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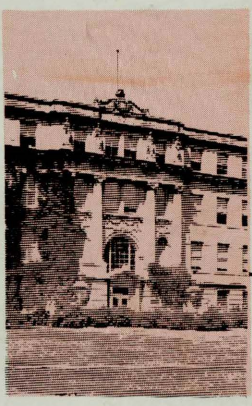
by C. T. Sun and K. C. Valanis



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On the Axially-Symmetric Deformation of a Hollow Circular Cylinder of Finite Length Under the Action of Axially-Symmetric Loading

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ON THE AXIALLY-SYMMETRIC DEFORMATION OF A HOLLOW CIRCULAR CYLINDER OF FINITE LENGTH UNDER THE ACTION OF AXIALLY-SYMMETRIC LOADING

by

C. T. Sun and K. C. Valanis

The boundary value problem of a circular cylinder under various loading conditions has been investigated by many authors. As early as in 1833, Lamé [1] obtained the solution of a long hollow cylinder loaded with constant internal and/or external pressure. In more recent years the mixed boundary value problems concerning the elastic deformation of a finite length cylinder have been studied in a number of papers [2-8]. However, none of the above mentioned papers gives the exact solution which satisfies the field equations of elasticity and the boundary conditions on the curved surface as well as at the ends of the cylinder. The exact solution of the axially-symmetric deformation of a solid cylinder of finite length was first achieved by Valov [9]. He introduced two sets of solutions which satisfy the field equations of elasticity. One set of the solution is given in the form of the modified Bessel functions of r of the first kind and sine and cosine functions of z. Another solution is expressed in terms of Bessel functions of r of the first kind and hyperbolic functions of z. In order to satisfy all the boundary conditions it is necessary to expand the modified Bessel functions in terms of Bessel functions and the hyperbolic functions in terms of trigonometric functions. All the boundary conditions, then will be satisfied by equating the Fourier coefficients. The final solution was given in the form of an infinite series.

In this report we solve the fundamental mixed boundary value problem of the theory of elasticity for the axially-symmetric deformations of a hollow cylinder of finite length. The approach used here is similar to that employed by Valov.

Take the cylindrical coordinate system as shown in Fig. 1.

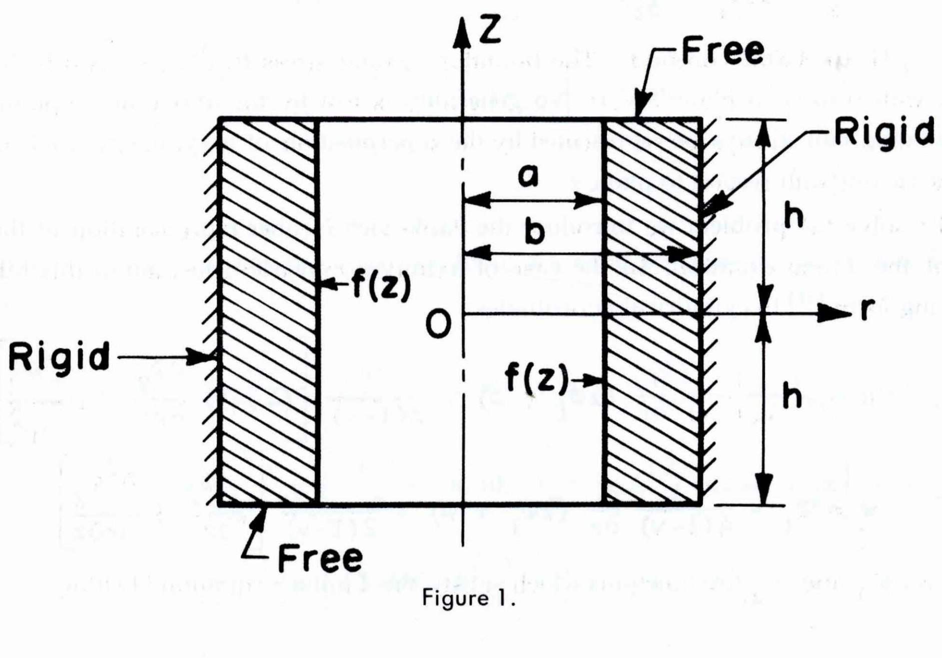

Figure 1.

It is required to find the radial displacement u(r, z) and the axial displacement w(r,z) which in the interior of the hollow cylinder $a \leq r \leq b$, $-h \leq z \leq h$ satisfy the Lamé differential equations

$$\frac{2(1-\nu)}{1-2\nu}\frac{\partial e}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = 0$$

$$\frac{2(1-\nu)}{1-2\nu}\frac{\partial e}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial r \partial z} = 0 \qquad (1)$$

and which on the surface of the cylinder satisfy the conditions

$$w(b,z) = 0 \qquad \tau_{rz}(a,z) = 0 \qquad \tau_{rz}(r,\pm h) = 0 \qquad (2)$$

$$u(b,z) = 0 \qquad \sigma_z(r,\pm h) = 0 \qquad \sigma_r(a,z) = f(z). \qquad (3)$$

Here ν is Poisson's ratio and e is the dilatation, i e.,

$$e = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}. \qquad (4)$$

The stress tensor components can be expressed in terms of u and w by the following relations:

$$\sigma_r = (\lambda + 2G)\frac{\partial u}{\partial r} + \lambda\left(\frac{u}{r} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_\theta = (\lambda + 2G)\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)$$

$$\sigma_z = (\lambda + 2G)\frac{\partial w}{\partial z} + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) \qquad (5)$$

$$\tau_{rz} = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)$$

where λ, G are Lamé constants. The boundary radial stress f(z) is assumed to be symmetric with respect to plane $z = 0$. No generality is lost by the above assumption, since any loading can always be represented by the superposition of a symmetric and an anti-symmetric one with respect to plane $z = 0$.

To solve the problem we introduce the Papkovich-Neuber representation of the solution of the Lamé equation. In the case of axially-symmetric deformation this takes the following form (9) in cylindrical coordinates.

$$u = -\frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(z\phi_1 + \phi) - \frac{1}{2(1-\nu)}\left[(4\nu-1)\frac{\partial\phi_2}{\partial r} + r\frac{\partial^2\phi_2}{\partial r^2}\right]$$

$$w = \phi_1 - \frac{1}{4(1-\nu)}\frac{\partial}{\partial z}(z\phi_1 + \phi) - \frac{1}{2(1-\nu)}\left[2\frac{\partial\phi_2}{\partial z} + \frac{\partial^2\phi_2}{\partial r\partial z}\right] \qquad (6)$$

where ϕ, ϕ_1 and ϕ_2 are functions which satisfy the Laplace equation. Letting

$$\phi = \left[C_n^{(1)} I_0(\alpha_n r) + C_n^{(2)} K_0(\alpha_n r)\right] \cos \alpha_n z ,$$

$$\phi_1 = 0, \tag{7}$$

$$\phi_2 = \left[C_n^{(3)} I_0(\alpha_n r) + C_n^{(4)} K_0(\alpha_n r)\right] \cos \alpha_n z ,$$

then in view of equation (6) we obtain the solutions of equation (1) in the following form:

$$u_n^{(1)}(r,z) = -\frac{\alpha n}{2(1-\nu)} \left[\frac{1}{2} C_n^{(1)} I_1(\alpha_n r) + C_n^{(3)} \left\{(4\nu-2) I_1(\alpha_n r) + \right.\right.$$

$$\left. + \alpha_n r \; I_0(\alpha_n r)\right\} - \frac{1}{2} C_n^{(2)} K_1(\alpha_n r) - C_n^{(4)} \left\{(4\nu-2) K_1(\alpha_n r) - \right.$$

$$\left.\left. - \alpha_n r K_0(\alpha_n r)\right\}\right] \cos \alpha_n z \tag{8}$$

$$w_n^{(1)}(r,z) = \frac{\alpha n}{2(1-\nu)} \left[\frac{1}{2} C_n^{(1)} I_0(\alpha_n r) + C_n^{(3)} \left\{2 I_0(\alpha_n r) + \alpha_n r I_1(\alpha_n r)\right\} + \right.$$

$$\left. + \frac{1}{2} C_n^{(2)} K_0(\alpha_n r) + C_n^{(4)} \left\{2 K_0(\alpha_n r) - \alpha_n r K_1(\alpha_n r)\right\}\right] \sin \alpha_n z$$

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where $C_n^{(1)}, C_n^{(2)}, C_n^{(3)}, C_n^{(4)}$ and α_n are arbitrary constants and $I_0(\alpha_n r)$, $I_1(\alpha_n r)$, and $K_0(\alpha_n r)$, $K_1(\alpha_n r)$ are modified Bessel functions of the first and second kind respectively. Further, letting

$$\phi = \left[A_m^{(1)} J_0(\lambda_m r) + A_m^{(2)} Y_0(\lambda_m r)\right] \cosh \lambda_m z$$

$$\phi_1 = \left[A_m^{(3)} J_0(\lambda_m r) + A_m^{(4)} Y_0(\lambda_m r)\right] \sinh \lambda_m z \tag{9}$$

$$\phi_2 = 0$$

then again in view of equation (6) we have a second solution of Lamé equation (1) as follows:

$$u_m^{(2)}(r,z) = \frac{\lambda m}{4(1-\nu)} \left[A_m^{(3)} z \sinh \lambda_m z + A_m^{(1)} \cosh \lambda_m z\right] J_1(\lambda_m r) +$$

$$+ \frac{\lambda m}{4(1-\nu)} \left[A_m^{(4)} z \sinh \lambda_m z + A_m^{(2)} \cosh \lambda_m z\right] Y_1(\lambda_m r)$$

$$w_m^{(2)}(r,z) = \frac{1}{4(1-\nu)} \left[(3-4\nu) A_m^{(3)} \sinh \lambda_m z - A_m^{(3)} \lambda_m z \cosh \lambda_m z - \right.$$

$$- A_m^{(1)}\lambda_m \sinh \lambda_m z \Big] J_0(\lambda_m r) +$$

$$+ \frac{1}{4(1-\nu)} \Big[(3-4\nu) A_m^{(4)} \sinh \lambda_m z - A_m^{(4)}\lambda_m z \cosh \lambda_m z -$$

$$- A_m^{(2)}\lambda_m \sinh \lambda_m z \Big] Y_0(\lambda_m r) \tag{10}$$

where $A_m^{(1)}, A_m^{(2)}, A_m^{(3)}, A_m^{(4)}$ and λ_m are arbitrary constants and $J_0(\lambda_m r)$, $J_1(\lambda_m r)$, and $Y_0(\lambda_m r)$, $Y_1(\lambda_m r)$ are Bessel functions of first and second kind respectively.

Noting that $u = C_0 r + \frac{D_0}{r}$, $w = 0$ are also solutions of equation (1), the complete solutions u(r,z) and w(r,z) are then expressed in a series form as:

$$\begin{aligned} u(r,z) &= C_0 r + \frac{D_0}{r} + \sum_{n=1}^{\infty} u_n^{(1)} + \sum_{m=1}^{\infty} u_m^{(2)} \\ w(r,z) &= \sum_{n=1}^{\infty} w_n^{(1)} + \sum_{m=1}^{\infty} w_m^{(2)} \end{aligned} \tag{11}$$

Since f(z) is assumed to be symmetric with respect to plane z = 0 it follows that u(r,z) is an even function of z and w(r,z) is an odd function of z as shown in equations (9) and (10).

With the help of equations (11) and (5), the first and second boundary conditions (2) lead to the following equations:

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{2(1-\nu)} \Big[\frac{1}{2} C_n^{(1)} I_0(\alpha_n b) + C_n^{(3)} \Big\{ 2I_0(\alpha_n b) + \alpha_n b I_1(\alpha_n b) \Big\} +$$

$$+ \frac{1}{2} C_n^{(2)} K_0(\alpha_n b) + C_n^{(4)} \Big\{ 2K_0(\alpha_n b) - \alpha_n b K_1(\alpha_n b) \Big\} \Big] \operatorname{Sin} \alpha_n z +$$

$$+ \sum_{m=1}^{\infty} \Big[\frac{J_0(\lambda_m b)}{4(1-\nu)} \Big\{ (3-4\nu) A_m^{(3)} - \lambda_m A_m^{(1)} \Big\} + \frac{Y_0(\lambda_m b)}{4(1-\nu)} \Big\{ (3-4\nu) A_m^{(4)} -$$

$$- \lambda_m A_m^{(2)} \Big\} \Big] \sin h\, \lambda_m z + \sum_{m=1}^{\infty} \Big[\frac{-J_0(\lambda_m b)}{4(1-\nu)} \lambda_m A_m^{(3)} - \frac{Y_0(\lambda_m b)}{4(1-\nu)} \lambda_m A_m^{(4)} \Big] x$$

$$x\, z \cos h\, \lambda_m z = 0 \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{\alpha_n^2}{2(1-\nu)} \Big[(C_n^{(1)} + 4\nu C_n^{(3)}) I_1(\alpha_n a) + 2C_n^{(3)} \alpha_n a I_0(\alpha_n a) -$$

$$- (C_n^{(2)} + 4\nu C_n^{(4)}) K_1(\alpha_n a) + 2C_n^{(4)} \alpha_n a K_0(\alpha_n a)\Big] \; \mathrm{Sin}\, \alpha_n z +$$

$$+ \sum_{m=1}^{\infty} \left[\frac{\lambda_m J_1(\lambda_m a)}{2(1-\nu)} \left\{ -(1-2\nu)\, A_m^{(3)} + \lambda_m A_m^{(1)} \right\} + \frac{\lambda_m Y_1(\lambda_m a)}{2(1-\nu)} \times \right.$$

$$\left. \times \left\{ -(1-2\nu)\, A_m^{(4)} + \lambda_m A_m^{(2)} \right\} \right] \sin h\, \lambda_m z + \sum_{m=1}^{\infty} \frac{\lambda m^2}{2(1-\nu)} \times$$

$$\times \left[J_1(\lambda_m a) A_m^{(3)} + Y_1(\lambda_m a)\, A_m^{(4)} \right] z \cosh \lambda_m z = 0 \qquad (13)$$

The above two equations will be satisfied if the following relations are satisfied for all m and n:

$$\frac{1}{2} I_0(\alpha_n b)\, C_n^{(1)} + \left\{ 2I_0(\alpha_n b) + \alpha_n b I_1(\alpha_n b) \right\} C_n^{(3)} + \frac{1}{2} K_0(\alpha_n b)\, C_n^{(2)} +$$

$$+ \left\{ 2K_0(\alpha_n b) - \alpha_n b K_1(\alpha_n b) \right\} C_n^{(4)} = 0 \qquad (14)$$

$$I_1(\alpha_n a)(C_n^{(1)} + 4\nu C_n^{(3)}) + 2\alpha_n a I_0(\alpha_n a)\, C_n^{(3)} - K_1(\alpha_n a)(C_n^{(2)} + 4\nu C_n^{(4)}) +$$

$$+ 2\alpha_n a K_0(\alpha_n a)\, C_n^{(4)} = 0 \qquad (15)$$

together with

$$J_0(\lambda_m b) \left\{ (3-4\nu) A_m^{(3)} - \lambda_m A_m^{(1)} \right\} + Y_0(\lambda_m b) \left\{ (3-4\nu) A_m^{(4)} - \lambda_m A_m^{(2)} \right\} = 0$$

$$J_0(\lambda_m b) A_m^{(3)} + Y_0(\lambda_m b) A_m^{(4)} = 0$$

$$J_1(\lambda_m a) \left\{ -(1-2\nu) A_m^{(3)} + \lambda_m A_m^{(1)} \right\} + Y_1(\lambda_m a) \left\{ -(1-2\nu) A_m^{(4)} + \lambda_m A_m^{(2)} \right\} = 0$$

$$J_1(\lambda_m a) A_m^{(3)} + Y_1(\lambda_m a) A_m^{(4)} = 0 \qquad (16)$$

The four relations in equation (16) are homogeneous. In order to have non-trivial solutions for $A_m^{(1)}$, $A_m^{(2)}$, $A_m^{(3)}$, and $A_m^{(4)}$ we must equate the determinant of the coefficients of $A_m^{(1)}$, $A_m^{(2)}$, $A_m^{(3)}$, and $A_m^{(4)}$ of equation (16) to be zero. This leads to the following characteristic equation:

$$J_1(\lambda_m a) Y_0(\lambda_m b) - Y_1(\lambda_m a) J_0(\lambda_m b) = 0 \qquad (17)$$

Hence, the eigenvalues λ_m, which are the solutions of equation (17), are determined. From equation (16), we have two independent relations as follows:

$$A_m^{(2)} = \beta_m A_m^{(1)}$$
$$A_m^{(4)} = \beta_m A_m^{(3)} \tag{18}$$

where

$$\beta_m = -\frac{J_0(\lambda_m b)}{Y_0(\lambda_m b)} = -\frac{J_1(\lambda_m a)}{Y_1(\lambda_m a)} \tag{19}$$

In order to satisfy the last boundary condition (2), and with the help of equation (18), we must have

$$\sum_{n=1}^{\infty} \frac{\alpha_n^2}{2(1-\nu)} \Big[(C_n^{(1)} + 4\nu C_n^{(3)}) I_1(\alpha_n r) + 2C_n^{(3)} \alpha_n r I_0(\alpha_n r) -$$
$$- (C_n^{(2)} + 4\nu C_n^{(4)}) K_1(\alpha_n r) + 2C_n^{(4)} \alpha_n r K_0(\alpha_n r) \Big] \operatorname{Sin} \alpha_n h +$$
$$+ \sum_{m=1}^{\infty} \frac{\lambda_m}{2(1-\nu)} \Big[-(1-2\nu) \operatorname{Sin} h\lambda_m h \cdot A_m^{(3)} + \lambda_m h \cos h\, \lambda_m h \cdot A_m^{(3)} +$$
$$+ \lambda_m \operatorname{Sin} h\, \lambda_m h \cdot A_m^{(1)} \Big] \Big[J_1(\lambda_m r) + \beta_m Y_1(\lambda_m r) \Big] = 0 \tag{20}$$

Equation (20) will be satisfied if we set

$$\operatorname{Sin} \alpha_n h = 0 \tag{21}$$

and

$$\Big\{ -(1-2\nu) \operatorname{Sin} h\, \lambda_m h + \lambda_m h \cos h\, \lambda_m h \Big\} A_m^{(3)} +$$
$$+ \lambda_m \operatorname{Sin} h\, \lambda_m h \cdot A_m^{(1)} = 0 \tag{22}$$

From equation (21), we must have

$$\alpha_n h = n\pi \qquad (n=1,2,3\ldots)$$

from which the eigenvalues

$$\alpha_n = \frac{n\pi}{h} \tag{23}$$

are determined. From equation (22), we have

$$A_m^{(1)} = \eta_m A_m^{(3)} \tag{24}$$

where

$$\eta_m = \frac{1-2\nu}{\lambda_m} - h \coth \lambda_m h \tag{25}$$

In view of equations (18) and (24), we have determined $A^{(1)}_m$, $A^{(2)}_m$ and $A^{(4)}_m$ in terms of $A^{(3)}_m$. From equations (14), (15) $C^{(2)}_n$ and $C^{(4)}_n$ can be eliminated. We have

$$\begin{aligned} C^{(2)}_n &= \delta^{(2)}_n C^{(1)}_n + \gamma^{(2)}_n C^{(3)}_n \\ C^{(4)}_n &= \delta^{(4)}_n C^{(1)}_n + \gamma^{(4)}_n C^{(3)}_n \end{aligned} \tag{26}$$

where

$$\delta^{(2)}_n = \frac{1}{\Delta_n}\Big[2I_1(\alpha_n a)K_0(\alpha_n b) + 2\nu I_0(\alpha_n b)K_1(\alpha_n a) - \alpha_n b I_1(\alpha_n a)K_1(\alpha_n b) - \\ -\alpha_n a I_0(\alpha_n b)K_0(\alpha_n a)\Big]$$

$$\begin{aligned} \gamma^{(2)}_n = \frac{2}{\Delta_n}\Big[&\Big\{2\nu I_1(\alpha_n a) + \alpha_n a I_0(\alpha_n a)\Big\}\Big\{2K_0(\alpha_n b) - \alpha_n b K_1(\alpha_n b)\Big\} - \\ &- \Big\{-2\nu K_1(\alpha_n a) + \alpha_n a K_0(\alpha_n a)\Big\}\Big\{2I_0(\alpha_n b) + \alpha_n b I_1(\alpha_n b)\Big\}\Big] \end{aligned} \tag{27}$$

$$\delta^{(4)}_n = -\frac{1}{2\Delta_n}\Big[I_0(\alpha_n b)K_1(\alpha_n a) + I_1(\alpha_n a)K_0(\alpha_n b)\Big]$$

$$\gamma^{(4)}_n = -\frac{1}{\Delta_n}\Big[2I_0(\alpha_n b)K_1(\alpha_n a) + 2\nu I_1(\alpha_n a)K_0(\alpha_n b) + \alpha_n b I_1(\alpha_n b)K_1(\alpha_n a) + \\ + \alpha_n a I_0(\alpha_n a)K_0(\alpha_n b)\Big]$$

$$\Delta_n = 2(1-\nu)K_0(\alpha_n b)K_1(\alpha_n a) - \alpha_n b K_1(\alpha_n a)K_1(\alpha_n b) + \alpha_n a K_0(\alpha_n a)K_0(\alpha_n b)$$

With the help of equations (18), (24), and (27), the remaining boundary conditions (3) lead to the following three equations:

$$C_0 b + \frac{D_0}{b} + \sum_{n=1}^{\infty} R^{(1)}_n(n)\text{Cos}\,\alpha_n z + \sum_{m=1}^{\infty} F_1(z, A^{(3)}_m) = 0 \tag{28}$$

$$\frac{4\nu G}{1-2\nu} C_0 + \sum_{n=1}^{\infty} F_2(r,n) + \sum_{m=1}^{\infty} Z_m(A_m^{(3)})\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] = 0 \qquad (29)$$

$$\frac{2G}{1-2\nu} C_0 - 2G \frac{D_0}{a^2} + \sum_{n=1}^{\infty} R_n^{(2)}(n) \operatorname{Cos} \alpha_n z + \sum_{m=1}^{\infty} F_3(z, A_m^{(3)})$$

$$= f_0 + \sum_{n=1}^{\infty} f_n \operatorname{Cos} \alpha_n z \qquad (30)$$

where

$$R_n^{(1)}(n) = \frac{-\alpha_n}{2(1-\nu)}\left[\left\{\frac{1}{2} I_1(\alpha_n b) - \frac{1}{2}\delta_n^{(2)} K_1(\alpha_n b) + \delta_n^{(4)}\left(\alpha_n b K_0(\alpha_n b) - \right.\right.\right.$$

$$\left.\left. - (4\nu-2)K_1(\alpha_n b)\right)\right\} C_n^{(1)} + \left\{(4\nu-2) I_1(\alpha_n b) + \alpha_n b I_0(\alpha_n b) - \right.$$

$$\left. - \frac{1}{2}\gamma_n^{(2)} K_1(\alpha_n b) - (4\nu-2)\gamma_n^{(4)} K_1(\alpha_n b) + \alpha_n b \gamma_n^{(4)} K_0(\alpha_n b)\right\} C_n^{(3)}\Big]$$

$$F_1(z, A_m^{(3)}) = \frac{\lambda m}{4(1-\nu)}\left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b)\right] \times$$

$$\times A_m^{(3)} \left[z \operatorname{Sin h} \lambda_m z + \eta_m \operatorname{Cos h} \lambda_m z\right]$$

$$F_2(r,n) = \frac{\alpha_n^2}{2(1-\nu)}\left[\left\{G C_n^{(1)} + 4G(1+\nu) C_n^{(3)}\right\} I_0(\alpha_n r) + \right.$$

$$+ 2G C_n^{(3)} \alpha_n r I_1(\alpha_n r) + \left\{G(\delta_n^{(2)} C_n^{(1)} + \gamma_n^{(2)} C_n^{(3)}) + \right.$$

$$\left. + 4G(1+\nu)\ (\delta_n^{(4)} C_n^{(1)} + \gamma_n^{(4)} C_n^{(3)})\right\} K_0(\alpha_n r) -$$

$$\left. -2G\ (\delta_n^{(4)} C_n^{(1)} + \gamma_n^{(4)} C_n^{(3)})\ \alpha_n r K_1(\alpha_n r)\right]$$

$$Z_m(A_m^{(3)}) = \frac{G\lambda m}{2(1-\nu)} \left(\operatorname{cos h} \lambda mh + \frac{\lambda mh}{\operatorname{sin h} \lambda mh}\right) A_m^{(3)}$$

$$R_n^{(2)}(n) = \frac{-\alpha n G}{2(1-\nu)}\left[\left\{\alpha_n I_0(\alpha_n a) - \frac{I_1(\alpha_n a)}{a}\right\} + \left\{\alpha_n K_0(\alpha_n a) + \frac{K_1(\alpha_n a)}{a}\right\}\delta_n^{(2)}\right.$$

$$+ 2 \left\{(2\nu-1)\alpha_n K_0(\alpha_n a) - 2(1-2\nu) \frac{K_1(\alpha_n a)}{a} - \alpha_n^2 a K_1(\alpha_n a)\right\} \delta_n^{(4)}\Big] C_n^{(1)}$$

$$- \frac{\alpha n G}{2(1-\nu)} \left[2 \left\{(2\nu-1)\alpha_n I_0(\alpha_n a) + 2(1-2\nu) \frac{I_1(\alpha_n a)}{a} + \alpha_n^2 a\, I_1(\alpha_n a)\right\} + \right.$$

$$+ \left\{\alpha_n K_0(\alpha_n a) + \frac{K_1(\alpha_n a)}{a}\right\} \gamma_n^{(2)} +$$

$$\left. + 2 \left\{(2\nu-1)\alpha_n K_0(\alpha_n a) - 2(1-2\nu) \frac{K_1(\alpha_n a)}{a} - \alpha_n^2 a K_1(\alpha_n a)\right\} \gamma_n^{(4)}\right] C_n^{(3)}$$

$$F_3(z, A_m^{(3)}) = \frac{\lambda m^2 G}{2(1-\nu)} \left[J_0(\lambda_m a) + \beta_m Y_0(\lambda_m a)\right] A_m^{(3)} \cdot z \sin h\, \lambda_m z +$$

$$+ \frac{\lambda m G}{2(1-\nu)} \left[J_0(\lambda_m a) + \beta_m Y_0(\lambda_m a)\right] (1 - \lambda_m h \coth \lambda_m h)\, A_m^{(3)} \cdot$$

$$\cos h\, \lambda_m z$$

$$f(z) = f_0 + \sum_{n=1}^{\infty} f_n \cos \alpha_n z$$

In order to equate the Fourier coefficients of the functions of equations (28), (29) and (30) and thereby to determine the unknown constants C_0 D_0, $C_n^{(1)}$, $C_n^{(3)}$ and $A_m^{(3)}$ it is necessary to expand $F_1(z, A_m^{(3)})$, $F_3(z, A_m^{(3)})$ in terms of cosine series and $F_2(r,n)$ in terms of the eigenfunctions $J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)$. We have

$$F_1(z, A_m^{(3)}) = F_0^{(1)}(A_m^{(3)}) + \sum_{n=1}^{\infty} F_n^{(1)}(A_m^{(3)}) \cos \alpha_n z \qquad (-h \leq z \leq h)$$

$$F_2(r,n) = \sum_{m=1}^{\infty} F_m^{(2)}(n) \left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] \qquad (a \leq r \leq b) \quad (31)$$

$$F_3(z, A_m^{(3)}) = F_0^{(3)}(A_m^{(3)}) + \sum_{n=1}^{\infty} F_n^{(3)}(A_m^{(3)}) \cos \alpha_n z \qquad (-h \leq z \leq h)$$

In view of the orthogonality of the eigenfunctions[10], it is not difficult to evaluate the Fourier coefficients and thereby to obtain the expressions:

$$F_0^{(1)}(A_m^{(3)}) = \frac{-\nu}{2(1-\nu) h\, \lambda m} \left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b)\right] \operatorname{Sin} h\, \lambda_m h \cdot A_m^{(3)}$$

$$F_n^{(1)}(A_m^{(3)}) = \frac{(-1)^{n-1} \lambda m}{(1-\nu) h} \left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b)\right] \frac{\alpha_n^2 - \nu(\lambda_m^2 + \alpha_n^2)}{(\lambda_m^2 + \alpha_n^2)^2} \times$$

$$\times \operatorname{Sin} h\, \lambda_m h \cdot A_m^{(3)}$$

$$F_m^{(2)}(n) = \frac{\alpha_n^2}{2(1-\nu)} \Big[G \Big\{ I^*_{0,mn} + \delta_n^{(2)} K^*_{0,mn} + 4(1+\nu)\delta_n^{(4)} K^*_{0,mn} -$$

$$- 2\delta_n^{(4)} \alpha_n K^*_{1,mn} \Big\} C_n^{(1)} + G \Big\{ 4(1+\nu) I^*_{0,mn} + \gamma_n^{(2)} K^*_{0,mn} +$$

$$+ 2\alpha_n I^*_{1,mn} + 4(1+\nu)\gamma_n^{(4)} K^*_{0,mn} - 2\gamma_n^{(4)} \alpha_n K^*_{1,mn} \Big\} C_n^{(3)} \Big]$$

$$F_0^{(3)}(A_m^{(3)}) = 0 \tag{32}$$

$$F_n^{(3)}(A_m^{(3)}) = \frac{2(-1)^{n-1} G \lambda_m^2 \alpha_n^2}{(1-\nu)h} \Big[J_0(\lambda_m a) + \beta_m Y_0(\lambda_m a) \Big] \frac{\mathrm{Sin\,h}\ \lambda mh}{(\lambda_m^2 + \alpha_n^2)^2} A_m^{(3)}$$

where

$$I_0(\alpha_n r) = \sum_{m=1}^{\infty} I^*_{0,mn} \Big[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r) \Big]$$

$$r I_1(\alpha_n r) = \sum_{m=1}^{\infty} I^*_{1,mn} \Big[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r) \Big] \qquad (a \le r \le b) \tag{33}$$

$$K_0(\alpha_n r) = \sum_{m=1}^{\infty} K^*_{0,mn} \Big[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r) \Big]$$

$$r K_1(\alpha_n r) = \sum_{m=1}^{\infty} K^*_{1,mn} \Big[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r) \Big]$$

The evaluation of $I^*_{0,mn}$, $I^*_{1,mn}$, $K^*_{0,mn}$ and $K^*_{1,mn}$ will be shown in the Appendix.

Now substituting the series (32), and the expansion of unity

$$1 = \sum_{m=1}^{\infty} L_m^{(1)} \Big[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r) \Big] \qquad (a \le r \le b) \tag{34}$$

where the expression of constant $L_m^{(1)}$ will be given in the Appendix, into equations (28), (29) and (30) and equating the Fourier coefficients, we have five equations:

$$C_0 b + \frac{D_0}{b} - \frac{\nu}{2(1-\nu)h} \sum_{m=1}^{\infty} L_m^{(2)} \cdot A_m^{(3)} = 0 \tag{35}$$

$$\varepsilon_n^{(1)} C_n^{(1)} + \mu_n^{(1)} C_n^{(3)} + \frac{(-1)^{n-1}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(1)} \cdot A_m^{(3)} = 0 \quad (n=1,2,3\ldots) \tag{36}$$

$$\frac{4\nu}{1-2\nu} L_m^{(1)} C_0 + \sum_{n=1}^{\infty} (H_{mn}^{(2)} C_n^{(1)} + H_{mn}^{(3)} C_n^{(3)}) +$$

$$+ L_m^{(3)} A_m^{(3)} = 0 \quad (m=1,2,3\ldots) \tag{37}$$

$$\frac{C_0}{1-2\nu} - \frac{D_0}{a^2} = \frac{f_0}{2G} \tag{38}$$

$$\epsilon_n^{(2)} C_n^{(1)} + \mu_n^{(2)} C_n^{(3)} + \frac{2(-1)^{n-1}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(4)} \cdot A_m^{(3)} = \frac{f_n}{G} \quad (n=1,2,3\ldots) \tag{39}$$

where

$$L_m^{(2)} = \frac{1}{\lambda m} \left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b) \right] \text{Sin h } \lambda mh \tag{40}$$

$$\epsilon_n^{(1)} = \frac{-\alpha n}{2(1-\nu)} \left[\frac{1}{2} I_1(\alpha_n b) - \frac{1}{2} \delta_n^{(2)} K_1(\alpha_n b) + \delta_n^{(4)} \left\{ \alpha_n b K_0(\alpha_n b) - \right.\right.$$

$$\left.\left. - (4\nu-2) K_1(\alpha_n b) \right\} \right] \tag{41}$$

$$\mu_n^{(1)} = \frac{-\alpha_n}{2(1-\nu)} \left[(4\nu-2) I_1(\alpha_n b) + \alpha_n b I_0(\alpha_n b) - \frac{1}{2} \gamma_n^{(2)} K_1(\alpha_n b) - \right.$$

$$\left. - (4\nu-2) \gamma_n^{(4)} K_1(\alpha_n b) + \alpha_n b \gamma_n^{(4)} K_0(\alpha_n b) \right]$$

$$H_{mn}^{(1)} = \frac{\lambda m \left\{ \alpha_n^2 - \nu(\lambda_m^2 + \alpha_n^2) \right\}}{(\lambda_m^2 + \alpha_n^2)^2} \left| J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b) \right] \text{Sin h } \lambda mh \tag{42}$$

$$H_{mn}^{(2)} = \frac{\alpha n^2}{2(1-\nu)} \left[I^*_{0,mn} + \delta_n^{(2)} K^*_{0,mn} + 4(1+\nu) \delta_n^{(4)} K^*_{0,mn} - \right.$$

$$\left. - 2\delta_n^{(4)} \alpha_n K^*_{1,mn} \right] \tag{43}$$

$$H_{mn}^{(3)} = \frac{\alpha n^2}{2(1-\nu)} \left[4(1+\nu) I^*_{0,mn} + 2\alpha_n I^*_{1,mn} + \gamma_n^{(2)} K^*_{0,mn} + \right.$$

$$\left. + 4(1+\nu) \gamma_n^{(4)} K^*_{0,mn} - 2\gamma_n^{(4)} \alpha_n K^*_{1,mn} \right] \tag{44}$$

$$L_m^{(3)} = \frac{\lambda m}{2(1-\nu)} \left(\cos\text{ h } \lambda mh + \frac{h\lambda m}{\sin\text{ h } \lambda mh} \right) \tag{45}$$

$$\epsilon_n^{(2)} = - \frac{\alpha n}{2(1-\nu)} \left[\left\{ \alpha_n I_0(\alpha_n a) - \frac{I_1(\alpha_n a)}{a} \right\} + \left\{ \alpha_n K_0(\alpha_n a) + \right.\right.$$

$$+ \frac{K_1(\alpha_n a)}{a}\Big\} \delta_n^{(2)} + 2 \Big\{(2\nu-1)\alpha_n K_0(\alpha_n a) - 2(1-2\nu) \frac{K_1(\alpha_n a)}{a} -$$

$$- \alpha_n^2 a K_1(\alpha_n a)\Big\} \delta_n^{(4)}\Big] \tag{46}$$

$$\mu_n^{(2)} = - \frac{\alpha n}{2(1-\nu)} \Big[2 \Big\{(2\nu-1)\alpha_n I_0(\alpha_n a) + 2(1-2\nu) \frac{I_1(\alpha_n a)}{a} +$$

$$+ \alpha_n^2 a I_1(\alpha_n a)\Big\} + \Big\{\alpha_n K_0(\alpha_n a) + \frac{K_1(\alpha_n a)}{a}\Big\} \gamma_n^{(2)} +$$

$$+ 2 \Big\{(2\nu-1)\alpha_n K_0(\alpha_n a) - 2(1-2\nu) \frac{K_1(\alpha_n a)}{a} - \alpha_n^2 a K_1(\alpha_n a)\Big\} \gamma_n^{(4)}\Big]$$

$$H_{mn}^{(4)} = \frac{\alpha n^2 \lambda m^2}{(\lambda m^2 + \alpha n^2)^2} \Big[J_0(\lambda_m a) + \beta_m Y_0(\lambda_m a)\Big] \text{ Sin h } \lambda_m h \tag{47}$$

From equations (35) and (38), we obtain C_0 and D_0 in terms of $A_m^{(3)}$ and f_0

$$C_0 = \frac{\nu(1-2\nu)b}{2(1-\nu)\left[b^2(1-2\nu) + a^2\right]h} \sum_{m=1}^{\infty} L_m^{(2)} A_m^{(3)} + \frac{1-2\nu}{(\frac{b}{a})^2(1-2\nu) + 1} \frac{f_0}{2G} \tag{48}$$

$$D_0 = \frac{\nu b}{2(1-\nu)\left[(\frac{b}{a})^2(1-2\nu) + 1\right]h} \sum_{m=1}^{\infty} L_m^{(2)} A_m^{(3)} -$$

$$- \frac{(1-2\nu)b^2}{(\frac{b}{a})^2(1-2\nu) + 1} \frac{f_0}{2G} \tag{49}$$

With the help of equations (36) and (39), we express $C_n^{(1)}$ and $C_n^{(3)}$ in terms of $A_m^{(3)}$ and f_n

$$C_n^{(1)} = \frac{1}{L_n^{(4)}} \Big[\frac{2(-1)^{n-1}\mu_n^{(1)}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(4)} A_m^{(3)} - \frac{(-1)^{n-1}\mu_n^{(2)}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(1)} A_m^{(3)} -$$

$$- \frac{\mu_n^{(1)} f_n}{G}\Big] \tag{50}$$

$$C_n^{(3)} = \frac{1}{L_n^{(4)}} \Big[\frac{-2(-1)^{n-1}\epsilon_n^{(1)}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(4)} A_m^{(3)} + \frac{(-1)^{n-1}\epsilon_n^{(2)}}{(1-\nu)h} \sum_{m=1}^{\infty} H_{mn}^{(1)} A_m^{(3)} +$$

$$+ \frac{\epsilon_n^{(1)} f_n}{G}\Big] \tag{51}$$

where

$$L_n^{(4)} = \epsilon_n^{(1)} \mu_n^{(2)} - \epsilon_n^{(2)} \mu_n^{(1)}$$

Now, substituting C_0, $C_n^{(1)}$, $C_n^{(3)}$ into equation (37) we obtain an infinite systems of infinite algebraic equations for the unknown constants $A_m^{(3)}$.

$$A_m^{(3)} = - \frac{2\nu^2 b}{(1-\nu)\left[b^2(1-2\nu) + a^2\right] h} \frac{L_m^{(1)}}{L_m^{(3)}} \sum_{i=1}^{\infty} L_i^{(2)} A_i^{(3)} + \frac{1}{(1-\nu)h} \sum_{i=1}^{\infty} T_{mi} A_i^{(3)} - \frac{2\nu}{(\frac{b}{a})^2(1-2\nu) + 1} \frac{L_m^{(1)}}{L_m^{(3)}} \frac{f_0}{G} + \Psi_m \tag{52}$$

(m=1,2,3...)

where

$$T_{mi} = \frac{1}{L_m^{(3)}} \sum_{n=1}^{\infty} \frac{1}{L_n^{(4)}} \Big[-2(-1)^{n-1} \mu_n^{(1)} H_{mn}^{(2)} H_{in}^{(4)} + (-1)^{n-1} \mu_n^{(2)} H_{mn}^{(2)} H_{in}^{(1)} + 2(-1)^{n-1} \epsilon_n^{(1)} H_{mn}^{(3)} H_{in}^{(4)} - (-1)^{n-1} \epsilon_n^{(2)} H_{mn}^{(3)} H_{in}^{(1)} \Big] \tag{53}$$

$$\Psi_m = \frac{1}{L_m^{(3)}} \sum_{n=1}^{\infty} \frac{1}{L_n^{(4)}} \left(H_{mn}^{(2)} \mu_n^{(1)} - H_{mn}^{(3)} \epsilon_n^{(1)}\right) \frac{f_n}{G} \tag{54}$$

(m=1,2,3...)

Therefore, the boundary conditions have been satisfied, and u(r,z), w(r,z) of equation (11) represents the solution of the mixed boundary value problem of a finite hollow cylinder under axially-symmetric loading. The eigenvalues in the radial direction λ_m are determined from the characteristic equation (17) and the ones in axial direction α_n are given in equation (21). The constants C_0, D_0, $C_n^{(1)}$ and $C_n^{(3)}$ are uniquely expressed in terms of $A_m^{(3)}$ in equations (46) to (49) respectively. The constants $A_m^{(1)}$, $A_m^{(2)}$ and $A_m^{(4)}$ are obtained from equations (18) and (24). From equation (26), we have the constants $C_n^{(2)}$ and $C_n^{(4)}$ which are given in terms of $C_n^{(1)}$ and $C_n^{(3)}$. The constant $A_m^{(3)}$ is determined from infinite system (52). It was shown by Valov[9] that the infinite system (52) is bounded. Then equation (11), which gives the solution of the problem, converges uniformly in the interior of the cylinder $-h < z < h$, $a < r < b$.

The stresses can be evaluated readily by using the stress-displacement relations given by equation (5). The actual formulae are omitted because they are too lengthy.

Finally it is interesting to observe that, in the case of $\nu = 0$ and the finite cylinder is under uniform pressure, the solution will be identical with the one obtained by Lamé[1] in 1833.

APPENDIX

The evaluation of constants $I^*_{0,mn}$, $I^*_{1,mn}$, $K^*_{0,mn}$, $K^*_{1,mn}$ and $L^{(1)}_m$. From pp. 434 Ref. (10), we have

$$D_{mn} = \int_a^b r y_0(\lambda_m r) y_0(\lambda_n r)\, dr = 0 \qquad m \neq n$$

$$D_{mm} = \int_a^b r y_0^2(\lambda_m r)\, dr$$

$$= \frac{b^2}{2}\left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b)\right]^2 - \frac{a^2}{2}\left[J_0(\lambda_m a) + \beta_m Y_0(\lambda_m a)\right]^2 \qquad (55)$$

where

$$y_0(\lambda_m r) = J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)$$

is the solution of the differential equation

$$\frac{d^2 y_0}{dr^2} + \frac{1}{r}\frac{dy_0}{dr} + \lambda_m^2\, y_0 = 0$$

and satisfies the boundary conditions

$$J_0(\lambda_m b) + \beta_m Y_0(\lambda_m b) = 0 \qquad (\text{i.e. } y_0(\lambda_m b) = 0)$$

$$J_1(\lambda_m a) + \beta_m Y_1(\lambda_m a) = 0 \qquad \left(\text{i.e. } \left.\frac{dy_0}{dr}\right|_{r=a} = 0\right)$$

as shown in the second and fourth equations of (16).

To determine the constants $I^*_{0,mn}$, $I^*_{1,mn}$, $K^*_{0,mn}$ and $K^*_{1,mn}$ in equation (33) and $L^{(1)}_m$ in equation (34), we multiply (33) and (34) by $r\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right]$ and then integrate from a to b, getting

$$I^*_{0,mn} = \frac{\int_a^b r I_0(\alpha_n r)\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] dr}{Dmm}$$

$$I^*_{1,mn} = \frac{\int_a^b r^2 I_1(\alpha_n r)\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] dr}{Dmm} \qquad (56)$$

$$K^*_{0,mn} = \frac{\int_a^b rK_0(\alpha_n r)\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] dr}{Dmm}$$

$$K^*_{1,mn} = \frac{\int_a^b r^2 K_1(\alpha_n r)\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] dr}{Dmm}$$

$$L^{(1)}_m = \frac{\int_a^b r\left[J_0(\lambda_m r) + \beta_m Y_0(\lambda_m r)\right] dr}{Dmm}$$

$$= \frac{b}{\lambda mDmm}\left[J_1(\lambda_m b) + \beta_m Y_1(\lambda_m b)\right] \qquad (57)$$

The integrals of equation (56) can not be evaluated by an elementary method. However, we can obtain their values by some method of numerical integration such as the trapezoidal rule. A numerical evaluation of the solution will be given in the very near future.

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