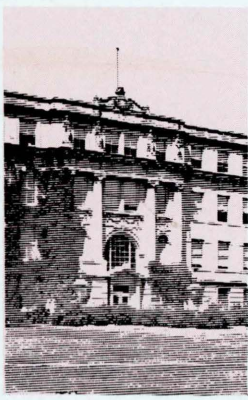
REPORT
55

AXIALLY SYMMETRIC WAVE PROPAGATION OF A FINITE SOLID CYLINDER

by C. T. Sun and K. C. Valanis



ENGINEERING RESEARCH INSTITUTE
IOWA STATE UNIVERSITY • AMES, IOWA

ENGINEERING
RESEARCH
ENGINEERING
RESEARCH
ENGINEERING
RESEARCH
ENGINEERING
RESEARCH
ENGINEERING
RESEARCH
ENGINEERING
RESEARCH

REPORT 55

AXIALLY SYMMETRIC WAVE PROPAGATION OF A FINITE SOLID CYLINDER

by Dr. C. T. Sun, Assistant Professor, and
Dr. K. C. Valanis, Professor
Department of Engineering Mechanics
Iowa State University

Price: $2.00

March, 1967
ENGINEERING RESEARCH INSTITUTE
IOWA STATE UNIVERSITY AMES

College of Engineering

Iowa State University,
104 Marston Hall,
Ames, Iowa 50010

AXIALLY SYMMETRIC WAVE PROPAGATION OF A

FINITE SOLID CYLINDER

by

C. T. Sun and K. C. Valanis

Introduction. Based on the general theory developed by K. C. Valanis[1], the solution of wave propagation in a viscoelastic material can be reduced by the superposition principle to the solution of a static problem, plus an eigenvalue problem and an integrodifferential equation of the Volterra type involving time only. The static problem of a finite hollow cylinder under axially symmetric loading has been solved by C. T. Sun and K. C. Valanis[2]. The purpose of this report is to develop the solution of the eigenvalue problem of a finite solid cylinder. From the general theory then the complete solution of wave propagation in a finite viscoelastic cylinder will be followed immediately.

Analysis. Take the cylindrical coordinate system as shown in Fig. 1. It is required to find the radial displacement $u(r, z, t)$ and the axial displacement $w(r, z, t)$ which in the interior of the cylinder $0 \leq r \leq a$, $-h \leq z \leq h$ satisfy the Lamé differential equation

$$\begin{aligned} \frac{2(1-\nu)}{1-2\nu}\frac{\partial e}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z} &= \frac{\rho}{\mu}\frac{\partial^2 u}{\partial t^2} \\ \frac{2(1-\nu)}{1-2\nu}\frac{\partial e}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] &= \frac{\rho}{\mu}\frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{1}$$

and which on the surface of the cylinder satisfy the homogeneous boundary conditions

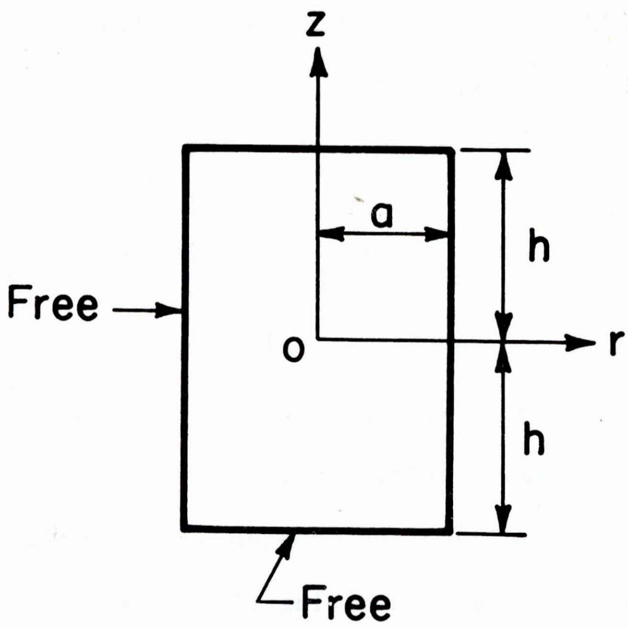


Figure 1

$$\tau_{rz}(a, z, t) = 0, \quad \tau_{rz}(r, \pm h, t) = 0$$
$$\sigma_z(r, \pm h, t) = 0, \quad \sigma_r(a, z, t) = 0 \tag{2}$$

Here ν is Poisson's ratio, μ is the shearing modulus, ρ is the density of the material and e is the dilatation i.e.

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} \tag{3}$$

The initial conditions are

$$u(r, z, 0) = 0$$
$$w(r, z, 0) = 0 \tag{4}$$
$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} = 0 \text{ at } t = 0$$

The stress tensor components can be expressed in terms of u and w by the following relations:

$$\frac{\partial^2 \phi_n}{\partial r^2} + \frac{1}{r}\frac{\partial \phi_n}{\partial r} + \frac{\partial^2 \phi_n}{\partial z^2} + \frac{k_n^2}{c_1^2}\phi_n = 0$$
$$\frac{\partial^2 \Psi_n}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi_n}{\partial r} - \frac{\Psi_n}{r^2} + \frac{\partial^2 \Psi_n}{\partial z^2} + \frac{k_n^2}{c_2^2}\Psi_n = 0 \qquad (10)$$

and $f_n(t)$ satisfies the Volterra integral equation

$$f_n(t) - k_n^2 \int_o^t F_n(t-\tau) f_n(\tau) d\tau = \cos k_n t \qquad (11)$$

where

$$F_n(t) = \int_o^t \cos k_n(t-\tau) g(\tau) d\tau \qquad (12)$$

$g(t) = H(t) - G(t)$ where $G(t)$ is the relaxation function in shear of the viscoelastic material.

In view of eqs. (6) and (9)

$$u(r,z,t) = \sum_n u_n(r,z) f_n(t)$$
$$w(r,z,t) = \sum_n w_n(r,z) f_n(t) \qquad (13)$$

To solve eq. (10), we let

$$\phi_n = -A_n J_o(\alpha_n r) \cos \beta_n z$$
$$\Psi_n = -B_n J_1(\alpha_n r) \sin \gamma_n z \qquad (14)$$

where A_n and B_n are constants

$$\alpha_n^2 = \frac{k_n^2}{c_1^2} - \beta_n^2$$
$$\alpha_n^2 = \frac{k_n^2}{c_2^2} - \gamma_n^2 \qquad (15)$$

From eqs. (6), (9) and (13), we have

$$u_n^{(1)} = J_1(\alpha_n r)(A_n\alpha_n \cos \beta_n z + B_n\gamma_n \cos \gamma_n z)$$
$$w_n^{(1)} = J_o(\alpha_n r)(A_n\beta_n \sin \beta_n z - B_n\alpha_n \sin \gamma_n z) \tag{16}$$

where $J_o(\alpha_n r)$ and $J_1(\alpha_n r)$ are the Bessel functions of the first kind.

Further letting

$$\phi_m = -C_m J_o(p_m r) \cos q_m z$$
$$\Psi_m = -D_m J_1(s_m r) \sin q_m z \tag{17}$$

where C_m and D_m are constants

$$p_m^2 = \frac{k_m^2}{c_1^2} - q_m^2$$
$$s_m^2 = \frac{k_m^2}{c_2^2} - q_m^2 \tag{18}$$

Then, again in view of eqs. (6), (9) and (13), we have a second solution as follows:

$$u_m^{(2)} = \cos q_m z \left[C_m p_m J_1(p_m r) + D_m q_m J_1(s_m r)\right]$$
$$w_m^{(2)} = \sin q_m z \left[C_m q_m J_0(p_m r) - D_m s_m J_o(s_m r)\right] \tag{19}$$

Noting that $u_n^{(3)}(r)f_n(t) = E_n J_1(\frac{k_n}{c_1} r) f_n(t)$, $w = 0$ are also solutions of eq. (1), the complete solutions of $u(r,z,t)$ and $w(r,z,t)$ are then expressed in a series form as:

$$u(r,z,t) = \sum_{n=1}^{\infty} u_n^{(1)} f_n(t) + \sum_{m=1}^{\infty} u_m^{(2)} f_m(t) + \sum_{j=1}^{\infty} u_j^{(3)} f_j(t)$$
$$w(r,z,t) = \sum_{n=1}^{\infty} w_n^{(1)} f_n(t) + \sum_{m=1}^{\infty} w_m^{(2)} f_m(t) \tag{20}$$

Since it is assumed[2] that the problem is symmetric with respect to plane $z = 0$ it follows that u is an even function of z and w is an odd function of z as shown in eqs. (16) and (19).

With the help of eqs. (5) and (20) the first two boundary conditions (2) lead to the following equations

$$\sum_{n=1}^{\infty}\left[-2\alpha_n\beta_n A_n \sin\beta_n h + (\alpha_n^2 - \gamma_n^2)B_n \sin\gamma_n h\right] J_1(\alpha_n r) + \sum_{m=1}^{\infty}\left[-2q_m p_m C_m J_1(p_m r) + (s_m^2 - q_m^2)D_m J_1(s_m r)\right] \sin q_m h = 0 \quad (21)$$

$$\sum_{n=1}^{\infty}\left[-2\alpha_n\beta_n A_n \sin\beta_n z + (\alpha_n^2 - \gamma_n^2)B_n \sin\gamma_n z\right] J_1(\alpha_n a) + \sum_{m=1}^{\infty}\left[-2q_m p_m C_m J_1(p_m a) + (s_m^2 - q_m^2)D_m J_1(s_m a)\right] \sin q_m z = 0 \quad (22)$$

The above two equations will be satisfied if the following relations are satisfied for all m and n:

$$\sin q_m h = 0 \quad (23)$$

$$-2\alpha_n\beta_n \sin\beta_n h\, A_n + (\alpha_n^2 - \gamma_n^2) \sin\gamma_n h\, B_n = 0 \quad (24)$$

$$J_1(\alpha_n a) = 0 \quad (25)$$

$$-2q_m p_m J_1(p_m a)C_m + (s_m^2 - q_m^2)J_1(s_m a)D_m = 0 \quad (26)$$

From eqs. (23) and (25), we must have

$$q_m h = n\pi \quad (27)$$

$$\alpha_n a = \mu_n \quad (28)$$

where μ_n is the n^{th} root of eq. (25). The expressions of q_m and α_n, therefore, are determined by the following relations

$$q_m = \frac{m\pi}{h} \tag{29}$$

$$\alpha_n = \frac{\mu_n}{a} \tag{30}$$

From eqs. (24) and (26) we can express B_n in terms of A_n and D_m in terms of C_m as follows:

$$\begin{aligned} B_n &= \eta_n A_n \\ D_m &= \epsilon_m C_m \end{aligned} \tag{31}$$

where

$$\begin{aligned} \eta_n &= \frac{2\alpha_n \beta_n \sin \beta_n h}{(\alpha_n^2 - \gamma_n^2) \sin \gamma_n h} \\ \epsilon_m &= \frac{2 q_m p_m J_1(p_m a)}{(s_m^2 - q_m^2) J_1(s_m a)} \end{aligned} \tag{32}$$

With the help of eqs. (5), (20), (31) and (32) the last two boundary conditions (2) lead to the following two equations:

$$\sum_{n=1}^{\infty} \left[\left\{ (1 - \nu)\beta_n^2 + \nu\alpha_n^2 \right\} \cos \beta_n h - (1 - 2\nu)\alpha_n \gamma_n \eta_n \cos \gamma_n h \right]$$

$$A_n J_o(\alpha_n r) + \sum_{m=1}^{\infty} \left[\left\{ (1 - \nu) q_m^2 + \nu p_m^2 \right\} J_o(p_m r) - (1 - 2\nu) \right.$$

$$\left. q_m s_m \epsilon_m J_o(s_m r) \right] (-1)^{m-1} C_m + \nu \sum_{j=1}^{\infty} E_j \frac{k_j}{c_1} J_o\left(\frac{k_j}{c_1} r\right) = 0 \tag{33}$$

$$\sum_{n=1}^{\infty} \left[\left\{ (1 - \nu)\alpha_n^2 + \nu\beta_n^2 \right\} \cos \beta_n z + (1 - 2\nu)\alpha_n \gamma_n \eta_n \cos \gamma_n z \right]$$

$$J_o(\alpha_n a)A_n + \sum_{m=1}^{\infty} \left[\left[\left\{(1 - \nu)p_m^2 J_o(p_m a) + \nu q_m^2 J_o(p_m a) - (1 - 2\nu)p_m \frac{J_1(p_m a)}{a}\right\} + (1 - 2\nu)\left\{q_m s_m J_o(s_m a) - q_m \frac{J_1(s_m a)}{a}\right\} \epsilon_m\right] C_m \cos q_m z + \sum_{j=1}^{\infty} E_j \left[(1 - \nu)\frac{k_j}{c_1} J_o(\frac{k_j}{c_1} a) - (1 - 2\nu) \frac{J_1(\frac{k_j}{c_1} a)}{a}\right] = 0 \qquad (34)$$

In order to equate the Fourier coefficients of the functions of eqs. (33) and (34) and thereby to determine the unknown constants A_n, C_m, E_j and the eigenvalues k_n it is necessary to expand $J_o(p_m r)$, $J_o(s_m r)$, $J_o(\frac{k_j}{c_1} r)$ in the terms of $J_o(\alpha_n r)$ and $\cos \beta_n z$, $\cos \gamma_n z$ in terms of $\cos q_m z$. We have

$$J_o(p_m r) = \sum_{n=1}^{\infty} J^{*}_{o,mn} J_o(\alpha_n r)$$

$$J_o(s_m r) = \sum_{n=1}^{\infty} J^{**}_{o,mn} J_o(\alpha_n r) \qquad (35)$$

$$J_o(\frac{k_j}{c_1} r) = \sum_{n=1}^{\infty} J^{***}_{o,jn} J_o(\alpha_n r)$$

$$\cos \beta_n z = F^{(1)}_{on} + \sum_{m=1}^{\infty} F^{(1)}_{mn} \cos q_m z$$

$$\cos \gamma_n z = F^{(2)}_{on} + \sum_{m=1}^{\infty} F^{(2)}_{mn} \cos q_m z \qquad (36)$$

where

$$J^{*}_{o,mn} = \frac{2p_m J_1(p_m a)}{aJ_o(\alpha_n a)(p_m^2 - \alpha_n^2)}$$

$$J^{**}_{o,mn} = \frac{2 s_m J_1(s_m a)}{a J_o(\alpha_n a)(s_m^2 - \alpha_n^2)}$$

$$J^{***}_{o,mn} = \frac{2 \frac{k_m}{c_1} J_1(\frac{k_m}{c_1} a)}{a J_o(\alpha_n a)(\frac{k_m^2}{c_1^2} - \alpha_n^2)}$$

$$F^{(1)}_{on} = \frac{\sin \beta_n h}{\beta_n h}$$

$$F^{(2)}_{on} = \frac{\sin \gamma_n h}{\gamma_n h} \qquad (38)$$

$$F^{(1)}_{mn} = \frac{1}{h}\left[\frac{\sin (q_m + \beta_n)h}{q_m + \beta_n} + \frac{\sin (q_m - \beta_n)h}{q_m - \beta_n}\right]$$

$$F^{(2)}_{mn} = \frac{1}{h}\left[\frac{\sin (q_m + \gamma_n)h}{q_m + \gamma_n} + \frac{\sin (q_m - \gamma_n)h}{q_m - \gamma_n}\right]$$

Now substituting the series (37) and (38) into eqs. (35), (36) and this, in turn, into eqs. (33) and (34) and equating the Fourier coefficients, we have three equations:

$$\left[\left\{(1 - \nu)\beta_n^2 + \nu\alpha_n^2\right\} \cos \beta_n h - (1 - 2\nu)\alpha_n \gamma_n \eta_n \cos \eta_n h\right] A_n$$

$$+ \sum_{m=1}^{\infty} (-1)^{m-1} \left[\left\{(1 - \nu) q_m^2 + \nu p_m^2\right\} \frac{2 p_m J_1(p_m a)}{a J_o(\alpha_n a)(p_m^2 - \alpha_n^2)} -\right.$$

$$\left.(1 - 2\nu) q_m s_m \epsilon_m \frac{2 s_m J_1(s_m a)}{a J_o(\alpha_n a)(s_m^2 - \alpha_n^2)}\right] C_m + \nu \sum_{j=1}^{\infty} \frac{k_j}{c_1}$$

$$\frac{2 \frac{k_j}{c_1} J_1(\frac{k_j}{c_1} a)}{a J_o(\alpha_n a)(\frac{k_j^2}{c_1^2} - \alpha_n^2)} E_j = 0 \qquad (39)$$

$$\frac{1}{h}\left[\left\{(1-\nu)\alpha_n^{\,2}+\nu\beta_n^{\,2}\right\}\frac{\sin\beta_n h}{\beta_n}+(1-2\nu)\alpha_n\eta_n\sin\gamma_n h\right]$$

$$J_o(\alpha_n a)A_n+\left[(1-\nu)\frac{k_n}{c_1}J_o\left(\frac{k_n}{c_1}a\right)-(1-2\nu)\frac{J_1\left(\frac{k_n}{c_1}a\right)}{a}\right]E_n=0 \tag{40}$$

and

$$\frac{1}{h}\sum_{n=1}^{\infty}\left[\left\{(1-\nu)\alpha_n^{\,2}+\nu\beta_n^{\,2}\right\}\left\{\frac{\sin(q_m+\beta_n)h}{q_m+\beta_n}+\right.\right.$$

$$\left.\frac{\sin(q_m-\beta_n)h}{q_m-\beta_n}\right\}+(1-2\nu)\alpha_n\gamma_n\eta_n\left\{\frac{\sin(q_m+\gamma_n)h}{q_m+\gamma_n}+\right.$$

$$\left.\left.\frac{\sin(q_m-\gamma_n)h}{q_m-\gamma_n}\right\}\right]J_o(\alpha_n a)A_n+\left[\left\{(1-\nu)p_m^{\,2}J_o(p_m a)+\right.\right.$$

$$\left.\nu q_m^{\,2}J_o(p_m a)-(1-2\nu)p_m\frac{J_1(p_m a)}{a}\right\}+(1-2\nu)\epsilon_m$$

$$\left.\left\{q_m s_m J_o(s_m a)-\frac{q_m J_1(s_m a)}{a}\right\}\right]C_m=0 \tag{41}$$

From eqs. (40) and (41) we can express E_n and C_m in terms of A_n respectively as:

$$E_n=\lambda_n A_n \tag{42}$$

$$C_m=-\frac{1}{h}\frac{\sum_{c=1}^{\infty}T_{mi}A_i}{Q_m} \tag{43}$$

where

$$\lambda_n=\frac{\left[\left\{(1-\nu)\alpha_n^{\,2}+\nu\beta_n^{\,2}\right\}\frac{\sin\beta_n h}{\beta_n}+(1-2\nu)\alpha_n\eta_n\sin\gamma_n h\right]J_o(\alpha_n a)}{h\left[-(1-\nu)\frac{k_n}{c_1}J_o\left(\frac{k_n}{c_1}a\right)+(1-2\nu)\frac{J_1\left(\frac{k_n}{c_1}a\right)}{a}\right]} \tag{44}$$

$$Q_m = \left\{(1 - \nu)p_m^2 J_o(p_m a) + \nu q_m^2 J_o(p_m a) - (1 - 2\nu)p_m \frac{J_1(p_m a)}{a}\right\}$$

$$+ (1 - 2\nu)\epsilon_m \left\{q_m s_m J_0(s_m a) - \frac{q_m J_1(s_m a)}{a}\right\} \tag{45}$$

$$T_{mi} = \left[\left\{(1 - 2\nu)\alpha_i^2 + \nu\beta_i^2\right\}\left\{\frac{\sin(q_m + \beta_i)h}{q_m + \beta_i} + \frac{\sin(q_m - \beta_i)h}{q_m - \beta_i}\right\} + (1 - 2\nu)\alpha_i\gamma_i\eta_i\left\{\frac{\sin(q_m + \gamma_i)h}{q_m + \gamma_i} + \frac{\sin(q_m - \gamma_i)h}{q_m - \gamma_i}\right\}\right] J_o(\alpha_i a) \tag{46}$$

Now, substituting eqs. (42), (43), (44), (45) and (46) into eq. (39) we obtain an infinite system of infinite algebraic equations for the unknown constants A_n as follows:

$$\theta_n A_n - \frac{1}{h}\sum_{m=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{m-1}\frac{H_{mn}T_{mi}}{Q_m}A_i + \sum_{i=1}^{\infty}L_{ni}A_i = 0$$

$$(n = 1, 2, 3 \ldots) \tag{47}$$

where

$$\theta_n = \left\{(1 - \nu)\beta_n^2 + \nu\alpha_n^2\right\}\cos\beta_n h - (1 - 2\nu)\alpha_n\gamma_n\eta_n\cos\gamma_n h \tag{48}$$

$$H_{mn} = \left\{(1 - \nu)q_m^2 + \nu p_m^2\right\}\frac{2p_m J_1(p_m a)}{aJ_o(\alpha_n a)(p_m^2 - \alpha_n^2)} - (1 - 2\nu)$$

$$q_m s_m \epsilon_m \frac{2s_m J_1(s_m a)}{aJ_o(\alpha_n a)(s_m^2 - \alpha_n^2)} \tag{49}$$

$$L_{ni} = \frac{\nu k_i}{c_1}\,\frac{2\frac{k_i}{c_1}J_1\left(\frac{k_i}{c_1}a\right)}{aJ_o(\alpha_n a)\left(\frac{k_i^2}{c_1^2} - \alpha_n^2\right)}\lambda_i \tag{50}$$

Since eqs. (47) are homogeneous equations for A_n, in order to have non-trivial solutions for A_n, we must equate the determinent of the coefficients of A_n of eq. (47) to be zero. This leads to the following characteristic equation

$$\left| G_{ni} \right| = 0 \tag{51}$$

where

$$G_{ni} = \theta_n - \frac{1}{h}\sum_{m=1}^{\infty} (-1)^{m-1} \frac{H_{mn}T_{mi}}{Q_m} + L_{ni} \quad (i=n)$$

$$G_{ni} = - \frac{1}{h}\sum_{m=1}^{\infty} (-1)^{m-1} \frac{H_{mn}T_{mi}}{Q_m} + L_{ni} \quad (i \neq n) \tag{52}$$

Evaluation of the eigenvalues. To evaluate the eigenvalues k_n from the expansion of the determinent (51), we observe, from eq. (52), that G_{ni} is not only a function of k_n, but also a function of the other eigenvalues k_i ($i = 1, 2 \ldots n-1$) as well. Thus the eigenvalues themselves appear as parameters in the characteristic equation. In other words, we have n unknowns in one equation. This difficulty, however, can be overcome by the following consideration.

Let us denote the expansion of eq. (51) by

$$F(x, k_1, k_2, \ldots k_n) = 0 \tag{53}$$

where x stands for the random root. Since $k_1, k_2, \ldots k_n$ are eigenvalues, they must also satisfy eq. (53). Thus, if we put k_i for a particular i ($i = 1, 2, \ldots n$) equal to x in eq. (53), we obtain simultaneous n algebraic equations as follows

$$F(x, k_1, k_2, \ldots k_{i-1}, k_{i+1}, \ldots k_n)\Big|_{k_i=x} = 0 \qquad (54)$$

$$(i = 1, 2, \ldots n)$$

Equations (54) can be solved simultaneously for n unknowns k_1, k_2, ... k_n. In view of the expressions of λ_n, Q_m, T_{mi}, θ_n, H_{mn}, and L_{ni} from eqs. (44), (45), (46), (48), (49) and (50) in which the eigenvalues k_n involve in the arguments of the trigonometrical functions as well as in the Bessel functions, it is impossible to solve the simultaneous eqs. (54) analytically. However, we can solve it by some numerical means such as iteration procedure or trial and error method. The numerical evaluation of the eigenvalues will be published in a separate report.

The solution of the eigenvalue problem has been completed. All the boundary conditions are satisfied and therefore

$$u_n(r,z) = u_n^{(1)}(r,z) + u_n^{(2)}(r,z) + u_n^{(3)}(r,z)$$

$$w_n(r,z) = w_n^{(1)}(r,z) + w_n^{(2)}(r,z) \qquad (55)$$

represents the solution of the eigenvalue problem. The constants B_n, C_n, D_n and E_n are expressed uniquely in terms of A_n from eqs. (31), (42) and (43). The constants A_n will be determined from the initial conditions. The constants q_n and α_n are given in eqs. (29) and (30), β_n, γ_n, p_n and s_n are expressed in terms of α_n, q_n and the eigenvalues k_n from eqs. (15) and (18). The eigenvalues k_n will be evaluated from the characteristic eq. (51).

Note that from eq. (51) there are infinite set of roots k_n for each n. Consequently, for each α_n and q_n there is a denumerably infinite set of eigenvalues k_{nm}, p_{nm}, s_{nm}, β_{nm} and γ_{nm}. It is more meaningful, therefore, to express the eigenfunctions u and w in the form

$$u(r,z) = \sum_{n,m} A_{nm}(u_{nm}^{(1)} + u_{nm}^{(2)} + u_{nm}^{(3)})$$
$$w(r,z) = \sum_{n,m} A_{nm}(w_{nm}^{(1)} + w_{nm}^{(2)}) \tag{56}$$

where

$$u_{nm}^{(1)}(r,z) = J_1(\alpha_n r)(\alpha_n \cos \beta_{nm} z + \bar{B}_{nm}\gamma_{nm} \cos \gamma_{nm} z)$$
$$u_{nm}^{(2)}(r,z) = \bar{C}_{nm} \cos q_n z \, [p_{nm} J_1(p_{nm} r) + \bar{D}_{nm} q_n J_1(s_{nm} r)]$$
$$u_{nm}^{(3)}(r) = \bar{E}_{nm} J_1(\frac{k_{nm}}{c_1} r) \tag{57}$$
$$w_{nm}^{(1)}(r,z) = J_o(\alpha_n r)(\beta_{nm} \sin \beta_{nm} z - \bar{B}_{nm}\alpha_n \sin \gamma_{nm} z)$$
$$w_{nm}^{(2)}(r,z) = \bar{C}_{nm} \sin q_n z \, [q_n J_o(p_{nm} r) - \bar{D}_{nm} s_{nm} J_o(s_{nm} r)] \tag{58}$$

where

$$\bar{B}_{nm} = \frac{B_{nm}}{A_{nm}} \qquad \bar{C}_{nm} = \frac{C_{nm}}{A_{nm}}$$
$$\bar{D}_{nm} = \frac{D_{nm}}{C_{nm}} \qquad \bar{E}_{nm} = \frac{E_{nm}}{A_{nm}} \tag{59}$$

The ratio in (59) are given in eqs. (31), (42), and (43) respectively.

<u>Solution of Wave Propagation Problem</u>. Now we are in a position to complete the solution of wave propagation in a viscoelastic finite solid cylinder. For the sake of simplicity of notation we denote U and W as the static solution. u_{nm} and w_{nm} as the eigenfunctions

and u and w as the complete solution. Based on the general theory we have

$$u(r,z,t) = U(r,z) - \sum_{n,m} A_{nm}(u_{nm}^{(1)} + u_{nm}^{(2)} + u_{nm}^{(3)})\, f_{nm}(t)$$
$$w(r,z,t) = W(r,z) - \sum_{n,m} A_{nm}(w_{nm}^{(1)} + w_{nm}^{(2)})\, f_{nm}(t) \tag{60}$$

where $f_{nm}(t)$ satisfies the integral eq. (11). The constant A_{nm} can be determined from the initial conditions (4). In view of the orthogonality of the eigenfunctions u_{nm} and w_{nm} we have

$$A_{nm} = \frac{\int_v \left[U(u_{nm}^{(1)} + u_{nm}^{(2)} + u_{nm}^{(3)}) + W(w_{nm}^{(1)} + w_{nm}^{(2)})\right] dv}{\int_v \left[(u_{nm}^{(1)} + u_{nm}^{(2)} + u_{nm}^{(3)})^2 + (w_{nm}^{(1)} + w_{nm}^{(2)})^2\right] dv} \tag{61}$$

The integration of (61) is over the entire volume of the cylinder.

The complete solution for stress components can be readily obtained along similar lines from the superposition of the static solution and the eigen-value solution. The detailed expression is omitted since it is too lengthy.

Finally, it is interesting to observe that the same problem can also be solved from the following consideration. First we assume that both ends ($z = \pm h$) of the cylinder are lubricated. The boundary conditions are then, $\tau_{rz} = w = 0$ at $z = \pm h$ and $\tau_{rz} = 0$ at $r = a$. The conditions of σ_r at $r = a$ will be satisfied later. Next we consider that the curved surface ($r = a$) is lubricated. In this case the boundary conditions are $\tau_{rz} = u = 0$ at $r = a$ and $\tau_{rz} = 0$ at $z = \pm h$. The condition σ_z at $z = \pm h$ will be evaluated later. Evidently the solution of the original problem will be obtained by the superposition of the solutions for the two cases. In order to satisfy the boundary conditions $\sigma_r = 0$ at $r = a$ and $\sigma_z = 0$ at $z = \pm h$ we must equate the summation of σ_r at $r = a$ and σ_z at $z = \pm h$ of the two cases to be zero respectively. This will yield exactly the same characteristic equation as (51) for the determination of the eigenvalues k_n. This approach, however, will provide the physical clarity of the original problem.

ACKNOWLEDGEMENT

National Science Foundation support for this work, under Contract No. GK748, is gratefully acknowledged.

References

1. Valanis, K. C. "Exact and Variational Solutions to a General Viscoelasto-Kinetic Problem" Journal of Applied Mechanics Dec. 1966.
2. Sun, C. T. and Valanis, K. C. "On the Axially-Symmetric Deformation of a Hollow Circular Cylinder of Finite Length Under the Action of Axially-Symmetric Loading" Report 48. Engineering Research Institute, Iowa State University.