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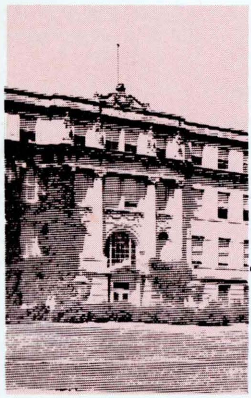

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Introduction. The problem of wave propagation in an elastic isotropic solid sphere has not been attacked in its full generality in the past. A restricted problem of steady oscillations has been investigated[1] but we are not aware of any work beyond this stage. This is all the more surprising because this problem has a variety of practical applications pertaining to seismic waves, as well as propagation of disturbances due to explosions on the earth's surface.

In this paper we solve the more complex problem of the visco-elastic sphere under axisymmetric loading; the material of the sphere may have general relaxation characteristics but its Poisson's ratio is restricted to having a constant value for all time. Evidently the solution of the elastic problem may be obtained as a special case. The method of solution is based on a superposition principle proposed by Valanis; this principle was discussed at length in previous papers[2,3]. However, for the sake of completeness, we give here an outline of its essential features.

The Superposition Principle. Since the elastic problem is a particular case of viscoelastic problem we proceed to treat the latter only, in its full generality. Let the interior of the body be denoted by D and its boundary by B. Let B_1 be part of B, and B_2 its compliment. The following wave problem is defined with respect to a cartesian coordinate system x_i for a linear, homogeneous, isotropic, viscoelastic body. A comma preceding a suffix denotes differentiation with respect to the corresponding co-ordinate, and a repeated index denotes summation over the range of values of the suffix. In the usual notation:

$$\sigma_{ij,j} = \frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

$$\sigma_{ij} = \delta_{ij}\, \lambda * \epsilon_{kk} + 2\mu * \epsilon_{ij} \tag{2}$$

$$\sigma_{ij}\, n_j = P_i \text{ on } B_1 \tag{3}$$

$$u_i = 0 \text{ on } B_2 \tag{4}$$

$$u_i = \frac{\partial u_i}{\partial t} = 0, \; t = 0 \tag{5}$$

where for convenience of notation,

$$f_1 * f_2 = \int_{0-}^{t} f_1(t-\tau)\, \frac{\partial f_2}{\partial \tau}\, d\tau \tag{6}$$

With the assumption of constant Poisson's ratio

$$\lambda(t) = \lambda_o\, G(t), \; \mu(t) = \mu_o\, G(t) \tag{7}$$

where λ_o and μ_o are constants. Thus using eq. (2) in eqs. (1) and (3) we have the following displacement problem:

$$(\lambda_o + \mu_o) G * u_{k,ki} + \mu_o\, G * u_{i,kk} = p\, \frac{\partial^2 u_i}{\partial t^2} \text{ in D} \tag{8}$$

$$\lambda_o\, G * u_{k,k}\, n_i + \mu_o\, G * (u_{i,k} + u_{k,i})\, n_k = P_i \text{ on } B_1 \tag{9}$$

$$u_i = 0 \text{ on } B_2 \tag{10}$$

$$u_i = \frac{\partial u_i}{\partial t} = 0, \; t = 0 \tag{11}$$

To illustrate an essential feature of the problem, take Laplace Transform of eq. (9) to find

$$\lambda_o \, p\overline{G} \, \overline{u}_{k,k} \, n_i + \mu_o \, p\overline{G} \, (\overline{u}_{i,k} + \overline{u}_{k,i}) \, n_k = \overline{P}_i \tag{12}$$

Thus, in view of eqs. (8), (10), (11) and (12), if u_i is a solution of the above problem, then $\overline{A} \, \overline{u}_i$ is a solution with $\overline{A} \, \overline{P}_i$ replacing $\overline{P}_i$ in eq. (12). We write eq. (12) in the form:

$$\lambda_o \, \overline{u}_{k,k} \, n_i + \mu_o(\overline{u}_{i,k} + \overline{u}_{k,i}) \, n_k = \frac{\overline{P}_i}{p\overline{G}} \tag{13}$$

Therefore without great loss of generality we can replace the right hand side of eq. (13) in the real plane by $P_i(x_k)H(t)$ and thus obtain canonical solution*. The actual solution will be obtained from the relation,

$$\overline{u}_{i(act)} = \overline{u}_{i(can)} \frac{\overline{g}}{\overline{G}} = p^2 \, \overline{u}_{i(can)} \, \overline{Jg} \tag{14}$$

where

$$P_i = P_i(x_k) \, g(t) \tag{15}$$

$$p\overline{G} = 1/p\overline{J} \tag{16}$$

and J(t) is essentially the creep function in shear.

For the sake of conciseness we introduce at this point the following notation:

---

* This inevitably implies that the most general form P_i can take is given by eq. (15).

$$L_o \left\{u_i\right\} = (\lambda_o + \mu_o)\, u_{k,ki} + \mu_o u_{i,kk} \tag{17}$$

$$B_o \left\{u_i\right\} = \lambda_o u_{k,k}\, n_i + \mu_o (u_{i,k} + u_{k,i})\, n_k \tag{18}$$

$$L_G \left\{u_i\right\} = (\lambda_o + \mu_o)\, G^* u_{k,ki} + \mu_o G^*\, u_{i,kk} \tag{19}$$

We are now in a position to formulate the canonical viscoelastic problem as follows:

$$L_G \left\{u_i\right\} = \frac{\partial^2 u_i}{\partial t^2} \text{ in } D. \tag{20}$$

$$B \left\{u_i\right\} = P_i(x_k)\, H(t) \text{ on } B_1. \tag{21}$$

$$u_i = 0 \text{ on } B_2 \tag{22}$$

$$\partial u_i/\partial t = u_i = 0,\ t = 0. \tag{23}$$

It is assumed that $P_i(x_k)$ do not give rise to any resultant moments but they may possibly give rise to a resultant force R_i.

To derive the stresses we make use of eqs. (2) and (14) in view of which,

$$\overline{\sigma}_{ij} = p\overline{g} \left\{\lambda_o\, \overline{u}_{(can)k,k}\, \delta_{ij} + \mu_o (\overline{u}_{(can)i,j} + \overline{u}_{(can)j,i})\right\} \tag{24}$$

<u>Thus to obtain the stress distribution from the canonical solution we employ elastic stress-strain relations</u>.

In particular if $g(t) = H(t)$, i.e., the surface loading on the body is a step function of time, then one obtains from eq. (24) the simple result

$$\sigma_{ij} = \lambda_o\, u_{(can)k,k}\, \delta_{ij} + \mu_o \left\{u_{(can)i,j} + u_{(can)j,i}\right\} \tag{25}$$

The superposition principle consists in defining (i) a static solution U_i which satisfies the conditions:

$$L_o \left\{U_i\right\} = 0 \text{ in } D \tag{26}$$

$$B_o \left\{U_i\right\} = P_i(x_k) \text{ on } B_1,\ U_i = 0 \text{ on } B_2, \tag{27}$$

or, where B_2 in null,

$$L_o \left\{U_i\right\} = \rho\alpha_i \tag{26a}$$

$$B_o \left\{U_i\right\} = P_i(x_k) \text{ on } B_2 \tag{27a}$$

where α_i is the rigid body acceleration vector to the unbalanced resultant force R_i; (ii) a reduced solution V_i which satisfies the following conditions:

$$L_G \left\{V_i\right\} = \rho \frac{\partial^2 V_i}{\partial t^2} \text{ in } D \tag{28}$$

$$B_o \left\{V_i\right\} = 0 \text{ on } B_1,\ V_i = 0 \text{ on } B_2 \tag{29}$$

$$V_i = U_i,\ \frac{\partial V_i}{\partial t} = 0,\ t = 0. \tag{30}$$

Then, u_i, such that

$$u_i = U_i - V_i \tag{31}$$

is the solution to the canonical problem. The reduced solution can be further decomposed into an eigensolution and a solution to a Volterra integral equation of the second kind, through the substitution

$$V_i = \sum_n A_n\, v_i^{(n)}\, f_n(t) \tag{32}$$

The functions $v_i^{(n)}$ satisfy the conditions

$$L_o\left\{v_i^{(n)}\right\} + \rho\, k_n^2\, v_i^{(n)} = 0 \text{ in } D \tag{33}$$

$$B_o\left\{v_i^{(n)}\right\} = 0 \text{ on } B_1,\; v_i^{(n)} = 0 \text{ on } B_2 \tag{34}$$

The initial conditions are satisfied if

$$\sum_n A_n\, v_i^{(n)} = U_i \tag{35}$$

and f_n satisfies the integrodifferential equation

$$k_n^2\, G * f_n(t) + \frac{d^2 f_n}{dt^2} = 0 \tag{36}$$

with the initial conditions

$$f_n(0) = 1,\; \frac{df_n}{dt} = 0,\; t = 0 \tag{37}$$

It was shown by the author in previous papers[2] that the vector functions $v_i^{(n)}$ are orthogonal (and may thus be made orthonormal) so that in view of eq. (35)

$$A_n = \int_D v_i^{(n)}\, U_i\, dv \tag{38}$$

Thus in summary, the superposition principle reduces the solution of the wave problem to those of a static elastic problem, an elastic eigenvalue problem, and an integrodifferential equation of the Volterra type involving time only. Thus

$$u_i(x_k,t) = U_i(x_k) - \sum_n A_n\, v_i^{(n)}(x_k)\, f_n(t) \tag{39}$$

Definition of the problem. We consider a solid isotropic viscoelastic sphere of radius a, unstressed and unstrained and in a state of rest at time t = 0. At this time an impulsive axisymmetric normal stress is applied to a small region of the sphere's surface so as to consistute a reasonable approximation to (a) a concentrated load applied to a point on the surface, and (b) two concentrated loads diametrically opposed. In case (a) the sphere will acquire rigid body motion in addition to its being deformed, whereas in case (b) the center of the sphere will remain at rest. See Fig. 1.

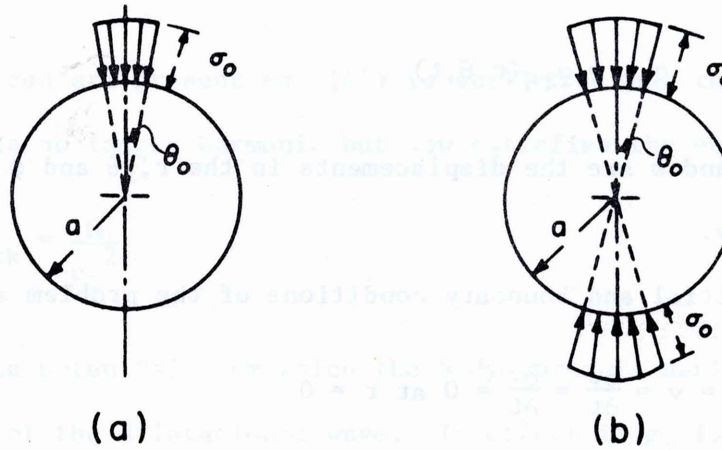

Figure 1

In terms of the spherical coordinates r, θ and ϕ. See Fig. 2.

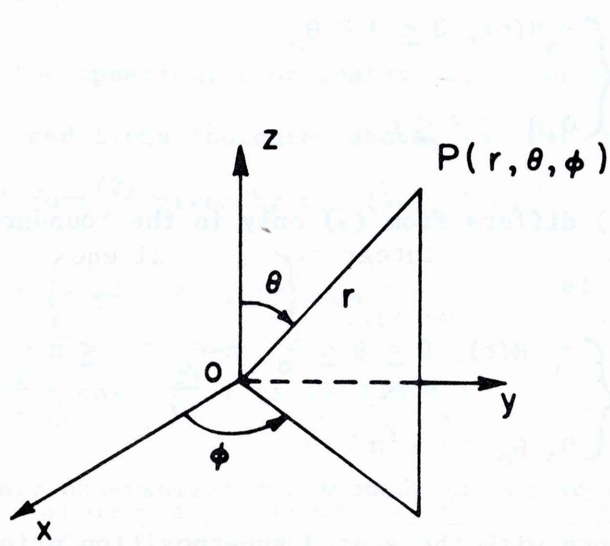

Figure 2

and in view of the prevailing axisymmetric conditions with respect to ϕ we have the following relations in the usual notation:

$$\left.\begin{aligned}
&w = 0, \text{ for all } r,\ \theta,\ \phi \text{ and } t. \\
&u = u(r,\theta,t),\ v = v(r,\theta,t) \\
&\sigma_{r\phi} = \sigma_{\theta\phi} = 0, \text{ for all } r,\ \theta,\ \phi \text{ and } t \\
&\sigma_r = \sigma_r(r,\theta,t),\ \sigma_\theta = \sigma_\theta(r,\theta,t),\ \sigma_\phi = \sigma_\phi(r,\theta,t), \\
&\sigma_{r\theta} = \sigma_{r\theta}(r,\theta,t)
\end{aligned}\right\} \quad (40a)$$

where u, v and w are the displacements in the r, θ and ϕ directions respectively.

The initial and boundary conditions of the problem are:

$$u = v = \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ at } t = 0 \quad (40b)$$

$$\sigma_{r\phi} = \sigma_{\theta\phi} \equiv 0, \quad (40c)$$

$$\sigma_{r\theta} = 0 \text{ at } r = a, \quad (40d)$$

$$\sigma_r = \begin{cases} \sigma_o H(t), & 0 \leq \theta \leq \theta_o \\ 0, & \theta < \theta \leq \pi \end{cases} \quad (40e)$$

Problem (b) differs from (a) only in the boundary condition (40e) which now is

$$\sigma_r = \begin{cases} \sigma_o\ H(t), & 0 \leq \theta \leq \theta_o,\ \pi-\theta_o \leq \theta \leq \pi \\ 0, & \theta_o < \theta < \pi - \theta_o \end{cases} \quad (40f)$$

In accordance with the stated superposition principle we obtain the static solutions first. Note that the eigenvalue problems (a) and (b) are identical.

Analysis of the Static Problems. We proceed with the analysis of the static problem (a). As stated in the general theory this is an elastic problem. In the absence of body forces the static problem is solved conveniently[4] by the use of the Papkovich-Neuber harmonic potentials ϕ and $\overline{\Psi}$ such that

$$\overline{u} = \mathrm{Grad}(\phi + \overline{r}\cdot\overline{\Psi}) - 4(1-\nu)\overline{\Psi} \tag{41}$$

When body forces are present eq. (41) is still true but the scalar potential ϕ is no longer harmonic but now satisfies the equation

$$\phi_{,kk} = \frac{\Omega}{c_1^{\,2}} \tag{42}$$

where Ω is the potential from which the body force is derivable and c_1 is the speed of the dilatational wave. In effect if g_i is the body force vector then

$$g_i = -\Omega_{,i} \tag{43}$$

In terms of the spherical coordinates (r, θ, ϕ) and in view of the axisymmetric conditions indicated above, eq. (41) may be written in a more special form[2] given by eq. (44), i.e.,

$$U = \frac{\partial\phi}{\partial r} + \left\{ r\,\frac{\partial\Psi}{\partial r} - (3-4\nu)\Psi \right\} \cos\theta \tag{44a}$$

$$V = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \cos\theta\,\frac{\partial\Psi}{\partial\theta} + (3-4\nu)\sin\theta\,\psi \tag{44b}$$

where Ψ is the only non-vanishing component of the vector $\overline{\Psi}$ $(0, 0, \Psi)$.

To simplify the presentation of the solution let

$$\phi = \phi_1 + \phi_2 \tag{45}$$

where

$$\nabla^2\phi_1 = 0$$

and ϕ_2 is the particular solution of eq. (42). At this point some elaboration on the boundary conditions of the static problem (a) is pertinent. In general, the stresses obtained from the particular solution ϕ_2 will not vanish on the boundary r = a. So the boundary conditions will inevitably have to be modified. In particular, in our case:

$$\phi_{2,kk} = -\beta z \tag{46}$$

where

$$\beta = -\frac{F}{(\frac{4}{3})\,\pi a^3(\lambda + 2\mu)} \tag{47}$$

and F is the total vertical unbalanced force acting on the surface of the sphere. The particular solution of eq. (46) is

$$\phi_2 = -\frac{\beta}{6} z^3 = -\frac{\beta}{6} r^3 \cos^3\theta \tag{48}$$

Let U_2, V_2, σ_{r2}, $\sigma_{r\theta 2}$ be quantities associated with ϕ_2. Then it transpires that

$$U_2 = \frac{\partial\phi_2}{\partial r} = -\frac{\beta}{2} r^2 \cos^3\theta \tag{49}$$

$$V_2 = \frac{1}{r}\frac{\partial\phi_2}{\partial\theta} = \frac{\beta}{2} r^2 \cos^2\theta \sin\theta \tag{50}$$

$$\sigma_{r2} = -\beta \left\{\lambda r \cos\theta + 2\mu r \cos^3\theta\right\} \tag{51}$$

$$\sigma_{r\theta 2} = 2\mu\beta r \cos^2\theta \sin\theta \tag{52}$$

It is obvious from eqs. (51) and (52) that σ_{r2} and $\sigma_{r\theta 2}$ do not vanish on the boundary r = a. So, to satisfy the static boundary conditions at r = a:

$$\sigma_r = \begin{cases} -\sigma_o, & 0 \le \theta \le \theta_o \\ o, & \theta_o < \theta \le \pi \end{cases} \tag{53}$$

$$\sigma_{r\theta} = 0 \tag{54}$$

we must set

$$\sigma_{r1} \begin{cases} -\sigma_o + \beta\left\{\lambda a \cos\theta + 2\mu a \cos^2\theta\right\}, & 0 \le \theta \le \theta_o \\ +\beta(\lambda a \cos\theta + 2\mu a \cos^3\theta), & \theta_o < \theta \le \pi \end{cases} \tag{55}$$

$$\sigma_{r\theta 1} = -2\mu\beta\, a \cos^2\theta \sin\theta \tag{56}$$

where σ_{r1} and $\sigma_{r\theta 1}$ are the stresses derived from the potentials ϕ_1 and Ψ. It further U_1 and V_1 are the displacements associated with ϕ_1 and Ψ then the solution of the static problem can be written as follows:

$$U = U_1 + U_2 \tag{57}$$

$$V = V_1 + V_2 \tag{58}$$

$$\sigma_r = \sigma_{r1} + \sigma_{r2}, \quad \sigma_{r\theta} = \sigma_{r\theta 1} + \sigma_{r\theta 2} \tag{59a,b}$$

The solution now to the static problem is straight forward; following Sternberg, et al.[5]

$$U_1 = -\sum_{n=-2}^{-\infty} a_n \frac{n+1}{r^{n+2}} P_n(x) - \sum_{n=-1}^{-\infty} b_n \frac{(n+1)(n+4-4\nu)}{r^{n+1}} P_{n+1}(x) \qquad (60)$$

$$V_1 = -\sum_{n=-2}^{-\infty} a_n \frac{\sin\theta\, P'_n(x)}{r^{n+2}} - \sin\theta \sum_{n=-1}^{-\infty} b_n \frac{(n-3) + 4-\nu}{r^{n+1}} P'_{n+1}(x) \qquad (61)$$

$$\sigma_{r1} = 2\mu \sum_{n=-3}^{-\infty} a_n (n+1)(n+2) \frac{P_n(x)}{r^{n+3}} + 2\mu \sum_{n=-2}^{-\infty} b_n (n+1) [(n+1)(n+4) - 2\nu] \frac{P_{n+1}(x)}{r^{n+2}} \qquad (62)$$

$$\sigma_{r\theta 1} = 2\mu \sum_{n=-3}^{-\infty} a_n \frac{n+2}{r^{n+3}} \sin\theta\, P'_n(x) + 2\mu \sum_{n=-2}^{-\infty} b_n (n^2 + 2n - 1 + 2\nu) \sin\theta \frac{P'_{n+1}(x)}{r^{n+2}} \qquad (63)$$

where $P_n(x)$ is the Legendre polynomial of order n; the argument x stands for cos θ and the prime indicates differentiation with respect to x. The coefficients a_n and b_n will be determined from the stress boundary conditions at r = a.

Let

$$\sigma_r(a,x) = f(x) = \sum_{n=0}^{\infty} \xi_n P_n(x) \qquad (64)$$

$$\sigma_{r\theta}(a,x) = g(x) = \sin\theta \sum_{n=1}^{\infty} \eta_n P'_n(x) \qquad (65)$$

Then it may be shown that

$$a_{-n-1} = \frac{(n^2+2n-1+2\nu)\xi_n + (n+1)(n^2-n-2-2\nu)\eta_n}{2(n-1)\left\{n^2+n+1+(2n+1)\nu\right\} a^{n-2}}$$

$$(n = 2, 3, 4 \ldots) \qquad (66)$$

$$b_{-n-2} = \frac{\xi_n + n\eta_n}{2\left\{n^2+n+1+(2n+1)\nu\right\} a^n}$$

$$(n = 0, 1, 2 \ldots) \qquad (67)$$

where

$$\xi_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx, \qquad (68)$$

$$\eta_n = \frac{2n+1}{2}\,\frac{(n-1)!}{n+1!}\int_{-1}^{1} \sin\theta g(x)P'_n(x)dx, \qquad (69)$$

$f(x)$ is given in eq. (55) and $g(x)$ is given in eq. (56); in particular

$$g(x) = -2\mu\beta a x^2 \qquad (70)$$

The solution of the static problem (b) is obtained along the same lines. In this case there is no body force and the solution can be obtained directly from the harmonic Papkovich-Neuber potentials ϕ and Ψ. The boundary conditions at $r = a$ are now:

$$\sigma_r = \begin{cases} -\sigma_o & 0 \le \theta \le \theta_o \\ 0 & \theta_o < \theta < \pi - \theta_o \\ -\sigma_o & \pi - \theta_o \le \theta \le \pi \end{cases} \qquad (71)$$

$$\sigma_{r\theta} = 0. \qquad (72)$$

In this case, the solution for U and V has the form given by eqs. (60) and (61) but of course the values of the coefficients a_n and b_n are different; their values are still obtained from eqs. (66) and (67) only now $\eta_n = 0$ for all n. The coefficients ξ_n are obtained from eq. (68) where f(x) now stands for the expression given in eq. (71). The stresses, naturally, have the same form as eqs. (62) and (63).

Analysis of the dynamic problem. The dynamic solution is derived most conveniently in terms of the displacement scalar and vector potentials ϕ and $\overline{\Psi}$. Again, because of the prevailing axially-symmetric conditions the vector $\overline{\Psi}$ has only one non-vanishing component which will be denoted by Ψ. In spherical curvilinear coordinates, ϕ and Ψ satisfy the following differential equations:

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \tag{73}$$

$$\nabla^2\Psi - \frac{\Psi}{r^2\sin^2\theta} = \frac{1}{c_2^2}\frac{\partial^2\Psi}{\partial t^2} \tag{74}$$

where

$$c_1^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}$$

and

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \tag{75}$$

The derivation of eq. (74) is given in Appendix I.

The solution, in terms of displacements u and v, may then be obtained from the relations

$$u = \frac{\partial\phi}{\partial r} + \frac{\Psi}{r}\cos\theta + \frac{1}{r}\frac{\partial\Psi}{\partial\theta} \tag{76}$$

$$v = \frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{\partial\Psi}{\partial r} - \frac{\Psi}{r} \tag{77}$$

The boundary conditions are now homogeneous, i.e.,

$$\sigma_r = \sigma_{r\theta} = 0 \text{ at } r = a \tag{78}$$

On the other hand the initial conditions are

$$u\Big|_{t=0} = U \tag{79}$$

$$v\Big|_{t=0} = V \tag{80}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ at } t = 0 \tag{81}$$

Following the general theory and in view of eq. (73)

$$\phi = \sum_{n=1}^{\infty} \Phi_n(r,\theta) f_n(t) \tag{82}$$

where

$$\Phi_n(r,\theta) = A_n r^{-\frac{1}{2}} J_{n+\frac{1}{2}}\left(\frac{k}{c_1} r\right) P_n(x), \tag{83}$$

P_n is the Legendre polynomial of order n and $J_{n+\frac{1}{2}}\left(\frac{k}{c_1} r\right)$ is spherical Bessel function of order n. Similarly in view of eq. (74)

$$\Psi = \sum_{n=1}^{\infty} \Psi_n(r,\theta)\, f_n(t) \tag{84}$$

where

$$\Psi_n(r,\theta) = B_n r^{-\frac{1}{2}} J_{n+\frac{1}{2}}\left(\frac{k}{c_2} r\right) P_n^{\,1}(x) \tag{85}$$

and $P_n^{\,1}(x)$ is the associated Legendre function of order n and degree 1.

Because of regularity requirements at $\theta = o$, the modified Lengdre function $Q_n^{\,1}(x)$ of degree one and order n is excluded from the solution.

Using eqs. (76) and (77) the displacements are obtained readily and are given in eqs. (86) and (87), i.e.,

$$u = \sum_{n=1}^{\infty} A_n u_n(r,k) P_n(x) f_n(t) \tag{86}$$

where

$$u_n(r,k) = r^{-\frac{1}{2}} \left[\frac{n}{r} J_{n+\frac{1}{2}} \left(\frac{k}{c_1} r\right) - \frac{k}{c_1} J_{n+3/2} \left(\frac{k}{c_1} r\right)\right] - \frac{B_n}{A_n} n(n+1) r^{-3/2} J_{n+\frac{1}{2}} \left(\frac{k}{c_2} r\right) \tag{86a}$$

and

$$v = \sum_{n=1}^{\infty} A_n v_n(r,k) P_n^{\,1}(x) f_n(t) \tag{87}$$

where

$$v_n = r^{-3/2} J_{n+\frac{1}{2}} \left(\frac{k}{c_1} r\right) - \frac{B_n}{A_n} \left[(n+1) r^{-3/2} J_{n+\frac{1}{2}} \left(\frac{k}{c_2} r\right) - \frac{k}{c_2} r^{-\frac{1}{2}} J_{n+3/2} \left(\frac{k}{c_2} r\right)\right] \tag{87a}$$

Using now isotropic elastic stress strain relations the stresses are found to be

$$\sigma_r = \sum_{n=1}^{\infty} 2\mu\, A_n F_n(r,k)\, P_n(x) f_n(t) \tag{88}$$

where

$$F_n(r,k) = \left[\left\{n(n-1) - \frac{1-\nu}{1-2\nu} \left(\frac{k}{c_1} r\right)^2\right\} J_{n+\frac{1}{2}} \left(\frac{k}{c_1} r\right)\right.$$

$$+ 2\frac{k}{c_1} r J_{n+3/2}\left(\frac{k}{c_1} r\right)\Bigg] r^{-5/2} - \frac{B_n}{A_n} n(n+1)$$

$$\left[(n-1) J_{n+\frac{1}{2}}\left(\frac{k}{c_2} r\right) - \frac{k}{c_2} r J_{n+3/2}\left(\frac{k}{c_2} r\right)\right] r^{-5/2} \quad (88a)$$

and

$$\sigma_{r\theta} = \sum_{n=1}^{\infty} \mu A_n G_n(r,k) P_n^{\ 1}(x) f_n(t) \quad (89)$$

where

$$G_n(r,k) = 2\left[(n-1) J_{n+\frac{1}{2}}\left(\frac{k}{c_1} r\right) - \frac{k}{c_1} r J_{n+3/2}\left(\frac{k}{c_1} r\right)\right] r^{-5/2}$$

$$+ \frac{B_n}{A_n}\Bigg[\left(-2n^2 + 2 + \frac{k^2}{c_2^{\ 2}} r^2\right) J_{n+\frac{1}{2}}\left(\frac{k}{c_2} r\right)$$

$$- 2\frac{k}{c_2} r J_{n+3/2}\left(\frac{k}{c_2} r\right)\Bigg] r^{-5/2} \quad (89a)$$

It remains now to satisfy the homogeneous boundary conditions given by eq. (78). These yield the relations:

$$F_n(a,k) = 0, \; G_n(a,k) = 0 \quad (90)$$

i.e.,

$$A_n \left[\left\{n(n-1) - \frac{1-\nu}{1-2\nu}\left(\frac{k}{c_1} a\right)^2\right\} J_{n+\frac{1}{2}}\left(\frac{k}{c_1} a\right) + 2\frac{k}{c_1} a J_{n+3/2}\left(\frac{k}{c_1} a\right)\right]$$

$$- B_n n(n+1)\left[(n-1) J_{n+\frac{1}{2}}\left(\frac{k}{c_2} a\right) - \frac{k}{c_2} a J_{n+3/2}\left(\frac{k}{c_2} a\right)\right] = 0 \quad (91)$$

and

$$2A_n\left[(n-1) J_{n+\frac{1}{2}}\left(\frac{k}{c_1} a\right) - \frac{k}{c_1} a J_{n+3/2}\left(\frac{k}{c_1} a\right)\right]$$

$$+ B_n\Bigg[\left(-2n^2 + 2 + \frac{k^2}{c_2^{\ 2}} a^2\right) J_{n+\frac{1}{2}}\left(\frac{k}{c_2} a\right) - 2\frac{k}{c_2} a$$

$$J_{n+3/2}\left(\frac{k}{c_2} a\right)\Bigg] = 0 \quad (92)$$

Equations (91) and (92) lead to the characteristic equation

$$2n(n+1)\left[(n-1)\,J_{n+\frac{1}{2}}\left(\frac{k}{c_2}a\right) - \frac{k}{c_2}\,a\,J_{n+3/2}\left(\frac{k}{c_2}a\right)\right]$$
$$\left[(n-1)\,J_{n+\frac{1}{2}}\left(\frac{k}{c_1}a\right) - \frac{k}{c_1}\,a\,J_{n+3/2}\left(\frac{k}{c_1}a\right)\right] + \left[\left\{n(n-1)\right.\right.$$
$$\left.\left. - \frac{1-\nu}{1-2\nu}\left(\frac{k}{c_1}a\right)^2\right\} J_{n+\frac{1}{2}}\left(\frac{k}{c_1}a\right) + 2\,\frac{k}{c_1}\,a\,J_{n+3/2}\left(\frac{k}{c_1}a\right)\right]$$
$$\left[\left(-2n^2 + 2 + \frac{k^2}{c_2^{\,2}}a^2\right) J_{n+\frac{1}{2}}\left(\frac{k}{c_2}a\right) - 2\,\frac{k}{c_2}\,a\,J_{n+3/2}\left(\frac{k}{c_2}a\right)\right] = 0$$

$$(n = 1, 2, 3 \ldots) \qquad (93)$$

Also either of eqs. (91) or (92) relates B_n to A_n.

It is important to observe that for each n eq. (93) possesses an infinity of roots k_m. It is, therefore, more meaningful to denote the roots of eq. (93) by k_{nm}. Thus eqs. (86) to (89) are now re-written in the form

$$u = \sum_{n,m=1}^{\infty} A_{nm} u_{nm} P_n(x) f_{nm}(t) \qquad (94)$$

$$v = \sum_{n,m=1}^{\infty} A_{nm} v_{nm} P_n^{\,1}(x) f_{nm}(t) \qquad (95)$$

$$\sigma_r = \sum_{n,m=1}^{\infty} 2\mu A_{nm} F_{nm} P_n(x) f_{nm}(t) \qquad (96)$$

$$\sigma_{r\theta} = \sum_{n,m=1}^{\infty} \mu A_{nm} G_{nm} P_n^{\,1}(x) f_{nm}(t) \qquad (97)$$

where

$$u_{nm} = u_n(r, k_{nm}) \qquad (98a)$$

$$v_{nm} = v_n(r, k_{nm}) \qquad (98b)$$

$$F_{nm} = F_n(r, k_{nm}) \tag{98c}$$

$$G_{nm} = G_n(r, k_{nm}) \tag{98d}$$

It now remains to determine the unknown constant A_{nm}. This is accomplished by satisfying the initial conditions of the dynamic problem given by eqs. (79) and (80). (Condition (81) is automatically satisfied by the appropriate of initial conditions on $f_{nm}(t)$ according to the general theory.) This concludes the analysis of the dynamic problem.

Solution of wave propagation problem (a). Before we proceed with the analysis it is more convenient to express the static solution in the slightly different form shown below

$$U = \sum_{j=0}^{\infty} r^j \left[a_{-j-2}(j+1)P_{j+1}(x) + b_{-j-1}j(-j+3-4\nu)P_{j-1}(x) \right] - \frac{\beta}{10} r^2 \left[3P_1(x) + 2P_3(x) \right] \tag{99}$$

$$V = -\sum_{j=0}^{\infty} r^j \sin\theta \left[a_{-j-2}P'_{j+1}(x) + b_{-j-1}(-j-4+4\nu) P'_{j-1}(x) \right] + \frac{\beta}{6} r^2 \sin\theta \left[2P_2(x) + P_o(x) \right] \tag{100}$$

Utilization of eq. (38) in the light of the previous discussion, and as a consequence of the spherical geometry

$$A_{nm} = \frac{\int_0^a \int_0^{\pi} (u_{nm} U + v_{nm} V) r^2 \sin\theta \, dr \, d\theta}{\int_0^a \int_0^{\pi} (u_{nm}u_{nm} + v_{nm}v_{nm}) r^2 \sin\theta \, dr \, d\theta} \tag{101}$$

Substitution in eq. (101) of U and V from eqs. (99) and (100) and of u_{nm} and v_{nm} from eqs. (86a), (87a), (98a) and (98b) and use of the following orthogonality conditions

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} 0, & m \neq n \\ \dfrac{1}{n+\frac{1}{2}}, & m = n \end{cases} \tag{102}$$

$$\int_{-1}^{1} \sin\theta\, P_n^{\,1}(x)P'_m(x)\,dx = \begin{cases} 0 & m \neq n \\ -\dfrac{n(n+1)}{n+\frac{1}{2}} & m = n \end{cases} \tag{103}$$

(See Appendix II) yield the following expression for A_{nm}

$$A_{nm} = \begin{cases} A_{nm}^{(1)} + A_{nm}^{(2)} & n = 1,\ 2,\ 3. \\ A_{nm}^{(1)} & n = 4,\ 5,\ \ldots,\ \infty \end{cases} \tag{104}$$

$$(m = 1,\ 2,\ 3,\ \ldots.,\ \infty)$$

where

$$A_{nm}^{(1)} = \frac{\left[1-n(n+1)\right]\ a_{-n-1}P_{nm}^{n-1} - (n+1)\left[n^2+4n(1-\nu)-2(1-2\nu)\right] b_{-n-2}q_{nm}^{n+1}}{R_{nm} + n(n+1)Q_{nm}} \tag{105}$$

$$A_{1m}^{(2)} = -\frac{2\beta}{5}\int_0^a r^4\, u_{1m}\,dr \tag{106a}$$

$$A_{2m}^{(2)} = -\frac{24\beta}{15}\int_0^a r^4 v_{2m}\,dr \tag{106b}$$

$$A_{3m}^{(2)} = -\frac{8\beta}{70}\int_0^a r^4 u_{3m}\,dr \tag{106c}$$

and

$$p^s{}_{nm} = \int_0^a r^2 u_{nm} r^s\,dr \tag{107a}$$

$$q^s{}_{nm} = \int_0^a r^2 v_{nm} r^s\,dr \tag{107b}$$

$$R_{nm} = \int_0^a r^2(u_{nm})^2 dr \qquad (107c)$$

$$Q_{nm} = \int_0^a r^2(v_{nm})^2 dr \qquad (107d)$$

Thus the solution of the wave propagation problem (a) can now be put in the explicit form in terms of the displacements

$$u(r,\theta,t) = U(r,\theta) - \sum_{n,m=1}^{\infty} A_{nm}u_{nm}(r)P_n(x)f_{nm}(t) \qquad (108)$$

$$v(r,\theta,t) = V(r,\theta) - \sum_{n,m=1}^{\infty} A_{nm}v_{nm}(r)P_n^1(x)f_{nm}(t) \qquad (109)$$

The stresses σ_r, $\sigma_{r\theta}$ can be expressed in similar fashion using eqs. (88) and (89). In particular,

$$\sigma_r = (\sigma_r)_{static} - 2\mu \sum_{n,m=1}^{\infty} A_{nm}F_{nm}(r)P_n(x)f_{nm}(t) \qquad (110)$$

$$\sigma_{r\theta} = (\sigma_{r\theta})_{static} - \mu \sum_{n,m=1}^{\infty} A_{nm}G_{nm}(r)P_n^1(x)f_{nm}(t) \qquad (111)$$

<u>Solution of wave propagation problem (b)</u>. This will be obtained by putting $\eta_n = 0$ for all n in eq. (66) and (67) and by taking $\beta = 0$. Naturally, ξ_n now will be different and will be obtained by inserting the appropriate loading function f(x) in eq. (68).

<u>Conclusion</u>. An exact solution has been obtained to the axially symmetric wave propagation problem of a viscoelastic sphere. The elastic problem is obtained as a special case by replacing $f_{nm}(t)$ by cos $k_{nm}t$ in the appropriate equations. To our knowledge this is the first time that such solution has been obtained.

Appendix I. It is well known that, in terms of the displacements, the solution to the wave propagation problem of isotropic elastic materials is obtainable from the scalar and vector potentials ϕ and $\overline{\Psi}$ where

$$\nabla^2\phi = \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} \tag{I1}$$

$$\nabla^2\overline{\Psi} = \frac{1}{c_2^2}\frac{\partial^2\overline{\Psi}}{\partial t^2} \tag{I2}$$

and

$$\overline{u} = \nabla\phi + \overline{\nabla} \times \overline{\Psi} \tag{I3}$$

The frame of reference of the above equations may be cartesian or curvilinear. However, when the component form is used, great care must be taken in expressing eq. (I2) in curvilinear coordinate systems. For instance, in certain cases (such as ours), symmetry conditions make two of the components of $\overline{\psi}$ vanish, in which event, if the frame of reference is cartesian, eq. (I2) is reduced to what appears to be a scalar equation in the remaining nonvanishing component ψ, i.e.,

$$\nabla^2\Psi = \frac{1}{c_2^2}\frac{\partial^2\Psi}{\partial t^2} \tag{I4}$$

Unfortunately, this has been misconstrued by some authors to mean that the left hand side of eq. (I4) is a Laplacian in all coordinate systems. This of course is not true, as will be demonstrated in the case of spherical coordinates.

Let x^k be cartesian coordinates and θ^k general curvilinear coordinates. Then the covariant base vectors $\bar{g}_k$ in the θ-system are given by the relation

$$\bar{g}_\alpha = \frac{\partial x^k}{\partial \theta^\alpha} \bar{i}_k \tag{I6}$$

where $\bar{i}_k$ are the cartesian unit vectors. The unit vectors $\bar{e}_\alpha$, tangent to the coordinates θ^α are obtained from (I6) through eq. (I7),

$$\bar{e}_\alpha = \frac{1}{h_\alpha} \bar{g}_\alpha \tag{I7}$$

where, $h_\alpha = \sqrt{g_{\alpha\alpha}}$ and $g_{\alpha\beta}$ is the metric of the curvilinear systems. Using eqs. (I6) and (I7) it is easily proved that

$$\frac{\partial \bar{e}_\alpha}{\partial \theta^\beta} = - \frac{1}{h_\alpha} \frac{\partial h_\alpha}{\partial \theta^\beta} \bar{e}_\alpha + \bar{e}_\gamma \Gamma_{\alpha\beta}^{\ \ \gamma} \tag{I8}$$

(α not summed; γ summed)

where $\Gamma_{\alpha\beta}^{\ \ \gamma}$ is the Christoffel symbol of the second kind.

In the case of spherical coordinates

$$\frac{\partial \bar{e}_\phi}{\partial r} = 0, \ \frac{\partial \bar{e}_\phi}{\partial \theta} = 0, \ \frac{\partial^2 \bar{e}_\phi}{\partial \phi^2} = - \bar{e}_\phi \tag{I9}$$

Symmetry conditions with respect to ϕ require that

$$\bar{\Psi} = \Psi \bar{e}_\phi \tag{I10}$$

Substituting eq. (I10) in eq. (I2) and using eq. (I9) one obtains the following differential equation for Ψ:

$$\nabla^2 \Psi - \frac{\Psi}{r^2 \sin^2 \theta} = \frac{1}{c_2^{\ 2}} \frac{\partial^2 \Psi}{\partial t^2} \tag{I11}$$

where the operator ∇^2 is given by eq. (75). Notice that the left hand side of eq. (I11) is not Laplacian.

Appendix II. The differential equation of the Legendre polynomial P_n is

$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)\,P_n = 0 \qquad \text{(II1)}$$

Multiplying eq. (II1) by $P_m(x)$ and integrating between the limits of -1 and +1, one obtains

$$\int_{-1}^{1} (1-x^2)P'_n(x)P'_m(x) = n(n+1)\int_{-1}^{1} P_n(x)P_m(x)dx \qquad \text{(II2)}$$

A well-known result(4) is

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \begin{cases} \frac{1}{n+\frac{1}{2}} & n=m \\ 0 & n\neq m \end{cases} \qquad \text{(II3)}$$

Thus:

$$\int_{-1}^{1} (1-x^2)P'_n(x)P'_m(x)dx = \begin{cases} \frac{n(n+1)}{n+\frac{1}{2}}, & n=m \\ 0, & n\neq m \end{cases} \qquad \text{(II4)}$$

We now make use of the classical formulae(6)

$$(1-x^2)P'_n(x) = -n(xP_n - P_{n-1}) \qquad \text{(II5)}$$

and

$$n(xP_n - P_{n-1}) = \sin\theta\; P_n^{\,1}(x) \qquad \text{(II6)}$$

where $P_n^{\,1}(x)$ is the associated Legendre function of order n and degree one. Substitution of eqs. (II5) and (II6) in eq. (II4) yields the desired result:

$$\int_{-1}^{1} \sin\theta\; P_n^{\,1}(x)P'_m(x) = \begin{cases} -\frac{n(n+1)}{n+\frac{1}{2}}, & m=n \\ 0, & m\neq n. \end{cases}$$

References

1. Christensen, R. M. and Gottenberg, W. G.: "The Dynamic Response of a Solid Viscoelastic Shphere to Translational and Rotational Excitation", Report EM 13-6, March, 1963, Space Technology Labs, Redondo Beach, California.

2. Valanis, K. C.: "Exact and Variational Solutions to a General Viscoelasto-Kinetic Problem", Journal of Appl. Mech., 33, 888, 1966.

3. Valanis, K. C.: "Some Exact Wave Propagation Solutions in Viscoelastic Materials", Mechanica Acta, (to be published).

4. Sternberg, Eli: "On the Integration of the Equations of Motion of Classical Theory of Elasticity", Arch. Rat. Mech. Anal., 6, 34, 1960.

5. Sternberg, Eli, et al., Proceedings, 2nd U.S. Congress of Appl. Mech., 209, 1951.

6. Erdelyi, A., et al., "Higher Transcendental Functions", McGraw Hill, New York, 1953.