REPORT

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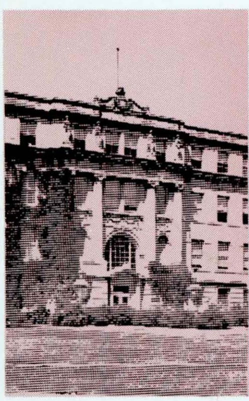

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1. Summary: The presence of rotation leads to a nonvanishing rate of growth of the discontinuity across an Alfven wave front. The two integrals obtained and the transversality completely determine the discontinuity.

2. Introduction and Basic Equations: In M.H.D., the discontinuity across an Alfven wave front does not grow; specifically its time rate of growth vanishes. We discuss here the effect of rotation on this discontinuity.

The linearised problem is studied by the use of the Fourier transform technique[1]. We use the theory of Singular surfaces[2] and the Ray Theory[3]. We study without linearising and taking the state ahead of the front arbitrary, but steady.

Cartesian tensor notation is used, with (x_i) denoting coordinates. The wave front $\Sigma(t)$ moves with a speed (G) normal to itself. The unit normal to the surface is denoted by (n_i). A Guassian system (u^α) $(\alpha = 1,2)$ is used to describe $\Sigma(t)$ and $(x_{i,\alpha})$ denote partial derivatives of (x_i) with respect to (u^α). The first and the second fundamental forms of the surface are denoted by $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$. If $f(x_i, t) = 0$ is the equation of the wave front, we denote the gradient vector by $(p_i = f_{,i} = pn_i)$. We can then show that[3,4,5]

$$Gp - 1 = 0 \qquad (1.1)$$

Assuming (G) depends on (x_i) and (n_i), we can write the Ray equations as[3,4,5]

$$V_i \equiv \frac{dx_i}{dt} = \frac{\partial(Gp)}{\partial\, p_i}$$

$$= Gn_i + (\delta_{ij} - n_i n_j)\frac{\partial G}{\partial n_j} , \qquad (1.2)$$

$$\frac{dp_i}{dt} = - p \frac{\partial G}{\partial x_i} . \qquad (1.3)$$

From (1.2), we have

$$V_\alpha = x_{i,\alpha} \frac{\partial G}{\partial n_i} . \qquad (1.4)$$

Let (d/dt) denote the derivative along the rays and $(\delta/\delta t)$, that following the normal trajectories. We then have[4,5]

$$\frac{dF}{dt} = \frac{\delta F}{\delta t} + V^\alpha F_{,\alpha} . \qquad (1.5)$$

We finally note the basic M.H.D. equations, taken as[1]

$$\frac{\partial V_i}{\partial t} + v_j v_{i,j} - \frac{\mu}{\rho} H_j H_{i,j} + \Omega_j(\Omega_i x_j - \Omega_j x_i) =$$

$$2\, e_{ijk}\Omega_j v_k + \frac{1}{\rho}\, p_{,i} = 0 , \qquad (1.6)$$

$$\frac{\partial H_i}{\partial t} + v_j H_{i,j} - H_j v_{i,j} = 0 , \qquad (1.7)$$

$$v_{i,i} = 0, \qquad H_{i,i} = 0 . \qquad (1.8,9)$$

Here (v_i), (H_i) and (Ω_i) are the vectors of fluid velocity, magnetic field and angular velocity; (μ) is the magnetic permeability and (ρ) is the density; lastly (p) is the total pressure (the sum of hydrostatic and Magnetic pressures).

3. <u>The Equation of Growth</u>: We assume that (p), (v_i) and (H_i) are continuous across $\Sigma(t)$ and the discontinuities are only in the gradients of the two vectors. Using a square bracket to denote the jumps, we write the discontinuities as

$$[v_{i,j}]\, n_j = \xi_i ,\ [H_{i,j}]\, n_j = \eta_i ,\ [v_{i,jk}]\, n_j n_k = \bar{\xi}_i ,$$

$$[H_{i,jk}]\, n_j n_k = \bar{\eta}_i . \tag{2.1}$$

The first order compatibility conditions give

$$(v_n - G)\, \xi_i = \frac{\mu H_n}{\rho}\, \eta_i ,\ (v_n - G)\, \eta_i = H_n \xi_i . \tag{2.2}$$

The suffix (n) denotes the normal component. The field variables in (2.2) and now-onwards only refer to values ahead.

The above relations lead to

$$G = v_n + \sqrt{\frac{\mu}{\rho}}\, H_n ,\ \xi_i = -\sqrt{\frac{\mu}{\rho}}\, \eta_i . \tag{2.3}$$

The equations for rays and those governing the variation of the normal (n_i) along the rays can be obtained as

$$V_i = v_i + \sqrt{\frac{\mu}{\rho}}\, H_i ,\ \frac{dn_i}{dt} = (n_j n_k V_{j,k} n_i - n_j V_{j,i}) . \tag{2.4}$$

Now differentiate (1.6) and (1.7) with respect to (x_i) and multiply by (n_i). The second order compatibility conditions give

$$-G\bar{\xi}_i + \frac{\delta \xi_i}{\delta t} + v_j(\xi_i n_j + g^{\alpha\beta} \xi_{i,\alpha} x_{j,\beta}) - \frac{\mu}{\rho} H_j(\bar{\eta}_i n_j$$

$$+ g^{\alpha\beta} \eta_{i,\alpha} x_{j,\beta}) + \xi_j v_{i,j} + v_{j,k} n_j n_k \xi_i - \frac{\mu}{\rho} (H_{j,k} n_j n_k \eta_i$$

$$+ H_{i,j} \eta_j) + 2e_{ijk} \Omega_j \xi_k = 0 , \tag{2.5}$$

$$-G\bar{\eta}_i + \frac{\delta\eta_i}{\delta t} + v_j(\bar{\eta}_i n_j + g^{\alpha\beta}\eta_{i,\alpha}x_{j,\beta}) - H_j(\bar{\xi}_i n_j + g^{\alpha\beta}\xi_{i,\alpha}x_{j,\beta})$$

$$+ v_{j,k}n_j n_k \eta_i + H_{i,j}\xi_j = H_{j,k}n_j n_k \xi_i - v_{i,j}\eta_j = 0 \; . \tag{2.6}$$

Using (1.4), (1.5), (2.3) and (2.4), these can be written as

$$\sqrt{\frac{\mu}{\rho}}\, H_n \left(\bar{\xi}_i + \sqrt{\frac{\mu}{\rho}}\, \bar{\eta}_i\right) = \frac{d\xi_i}{dt} + 2e_{ijk}\Omega_j\xi_k + V_{i,j}\xi_j$$

$$+ V_{j,k}n_j n_k \xi_i \; , \tag{2.7}$$

$$H_n\left(\bar{\xi}_i + \sqrt{\frac{\mu}{\rho}}\, \bar{\eta}_i\right) = \frac{d\eta_i}{dt} + V_{j,k}n_j n_k \eta_i - V_{i,j}\eta_j \; . \tag{2.8}$$

Eliminating the barred vectors, we obtain, by use of (2.3),

$$\frac{d\xi_i}{dt} + e_{ijk}\Omega_j\xi_k + \xi_i \frac{d \log G}{dt} = 0 \; . \tag{2.9}$$

Let $G\,\xi_i = \Psi_i$; then this equation can be written as

$$\frac{d\Psi_i}{dt} + e_{ijk}\Omega_j\Psi_k = 0 \; . \tag{2.10}$$

This is the final growth equation. Clearly, in absence of rotation, the rate vanishes. Rotation leads to a nonvanishing growth-rate of the discontinuity.

We always have one integral

$$\Psi_i\Psi_i = \text{constant}. \tag{2.11}$$

So the absolute magnitude of the discontinuity is constant; the discontinuity, though varying, never grows indefinitely.

We further note that the discontinuity is tangential. This property of transversality, obtained from (1.7,8), can be written as

$$\Psi_i n_i = 0 \; . \tag{2.12}$$

When (Ω_i) is constant, we obtain one more integral

$$\Omega_i \Psi_i = \text{constant}. \tag{2.13}$$

So for constant rotation, the three relations (2.11) (2.12) and (2.13) constitute the complete integral.

We now obtain another integral valid for arbitrary angular velocity (Ω_i) and arbitrary steady state ahead. Differentiate (2.12) along the rays and substitute from (2.4) and (2.10); we then get

$$e_{ijk} n_j \Omega_j \Psi_k + n_j \Psi_k V_{j,k} = 0 \tag{2.14}$$

Solving (2.11), (2.12) and (2.14) for (Ψ_i), we obtain

$$\Psi_i = \frac{\Psi_o}{P} P_i \tag{2.15}$$

where Ψ_o is the initial magnitude of (Ψ_i) and

$$P_i = \Omega_i - \Omega_j n_j n_i - e_{ijk} n_j n_p V_{p,k}, \; P_i P_i = P^2 \tag{2.15,a}$$

This gives the complete description of the discontinuity vector (Ψ_i) in terms of its initial magnitude and of the state ahead.

Any further discussion requires the knowledge of steady-state solutions of the state ahead. We note one simple solution for which the nature of the discontinuity can be more clearly explained. Let (Ω_i) (H_i) be constant vectors with only third component as non-vanishing. Thus we take the steady-state solutions as

$$v_i = (\Omega x_2, \; - \Omega x_1, \; 0), \; H_i(0, 0, H), \; \Omega_i(0,0,\Omega) \tag{2.16}$$

Let the initial wave-form be a plane normal to the 3-axis. It can then be shown that the front remains a plane normal to 3-axis. Then the solution of (2.10) is given as

$$\Psi_1 = \Psi_{01}\cos \Omega t + \Psi_{02} \sin \Omega t,\ \Psi_2 = -\Psi_{01} \sin \Omega t + \Psi_{02} \cos \Omega t,\ \Psi_3 = 0 \tag{2.17}$$

Thus the discontinuity vector rotates in the plane of the wavefront with the velocity of the fluid.

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