic. A perturbation about a given state gives a linear theory; absence of preferred directions and independence of position makes the model isotropic and homogeneous.

The term "wave" encompasses a large class of phenomena; it may be associated with dispersion and dissipation. This study is based on the theory of singular surfaces. It is postulated that there exist discontinuities in certain quantities across such surfaces that move. For a linear and homogeneous problem, transform technique provides a wealth of information. When anisotropy is present, inversions become complicated, though some information can still be extracted. For the non-homogeneous and nonlinear problems, such techniques do not work. The theory presented here does not provide the solution throughout the field; it is restricted to the singular surface. A full field picture can only be obtained by piecing together smooth solutions with such discontinuous ones. However, one can discuss the most general non-linear, nonhomogeneous, and anisotropic problem.

The choice of the subject-matter is, of course, governed by the author's current interests. References are not intended to be exhaustive; only the immediately relevant ones are noted. Further references can be obtained from those listed. Also the quoted reference is not necessarily the original source.

Professor B. R. Seth, a pioneer in the field of non-linear mechanics, has been an inspiration to a number of us. So we believe such a study will be highly appropriate in a volume dedicated to him.

2. Waves, Compatibility Conditions, and Rays.

2.1 Waves: Any mathematical model of a continuum is given by a system of partial differential equations. (They may be integro-differential equations; the general ideas presented here will be useful then too.) In continuum mechanics, the laws governing the conservation of mass, momentum, and energy form a common starting point. Each medium is then characterized by its constitutive laws. Conservation laws must together lead to a determinate system for the field variables entering therein.

Partial differential equations describe the variation of the field variables in space and time. Consider a moving surface $\Sigma(t)$ traversing the medium and further note that (some at least of) the field variables or their derivatives are discontinuous across this surface. The surface is called the singular surface or a wave-front. It is only such a wave that is being studied here. We designate such singular surface as "strong" or "shocks" when its speed of propagation is not determined from the given system itself; an additional hypothesis is needed to obtain it. When the speed of propagation is determined in terms of the field-variables ahead of the front, by the system itself, then we call it a "weak" one. However, the existence of such a discontinuity is an assumption common to both types. A fully dissipative system is excluded from our consideration. At least some of the characteristics of the partial differential equation must be real. Our aim is to obtain the normal speed of propagation of such a wave-front and to study the variation of the strength of the wave (defined in terms of the existing discontinuities) as the wave-front moves.

2.2 <u>Compatibility Conditions</u>: Thomas[1] recently was the first to give a systematic derivation of these relations. (See Truesdell[2] for a historical development.)

Assume that the discontinuities across a wave-front of a field variable and its normal derivatives are known. The geometric conditions then express the discontinuities in the various spatial derivatives in terms of these, their tangential derivatives and the geometry of the surface. The kinematical ones, involving spatial and time derivatives, also are given in terms of these, the normal speed of propagation and in terms of a type of convected derivative of these.

To be specific, we introduce a fixed orthogonal Cartesian system of coordinates (x_i). Let the surface $\Sigma(t)$ be described by a Gaussian system $(u^\alpha, \alpha = 1, 2)$, which are, in general, curvilinear. We represent $\Sigma(t)$ as

$$x_i = x_i(u^\alpha, t) \quad . \tag{2.2-1}$$

We use the Cartesian tensor notation for the spatial system (x_i); Latin indices (i, j, k ---) range over (1, 2, 3); they denote tensors; a comma preceding a Latin index denotes a partial differentiation. Greek indices $(\alpha, \beta, \ldots)$, ranging over (1, 2), denote surface tensors.

They are to be distinguished as covariant and contravariant; a comma preceding a Greek index thus denotes a co-variant derivative. Covariant derivatives of scalars, of course, reduce to partial derivatives. Note that spatial tensors are scalars with respect to the surface system.

Let (n_i) be the unit normal to $\Sigma(t)$, pointing into a region which we call the 'front' and which we denote as region (1). The other side of $\Sigma(t)$, called the 'rear', is denoted as region (2). Let (G) be the speed of propagation of $\Sigma(t)$ in the direction of (n_i). Let $(g_{\alpha\beta} = x_{i,\alpha} x_{i,\beta})$ and $(b_{\alpha\beta})$ be the first and second fundamental forms of $\Sigma(t)$; $\Omega = (b^{\alpha}_{\alpha}/2)$ and $b = \det(b_{\alpha}^{\beta})$ are then the mean and Gaussian curvatures. The formulas of Weingarten and of Mainardi-Codazzi are

$$n_{i,\alpha} = -b_{\alpha}^{\beta} x_{i,\beta} , \qquad (2.2\text{-}2)$$

$$b_{\alpha\beta,r} = b_{\alpha r,\beta} . \qquad (2.2\text{-}3)$$

We also introduce a convected time derivative in the direction of the normal as $(\delta P/\delta t)$, where (P) is any field variable defined on $\Sigma(t)$. This delta time derivative describes the rate of variation of (P) as observed by one who rides the wave-front and moves with it in the direction of the normal (n_i). Note the other formula derived by Thomas[1] giving this derivative of (n_i) as

$$\frac{\delta n_i}{\delta t} = -g^{\alpha\beta} G_{,\alpha} x_{i,\beta} . \qquad (2.2\text{-}4)$$

The discontinuity in a field variable (P) is denoted by a square bracket as

$$[P] = P_2 - P_1 . \qquad (2.2\text{-}5)$$

Thus the discontinuities in the various order normal derivatives are denoted as

$$[P] = A, [P_{,i}]n_i = B, [P_{,ij}]n_in_j = C, [P_{,ijk}] n_in_jn_k = D . \tag{2.2-6}$$

The compatibility conditions can now be stated as

$$[P_{,i}] = Bn_i + g^{\alpha\beta}A_{,\alpha}x_{i,\beta} , \tag{2.2-7,a}$$

$$[P_{,ij}] = Cn_in_j + g^{\alpha\beta}B_{,\alpha}(n_jx_{i,\beta} + n_ix_{j,\beta}) - Bb^{\alpha\beta}x_{i,\alpha}x_{j,\beta} , \quad \text{if } A = 0 , \tag{2.2-7,b}$$

$$[P_{,ijk}] = Dn_in_jn_k + g^{\alpha\beta}C_{,\alpha}(n_in_jx_{k,\beta} + n_jn_kx_{i,\beta} + n_kn_ix_{j,\beta}) - Cb^{\alpha\beta}(n_ix_{j,\alpha}x_{k,\beta} + n_jx_{k,\alpha}x_{i,\beta} + n_kx_{i,\alpha}x_{j,\beta}) , \quad \text{if } A = B = 0 , \tag{2.2-7,c}$$

$$\left[\frac{\partial P}{\partial t}\right] = -GB + \frac{\delta A}{\delta t} , \tag{2.2-8,a}$$

$$\left[\frac{\partial^2 P}{\partial x_i \partial t}\right] = (-GC + \frac{\delta B}{\delta t})n_i - g^{\alpha\beta}(GB)_{,\alpha}x_{i,\beta} , \quad \text{if } A = 0 , \tag{2.2-8,b}$$

$$\left[\frac{\partial^2 P}{\partial t^2}\right] = G^2C - 2G\frac{\delta G}{\delta t} - B\frac{\delta G}{\delta t} , \quad \text{if } A = 0 , \tag{2.2-8,c}$$

$$\left[\frac{\partial^3 P}{\partial x_i \partial x_i^2}\right] = G^2(Dn_i + g^{\alpha\beta}C_{,\alpha}x_{i,\beta}) - 2G\frac{\delta(Cni)}{\delta t} , \quad \text{if } A = B = 0 , \tag{2.2-8,d}$$

$$\left[\frac{\partial^3 P}{\partial x_i \partial x_j \partial t}\right] = -G\left\{Dn_in_j + g^{\alpha\beta}C_{,\alpha}(n_ix_{j,\beta} + n_jx_{i,\beta}) - Cb^{\alpha\beta}x_{i,\alpha}x_{j,\beta}\right\} + \frac{\delta(Cninj)}{\delta t} , \quad \text{if } A = B = 0 . \tag{2.2-8,e}$$

Relations are noted only as they are used in the present work. More general ones can be written along the same lines. Relation (2.2-7,a) is the resolution into components of the vector in directions normal and tangential. The basic assumption made here is that of Hadamard's Lemma, stated as: the tangential derivative of the discontinuity in a quantity is equal to the jump in the tangential derivatives of the quantity. The other basic result (2.2-8,a), follows from the definition of the delta-time derivative. (See also (2.3-2) below.) The remaining ones follow from a repeated application of these on the assumption that the order of differentiation is irrelevant.

These relations, applied to the partial differential equations describing a continuum, lead first to the equation that determines the normal speed of propagation (G) and then to the one that governs the growth of discontinuity.

2.3 Ray-Theory: The ideas presented here originate in the Theory of Optics. The theory, as applied to partial differential equations, is elaborated in Courant and Hilberts' book[3]. We set them forth now as needed for our purpose and in the language we are using.

Let $A(x_i, t)$ and $B(x_i + \Delta x_i, t + \Delta t)$ be two consecutive points, each lying on the successive positions of the wave-front $\Sigma(t)$ at times (t) and $(t + \Delta t)$; let (V_i) be the velocity of the wave-front in the direction of (Δx_i), so that $\Delta x_i = V_i \Delta t$. If $\Delta P = P(B) - P(A)$, we can write the convected derivative of (P) with respect to (V_i) as

$$\lim \frac{\Delta P}{\Delta t} = \frac{\partial P}{\partial t} + P_{,i} V_i \quad . \qquad (2.3\text{-}1)$$

Vector (V_i) is later identified as the Ray-velocity (explained subsequently, see (2.3-11) below) and the above convected derivative, denoted as (dP/dt), is called the ray-derivative. It is the derivative following the wave-front with velocity (V_i) in the direction given by (V_i) and it describes the time-rate of variation of (P) as apparent to an observer moving with the wavefront in the ray-direction with the ray-velocity.

In particular, if we move with the wave-front along a normal trajectory, we can set $V_i = Gn_i$; then the time-derivative, now called the convected normal derivative, is denoted by $(\delta P/\delta t)$. We thus have

$$\frac{\delta P}{\delta t} = \frac{\partial P}{\partial t} + P_{,i}Gn_i \; . \tag{2.3-2}$$

From their definitions, we can relate the two convected derivatives as

$$\frac{dP}{dt} = \frac{\delta P}{\delta t} + (V_i - Gn_i)P_{,i} \; . \tag{2.3-3}$$

Consider now the resolution into components of (V_i) as

$$V_i = (V_n)n_i + V^{\alpha}x_{i,\alpha} \, , \tag{2.3-4,a}$$

where

$$V_n = V_j n_j \, , \; V_{\alpha} = V_i x_{i,\alpha} \, , \; V^{\alpha} = g^{\alpha\beta}V_{\beta} \; . \tag{2.3-4,b}$$

We note that the suffix (n) is <u>not</u> a <u>tensor</u> <u>index</u>; it is reserved throughout to denote the normal component. Since clearly we have

$$V_n = G, \tag{2.3-5}$$

we can re-write (2.3-3) as

$$\frac{dP}{dt} = \frac{\delta P}{\delta t} + V^{\alpha} P_{,\alpha} \ . \qquad (2.3\text{-}6)$$

This relation plays a vital role in enabling us to integrate the equation of growth of a discontinuity. Compatibility conditions lead to a growth equation, which involves the normal convected derivative. This differential equation is a partial differential equation. The above relation reduces it to an ordinary one, as proved by Courant and Hilbert[3].

We now recapitulate a few ideas from the ray-theory. Let the wave-front be represented as

$$f(x_i, t) = 0 \ . \qquad (2.3\text{-}7)$$

Since it remains a wave-front at all times, its delta time derivative must vanish. So we can get

$$\frac{\partial f}{\partial t} + f_{,i} G n_i = 0 \ . \qquad (2.3\text{-}8)$$

If $(\partial f/\partial t) \neq 0$, which is true for a propagating wave-front, we can solve (2.3-7) for (t) and rewrite it as

$$f \equiv W(x_i) - t = 0 \ . \qquad (2.3\text{-}9)$$

The condition (2.3-8) now reads as

$$Gp - 1 = 0 \ , \qquad (2.3\text{-}10)$$

where

$$p_i = f_{,i} = W_{,i} = p n_i \, , \; p^2 = p_j p_j \ .$$

The normal speed of propagation (G) is, in general, a function of (x_i) and (n_i) (and hence of p_i). So (2.3-10) is a first order partial differential equation for the wave-front. Its solution is obtained by solving the system of ordinary differential equations, given by

$$V_i = \frac{dx_i}{dt}$$

$$= \frac{\partial Gp}{\partial p_i}$$

$$= Gn_i + (\delta_{ij} - n_i n_j) \frac{\partial G}{\partial nj}, \qquad (2.3\text{-}11)$$

$$U_i = \frac{dp_i}{dt}$$

$$= \frac{\partial (Gp)}{\partial x_i}, \qquad (2.3\text{-}12)$$

or

$$W_i = \frac{dn_i}{dt}$$

$$= (n_i n_j - \delta_{ij}) \frac{\partial G}{\partial xj} . \qquad (2.3\text{-}12,a)$$

In the above system, (x_i) and (n_i) are to be regarded as independent variables.

If (x_i, p_i) are the solutions of this system, the trajectories described by (x_i) are called rays. The variable (t) in the above system may be any parameter along the curve; we fix it as time. Then the equations (2.3-11) define the ray-velocity vector (V_i). The system (2.3-12,a) describes the time-rate of variation of the normal vector (n_i) as we move along the rays.

The crucial point is that one can proceed a long way in obtaining the integral of the differential equation governing the growth of the discontinuity, without actually solving this system of, in general highly non-linear, first order ordinary differential systems. The time-rates of variations of various geometric quantities needed in the work, are noted below.

First, consider the case when (G) is independent of both (n_i) and (x_i); we call the propagation here both isotropic and homogeneous. The rays here coincide with the normal trajectories; also the time-rate of variation of (n_i) vanishes as we move along the normal trajectories. Thus the normal vector is unchanged in direction. The successive positions of $\Sigma(t)$ form a system of parallel surfaces. It is now straightforward to obtain all the geometric properties of $\Sigma(t)$. We need, in particular, the mean curvature. This is given by

$$\Omega(t) = \frac{\Omega_o - b_o n}{1 - 2\Omega_o n + b_o n^2}, \; n = Gt, \qquad (2.3\text{-}13)$$

with the suffix zero denoting the initial value.

Next we consider homogeneous but anisotropic propagation. The normal velocity (G) now depends only on (n_i). Since (G) is independent of (x_i) again, the normal is unchanged in direction, but <u>now</u> this holds only as <u>we move along the rays</u>. This follows from (2.3-12,a), which reduces to

$$\frac{dn_i}{dt} = 0 \; . \qquad (2.3\text{-}14)$$

The rays are distinct from the normal trajectories. The ray-velocity (V_i) now has a tangential component, given by

$$V_\alpha = x_{i,\alpha} \frac{\partial G}{\partial n_i} \; . \qquad (2.3\text{-}15)$$

We always have, by definition,

$$\frac{\delta x_i}{\delta t} = G n_i \; . \tag{2.3-16}$$

Note that this delta-time-derivative is a normal derivative; so tangential differentiation, derivative with respect to (u^α), commutes with it (but not with the ray-derivative). On this assumption, we can obtain, starting from (2.3-16), the delta-time-derivatives of $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$. Using (2.3-6), we can then calculate the ray-derivatives of these. After some lengthy but straight-forward calculations, based on these and (2.2-3), we can prove[4]

$$\frac{d\,(\log b)}{dt} = G b_\alpha^{\ \alpha} - V^\alpha_{\ ,\alpha} \, , \tag{2.3-17}$$

where

$$b = \det\,(b_\alpha^{\ \beta}) = |b_\alpha^{\ \beta}| = \text{Gaussian curvature.}$$

Finally, consider the case of non-homogeneous and anisotropic propagation. The normal speed (G) now depends on both (x_i) and (n_i). We can obtain a few useful formulas again. A trivially obvious result is

$$\frac{dP}{dt} = P_{,i} V_i \, , \tag{2.3-18}$$

if (P) is independent of time.

Consider an element of area (Δa_i) of the wave-front. Let it be formed by two line-elements (dx_i) and (Δx_i), issuing from (x_i). Then we have

$$n_i \Delta a = \Delta a_i$$

$$= e_{ijk} dx_j \Delta x_k \,, \tag{2.3-19}$$

which is equivalent to

$$dx_j \Delta x_k - dx_k \Delta x_j = e_{ijk} \Delta a_i \,. \tag{2.3-19,a}$$

Let (V_i) be the velocity with which the element moves. We first have

$$\frac{d}{dt}(dx_i) = V_{i,j} dx_j \,. \tag{2.3-20}$$

From (2.3-19) and (2.3-20), we get

$$\frac{d}{dt}(\Delta a_i) = V_{j,j} da_i - V_{j,i} da_j \,, \tag{2.3-21}$$

or

$$\frac{d}{dt}(\log \Delta a) = V_{j,j} - V_{j,i} n_i n_j \,. \tag{2.3-21,a}$$

From (2.3-12) and (2.3-10), we can get

$$\frac{d \log G}{dt} = - \frac{d \log p}{dt}$$

$$= n_i \frac{\partial G}{\partial x_i} \,. \tag{2.3-22}$$

From these it follows that

$$\frac{d}{dt}(\log EG) = V_{j,j} \,, \tag{2.3-23}$$

where

$$E = \frac{\Delta a}{\Delta a_o} \,, \tag{2.3-23,a}$$

with suffix zero <u>always</u> denoting the initial value.

The above result is also obtained by the use of Gauss's theorem on divergence, as applied to a tubular region, bounded laterally by the rays and with faces as successive positions of (Σ) at times (t) and $(t + \Delta t)$[5]. We have noted the alternative derivation to bring out the intermediary results.

We conclude this section with a relation which establishes the connection between the different cases. Let $(\Delta\omega)$ be the spherical image of (Δa). The spherical image of a surface is a unit sphere centered at origin and spanned by the unit normal vector to the surface. Referring to principal directions of the wave front, it is now easy to establish

$$\Delta\omega = b \, \Delta a \, . \qquad (2.3\text{-}24)$$

3. <u>Wave Propagation and the Growth of the Discontinuities</u>.

3.1. <u>Isotropic</u>, <u>homogeneous propagation</u>: This is the simplest of the cases. The velocity of propagation (G) is an absolute constant. Again, the first application of these ideas is due to Thomas[6,7]. We illustrate the basic ideas with reference to isentropic motion of a gas. Let the wave-front move into a medium at rest and in a constant state. The result generalizes to three dimensions, the well-known result based on the theory of simple waves for one dimension, the development of a shock. Let the density (ρ) and the velocity (v_i) be continuous while their derivatives have discontinuities given by

$$[\rho_{,i}]n_i = \zeta , \; [v_{i,j}]n_j = \xi_i \; , \; [v_{i,jk}]n_j n_k = \overline{\xi}_i \; . \qquad (3.1\text{-}1)$$

From the equations of motion and of continuity and from those obtained by differentiating these, we obtain

$$G\zeta = \rho_o\xi_n \ , \ \rho_o G\xi_i = a_o^2\zeta n_i \ , \ (G^2\delta_{ij} - a_o^2 n_i n_j)\xi_j = 0, \quad (3.1\text{-}2,a,b,c)$$

$$(\rho_o G^2\delta_{ij} - a_o^2 n_i n_j)\bar{\bar{\xi}}_j = a_o^2 n_i(\frac{\delta\zeta}{\delta t} + 2\zeta\xi_n + \rho_o g^{\alpha\beta}\xi_{j,\alpha}x_{j,\beta})$$

$$+ G(\rho_o \frac{\delta\xi_i}{\delta t} + a_o^2 g^{\alpha\beta}\zeta_{,\alpha}x_{i,\beta} + \frac{c_o^2 a_o^2}{\rho_o}\zeta^2 n_i) \ , \quad (3.1\text{-}3)$$

with

$$\frac{dp}{d\rho} = a^2 \ , \ \frac{d^2p}{d\rho^2} = \frac{c^2a^2}{\rho} \ .$$

Two basic points are to be noted. First, the relations (3.1-2) give all the discontinuities in terms of any one; second, the coefficients of (ξ_i) in (3.1-2,c) are the <u>same</u> as those of ($\bar{\bar{\xi}}_i$) in (3.1-3). Their structure reveals that, on multiplying by (n_i), the left members of (3.1-3) vanish; this gives us the growth equation. The integration of this is now straight-forward. A detailed discussion of such an integration in the case of linear isotropic elasticity can be found in Thomas[7]. One can clearly see the development of caustics in the linear case and that of caustics and shocks in the non-linear problem.

If the wave-front is moving into a hypo-elastic medium[8], unstressed and at rest one gets two shear modes, completely analogous to those in linear elasticity, and one dilatational mode which can grow indefinitely. In gas-dynamics the growth to infinity of the strength of such a wave, is interpreted as the initiation of a shock. In solid mechanics, an analogous interpretation of the indefinite growth is as the initiation of fracture.

In non-linear, isotropic elasticity[9], we take the constitutive law as

$$t_{ij} = a\delta_{ij} + be_{ij} + ce_{ik}e_{kj} , \qquad (3.1\text{-}4)$$

with

$$2e_{ij} = u_{i,j} + u_{ji} - u_{k,i}u_{k,j} . \qquad (3.1\text{-}4,a)$$

Here (t_{ij}) and (e_{ij}) are the stress and strain tensors; the relation (3.1-4,a) defines the strain-tensor in terms of the displacement vector (u_i). The quantities (a), (b), (c) are, of course, arbitrary functions of the invariants (I) (II) and (III) of the strain-tensor (e_{ij}). We obtain three modes of propagation: two shear modes that do not grow and one dilatational one that can grow. Such a separation holds only for an initially unstrained medium; the condition for growth is $(k_o \gtrless o)$ for compressive and rarefraction waves, with (k_o) given by

$$k_o = \frac{3}{2} - \frac{\alpha_{11} + 2\beta_1 + 2\gamma_o}{2(\alpha_1 + \beta_o)} \qquad (3.1\text{-}5)$$

where

$$\alpha_1 = \frac{\partial a}{\partial I}\Big)_o , \alpha_{11} = \frac{\partial^2 a}{\partial I^2}\Big)_o , \beta_1 = \frac{\partial b}{\partial I}\Big)_o , \gamma_o = c\Big)_o ,$$

$$\beta_o = b\Big)_o . \qquad (3.1\text{-}5,a)$$

For a hyper-elastic material, with strain-energy given as

$$U = \frac{1}{2}(\lambda + 2\mu) I^2 - 2\mu II + \ell I^3 + m\, I\, II + n\, III , \qquad (3.1\text{-}6)$$

this critical parameter $(k_o$ reduces to

$$k_o = \frac{9}{2}\,\frac{3\ell}{\lambda + 2\mu} \,. \tag{3.1-7}$$

It is important to note that, if the wave front is moving into an unstrained medium (unstressed and at rest in hypo-elasticity), higher order terms in polynomial expansion of the strain-energy do not contribute anything more to (k_o). Bland[10] discusses the problem of one dimensional wave from the thermodynamical and mechanical view-points to obtain an identical condition for growth. It is of interest to note that our result includes his result.

In the study of ionisation fronts, thermal and electrical diffusivities play a more important role. The study of an inviscid but heat conducting gas with finite electrical conductivity reveals interesting results[11,12]. A weak discontinuity does exist, and it does grow indefinitely, but there is an exponential damping factor. A blast-wave of the gas-dynamical type exists, but vorticity and current depend on the electrical conductivity and the magnetic field.

A number of other interesting applications to water-waves and M.H.D. are studied by Kaul[13].

3.2. <u>Anisotropic Homogeneous Wave-Propagation</u>: This field has attracted much attention recently. Bazer[14] exploited a number of ideas from the theory of optics. Lighthill[15] studied the linear, homogeneous problem and drew interesting conclusions. Since it has found many applications[16, 17, 18, 19], we restate a few features of Lighthill's work. This enables us to compare our results.

Consider a general wave-equation of the form

$$F(\partial_o^{\,2r}\,,\ \partial_i^{\,2n_i})\ u = 0\ , \tag{3.2-1}$$

where (F) is a polynomial in its arguments and $\partial_o \equiv (\partial/\partial t)$, $\partial_i \equiv (\partial/\partial x_i)$.

Assuming time dependent source term of the type $\exp(i\omega t)$ on the right hand member of (3.2-1), the solution (u) can be represented in the form of a Fourier Integral as

$$u \sim \int \frac{N(\alpha,\omega)}{D(\alpha,\omega)} e^{i\alpha x} d\alpha \,, \qquad (3.2\text{-}2)$$

where

$$D \equiv F(\ (-\omega^2)^r\ ,\ (-\alpha_i{}^2)^{n_i}) \text{ and } \alpha x = \alpha_i x_i \ .$$

The major difficulty is in evaluating the integrals. For the two dimensional M.H.D. problem, Weitzner[20] obtains an explicit answer in terms of the roots of a quartic; but these are, by no means, easy to write down. Lighthill evaluates these by use of the method of steepest descent. More discussion is given by Ludwig[21] and Duff[22]. The crux of the result is that the asymptotic variation of (u) with respect to the distance from the source, is governed by the singularities of the surface D = 0 in the α space; this is also called as the reciprocal wavespeed locus.

This work prompted us to study the problem by the use of the theory of singular surfaces. A transform technique is applicable only to a system of linear partial differential equations with constant coefficients. The present method imposes no such restrictions. The differential equation governing the growth of the discontinuity along the normal trajectories involves tangential derivatives and hence is a partial differential equation[23]. So integration becomes possible only when assumptions are made to reduce it to an ordinary one, but then the problem reduces to that of isotropic propagation. However, as noted in Courant and Hilbert, it can be reduced to an ordinary one, if transformed to one that describes the variation of the strength along the rays.

To bring out how the exploitation of these ideas enables us to integrate the growth equation for the full non-linear problem, we illustrate by the use of a linear problem of anisotropic elasticity[24]. If $(C_{ijk\ell})$ is the tensor of elastic moduli and (ξ_i) the discontinuity in the second normal derivative of the displacement vector, we obtain first

$$a_{ik}\xi_k = \rho_o G^2 \xi_i , \qquad (3.2\text{-}3)$$

where

$$a_{ik} = C_{ijk\ell} n_j n_\ell .$$

Assuming for convenience (though not necessary) that the material is hyperelastic, we can take that (a_{ik}) as symmetric. Then (3.2-3) states that (ξ_i) is an eigenvector of (a_{ik}) with eigen-value $(\rho_o G^2)$. Thus the three eigen-values give the speeds of propagation, and using these we obtain the corresponding eigen-vectors. If (ℓ_i) is an eigen-vector, we can set $\xi_i = \ell_i \Psi$, where (Ψ) denotes the strength of the discontinuity. We designate an eigen-value together with its eigen-vector, as a mode of propagation. In the case of isotropy, they reduce to one dilatational and two shear waves. For the general anisotropic case, no such physical interpretation of the modes seems to be possible. Each mode is accompanied both by dilatation and rotation. When there is transverse isotropy, we obtain one purely rotational mode, the other two are again mixed. It is of interest to investigate, if any interpretation of these modes is possible in some physical terms, other than those of vorticity and divergence.

The equation governing the growth of the discontinuity looks as

$$(\rho_o G^2 \delta_{ij} - a_{ij})\bar{\xi}_j = A \frac{\delta \xi_i}{\delta t} + B^{\alpha} \xi_{i,\alpha} + C\xi_i \,, \tag{3.2-4}$$

where $(\bar{\xi}_i)$ is the discontinuity in one higher-order normal derivative of the displacement vector.

A comparative study of (3.2-3) and (3.2-4) reveals a feature common to all the problems. The coefficients of $(\bar{\xi}_i)$ in (3.2-4) are the same as those of (ξ_i) in (3.2-3). So multiplying (3.2-4) by the eigen-vector (ℓ_i) and summing over the repeated index, the left-hand side of (3.2-4) vanishes. This gives us the equation governing the growth of (Ψ) along the normal trajectories. It can now be proved that this reduces to an ordinary differential equation for (Ψ) along the rays. Varley[25] proves this for a more general case, but in the final discussion, assumptions are made to throw off all anisotropy. The quantities that appear in the final growth equation along the rays depend on (n_i) and other constants. But, by (2.3-12,a), this is also constant as we move along the rays. So a complete integration is possible, leading to

$$\Psi^2 b^{-1} = \text{constant}. \tag{3.2-5}$$

To relate our work to that of Lighthill's, we have to evaluate the Gaussian curvature (b). This appears to be lengthy for the general case. For the axially symmetric case, we obtain it in the form

$$b^{-1} = (\rho_1 \rho_2) = (\rho_{o1} + k_1 t)(\rho_{o2} + k_2 t) \; . \qquad (3.2\text{-}6)$$

The reciprocal wave-speed locus is now given by (2.3-10) in the p-space as $p^{-1} = G(n_i) = G(p_i/p)$. The curvatures of this locus turn out to be exactly proportional to (k_1), (k_2) in (3.2-6). So a complete correspondence is established between the results here and those of Lighthill.

Lighthill's result for the dispersive case (when (F) in (3.2-1) is a non-homogeneous polynomial) is not obtained by us.

As we stressed, our result can be extended to the non-linear problem too. In fact for the M.H.D. problem[26], the growth equation appears as

$$\frac{2}{\Psi}\frac{d\Psi}{dt} - \frac{1}{b}\frac{db}{dt} + D\Psi = 0 \; . \qquad (3.2\text{-}7)$$

If we linearise before taking jumps, we obtain the same equation with $D = 0$. In the non-linear problem it is not possible to discuss the asumptotic nature. The solution breaks down after a finite time, predicting the initiation of a shock. The strength of the wave is explicitly given by

$$\frac{\sqrt{b}}{\Psi} - \frac{\sqrt{b_o}}{\Psi_o} = \frac{1}{2} D \int_o^t \sqrt{b}\, dt \; . \qquad (3.2\text{-}7)$$

The integral is again easy to evaluate in the case of M.H.D., since there is axial symmetry.

Applications to non-linear Magneto-elasticity[27] and to initially stressed Hypo-elastic medium[28] are made on analogous lines. In the latter case the tensor (a_{ij}) of (3.2-3) is not symmetric, in general;

the eigenvalues and hence the speeds of propagation are not now necessarily real. However when they are real, (ξ_i) is parallel to the right-eigen-vector of (a_{ij}). We have to multiply (3.2-4) now by the left-eigen-vector to obtain the growth equation. Remaining analysis is similar though more lengthy.

Other interesting applications to water-waves in the presence of magnetic field are also made[29].

3.3. Non-homogeneous, Anisotropic Wave Propagation: If the wave-front is moving into a medium which is in a state of arbitrary steady motion, assumed to be known, then the normal speed of propagation (G) will depend on (n_i) and the field-variables ahead. By assumption, the latter are known functions of (x_i). Thus (G) depends on (x_i) and (n_i). This is the most general case of the three and includes the earlier ones as particular ones. But the result we are able to obtain now is not as explicit as in the earlier cases. We note the two applications we have made.

Consider a singular surface moving into a gaseous medium which is in a state of arbitrary steady motion[30]. To postpone the study of possible interactions, we take this flow ahead as smooth. We then obtain

$$\xi_i = n_i \Psi \ , \ G = a + u_j n_j \ , \ V_i = u_i + a n_i \ , \qquad (3.3\text{-}1,a)$$

$$\frac{\delta \Psi}{\delta t} + A^{\alpha} \Psi_{,\alpha} + B\Psi + C\Psi^2 = 0 \ . \qquad (3.3\text{-}1,b)$$

Here (A^{α}), (B) and (C) are functions of (n_i) and of field variables, density (ρ), velocity (u_i), ahead. This is again reduced to an ordinary differential equation along the rays as

$$\frac{d\Psi}{dt} + B\Psi + C\Psi^2 = 0 \; . \qquad (3.3\text{-}2)$$

After some calculations, we have been able to express (B) as a ray derivative of a certain quantity. The result of integration then comes as

$$\frac{1}{\Psi D} - \frac{1}{\Psi_o D_o} = \int_o^t \frac{c^2 + 2}{2D}\, dt \; , \qquad (3.3\text{-}3)$$

where

$$D = \left(\frac{EG^4}{a}\right)^{\frac{1}{2}} , \; a^2 = \frac{dp}{d\rho}\Big)_o \; , \; \frac{c^2 a^2}{\rho} = \frac{d^2 p}{d\rho^2}\Big)_o \; . \qquad (3.3\text{-}3\,,a)$$

The motion, of course, is assumed to be isentropic. A complete integration can be achieved only if the integrand in (3.3-3) is expressed as a ray-derivative of some quantity. We have not yet been able to do it. Some simplification results if we assume the adiabatic equation of state; (c^2) reduces to (γ-1), where (γ) is the ratio of specific heats. For spherical and cylindrical symmetries, it is possible to obtain more concrete results, even in the presence of gravitation. The study of the blast wave is also similar. These problems are of known importance in astrophysics. It is, however, worth noting that we have obtained a complete solution for the linearised non-homogeneous problem.

Another problem, we have currently studied, is of the rotating incompressible fluid in the presence of a magnetic field. The linearised problem reveals a number of modes [31]. Taylor first noted the curious result that rotation gives an incompressible fluid the property of transmitting waves. The application of the present method yields only one mode - that of Alfven mode. Assuming the medium ahead to be in an arbitrary steady motion, we obtain

$$\frac{d\Psi_i}{dt} + e_{ijk}\Omega_j\Psi_k = 0 \ , \qquad (3.3\text{-}4)$$

where $\Psi_i = G\xi_i$, $\xi_i = [v_{i,j}]n_j$ and (Ω_i) is the angular velocity vector.

In absence of rotation, the discontinuity remains constant as we move along the rays. The rotation only sustains it; it does not imbue it with the property of growth or decay. We obtain two integrals

$$\Psi_i\Psi_i = \text{constant}, \ \Omega_i\Psi_i = \text{constant} \ . \qquad (3.3\text{-}5)$$

The latter is true for constant (Ω_i). The first expresses the boundedness of the discontinuity; the second states the constancy of the angle between the discontinuity vector and the angular velocity vector. Assuming further, without loss of generality, that $(\Omega_i) = (\Omega, o, o)$, we get

$$\Psi_1 = \Psi_{o1}, \ \Psi_2 = \Psi_{o3} \sin \Omega t + \Psi_{o2} \cos \Omega t, \ \Psi_3 = \Psi_{o3} \cos \Omega t - \Psi_{o2} \sin \Omega t \ . \qquad (3.3\text{-}6)$$

This brings out the oscillatory nature of the discontinuity.

4. Conclusions.

We have presented a fairly complete view on how the theory of singular surfaces can provide the method of studying the growth of the discontinuities in a totally hyperbolic system. The method has the generality of studying partial differential equations, which have non-constant coefficients and which are non-linear. However the method has one severe limitation: our attention is limited to the wave front.

Taking this as a starting point, we believe that it is possible to continue the solution at least in some finite neighborhood of the surface. It is possible to join distinct smooth solutions with these. Further we have not entered into the problem of interactions. These are all fruitful fields of study.

Another limitation is worth noting. We have not been able to study dispersive phenomena. Recent papers by Whitham[33,34] provide a field study. Just as we restricted the meaning of the term "wave", we have to restrict attention only to a class of dispersive phenomena. This subject is under study.

The ideas can be extended to general relativity, too. A number of papers by Thomas[35,36] provide interest openings. It is possible to study the variation of strength along rays. This can expand on the types of initial media considered by Thomas.

We close this discussion after noting one feature common to all the results we have obtained. A weak discontinuity, governed by a non-linear system, is assumed to grow into a strong one. This provides an answer for the inevitable appearance of shocks in a non-linear system, as arising from the weak ones. But throughout, we have seen that, it is only the dilatational ones that grow; the transverse ones do not. But strong transverse discontinuities are known to exist viz. Alfven Shocks. It does not seem possible to explain their existence as an initial value problem of the growth of a weak one. Similarly one fails to obtain a strong contact discontinuity from a weak one. The latter arise only as a result of interaction of strong ones. It is only in hypo-elasticity that a possible growth of transverse wave is predicted, depending on the initial state.

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