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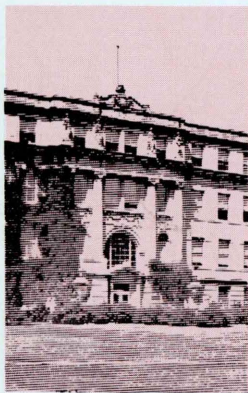
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**Numerical Solution of the Navier-Stokes
Equations for the Entrance Region of
Suddenly Accelerated Parallel Plates**

by Jerry D. Pearson
and
G. K. Serovy



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NUMERICAL SOLUTION OF THE NAVIER-STOKES EQUATIONS
FOR THE ENTRANCE REGION OF SUDDENLY ACCELERATED PARALLEL PLATES

by

Jerry Dean Pearson

A Dissertation Submitted to the
Graduate Faculty in Partial Fulfillment of
The Requirements for the Degree of
DOCTOR OF PHILOSOPHY

Major Subjects: Mechanical Engineering
Computer Science

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1966

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SYMBOLS

a	Radius
a_i	Acceleration vector
$A_{ij}\dots$	General tensor
B	Block Jacobi matrix
$B_{ij}\dots$	General tensor
d	Distance between plates
D	Drag
D_{ij}	Rate of deformation tensor
f_i	General vector
F_i	Specific body force
g	Body force potential function
g_i	General vector
h	Distance increment
i	Node number in the x-direction
I	Identity matrix
j	Node number in the y-direction
J	Jacobian
l	Entrance length parameter (see Equation 56)
$M+1$	Number of nodes in x-direction
n	Time plane number
n_i	External normal vector
$N+1$	Number of nodes in y-direction
O	Order of
p	Pressure

r	Relaxation factor or radial coordinate
R	Reynolds number
s	Distance
S	Surface area
t	Time
T	Difference matrix
T_{ij}	Stress tensor
u	Dimensionless velocity in x-direction
U	Characteristic velocity
u_i	Velocity vector
v	Dimensionless velocity in y-direction
V	Volume
v_i	Velocity vector
x	Dimensionless coordinate
x_i	Spatial coordinate
X_i	Material coordinate
y	Dimensionless coordinate
α	$h^2/\Delta\tau$
δ_{ij}	Kronecker delta tensor
Δx	Central difference operator
Δy	Central difference operator
$\Delta^2 x$	Second difference operator
$\Delta^2 y$	Second difference operator
θ	Spherical coordinate or ADI parameter
θ'	ADI parameter
θ''	ADI parameter

θ''	ADI parameter
$\lambda_{m,n}$	Eigenvalue
ϵ_s	Stream function convergence parameter
ϵ_v	Vorticity convergence parameter
ϵ_{ijk}	Epsilon tensor
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
σ	Spectral radius
τ	Dimensionless time
ϕ	Dimensionless stream function
ϕ_i	Stream function vector
ψ	Stream function
ω	Dimensionless vorticity
ω_i	Vorticity vector
Ω_i	Vorticity vector
Ω_{ij}	Vorticity tensor
$\frac{d}{dt}$	Material derivative
∂_i	Tensor del operator, $\frac{\partial}{\partial x_i}$
∇^2	Laplacian operator, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
∇_1^2	Laplacian operator, $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$

INTRODUCTION

The difficulties associated with the mathematical solution of the hydrodynamic equations for a viscous fluid are such that in the two hundred fifty years since these equations were first derived, only a very small number of particular cases have been solved exactly. This is true despite the fact that many great mathematicians have studied the problem.

Two forms of the viscous hydrodynamic equations have been derived. The most common is that form derived considering spatial, or what are commonly called Eulerian, coordinates. This form utilizes the velocity field as a function of position in space and time, $u_i(x_j, t)$, as the object of investigation. The Navier-Stokes equations are a set of equations of this form derived using particular assumptions as to the forces within the fluid. In principle, the Navier-Stokes equations along with the continuity equation and a knowledge of fluid properties allow the velocity field to be determined for any time $t > t_0$ where an initial velocity field is known, $u_i(x_j, t_0)$.

The second form of the hydrodynamic equations is that derived by considering material, or Lagrangian, coordinates. The object of investigation of the Lagrangian method is the position of a particle of fluid at a given time, $x_i(X_j, t)$, where X_j is a label associated with the particle. X_j is often the spatial coordinate of the particle at a particular time, t_0 .

The Eulerian and Lagrangian forms of the hydrodynamic equations are related by

$$\frac{\partial x_i(X_j, t)}{\partial t} = u_i(x_j, t)$$

Each of these forms of the hydrodynamic equations is a set of nonlinear partial differential equations. For the Eulerian or Navier-Stokes equations, the ratio of the characteristic magnitude of the nonlinear terms to the linear terms is equal to the ratio of the inertial forces to the viscous forces, that is, the Reynolds number. For the Lagrangian equations, the ratio of the nonlinear terms to the linear terms is the reciprocal of the Reynolds number. Thus the linear terms are dominant for small Reynolds numbers in the Navier-Stokes equations while the linear terms are dominant for large Reynolds numbers in the Lagrangian equations.

In problems where the Reynolds number is large, it is often most convenient to use the Lagrangian equations, even though they are more cumbersome. An example of this is presented in the paper by Monin (30) on turbulent motion of an incompressible viscous fluid. For low Reynolds numbers the Navier-Stokes equations are the most convenient.

As was previously noted, many years elapsed with only moderate success in the solution of the equations for flow of viscous fluids. The most notable advancements of a practical nature have been the boundary layer theory of Prandtl,

presented thoroughly by Schlichting (39), and various numerical techniques, the first of which was given by Thom (45) in 1933. The development of fast digital computers in the middle 1950's made the numerical methods much more feasible and a number of studies have been reported. This dissertation reviews several of these investigations and describes a new study of the time dependent Navier-Stokes equations by numerical techniques. The particular example considered is the unsteady laminar flow of a viscous fluid in the inlet section between two semi-infinite parallel plates. Results are compared with other available data.

NAVIER-STOKES EQUATIONS

Cartesian coordinate systems are used throughout this dissertation so that Cartesian tensor analysis is employed. A few of the basic notations are presented in the Appendix.

A Cartesian coordinate system is set up in a region such that the spatial coordinates of each point in the region are given by x_i . At a given time, $t = t_0$, each particle of fluid in the region is identified or labeled with its spatial coordinate at that time. This material coordinate is designated by X_i . Since the fluid is to be treated as a continuum, the term particle of fluid is used here in the same sense as a point in a region, not in the sense of a molecule of fluid. The position of a particle of fluid at time $t > t_0$ is then given by

$$x_i = x_i(X_j, t) .$$

The inverse relation,

$$X_j = X_j(x_i, t) ,$$

is assumed to exist. This assumption is physically justifiable since the fluid particles are distinct and maintain their identity. A given particle cannot be in two positions, nor can there be two particles at the same position. Mathematically, this implies that the Jacobian,

$$J = \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)},$$

and its inverse exist.

A tensor variable, $A_{ij\dots}$, may be a function of either the spatial coordinates,

$$A_{ij\dots} = A_{ij\dots}(x_k, t),$$

or the material coordinates,

$$A_{ij\dots} = A_{ij\dots}(X_k, t).$$

Which argument is meant will be clear from the context.

Define

$$\partial_j A = \frac{\partial A}{\partial x_j} = \frac{\partial A(x_i, t)}{\partial x_j}$$

$$\frac{\partial A}{\partial t} = \frac{\partial A(x_i, t)}{\partial t}$$

and

$$\frac{dA}{dt} = \frac{\partial A(X_i, t)}{\partial t}.$$

In the physical sense, $\frac{\partial A}{\partial t}$ is the change of the variable A with time at a given position, while $\frac{dA}{dt}$ is the change of A with time for a given particle of fluid. $\frac{dA}{dt}$ is often called the

material derivative with respect to time, as opposed to the spatial derivative with respect to time.

The velocity of a particle of fluid is the rate of change of position of that particle,

$$v_i = \frac{dx_i}{dt} = \frac{\partial x_i(X_j, t)}{\partial t} ,$$

and the acceleration of a particle is similarly given by

$$a_i = \frac{dv_i}{dt} = \frac{\partial^2 x_i(X_j, t)}{\partial t^2} .$$

For any tensor A,

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A(X_i, t)}{\partial t} \\ &= \frac{\partial A(x_i, t)}{\partial t} \frac{dt}{dt} + \frac{\partial A(x_i, t)}{\partial x_j} \frac{dx_j}{dt} \\ &= \frac{\partial A}{\partial t} + v_j \partial_j A . \end{aligned}$$

In particular,

$$\begin{aligned} a_i &= \frac{dv_i}{dt} \\ &= \frac{\partial v_i}{\partial t} + v_j \partial_j v_i \end{aligned}$$

The material derivative of the Jacobian, J , can also be written in terms of spatial derivatives. Since

$$\begin{aligned} J &= \frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)} \\ &= \det \left[\frac{\partial x_i}{\partial X_j} \right] \\ &= \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) , \end{aligned}$$

then,

$$\begin{aligned} \frac{dJ}{dt} &= \epsilon_{ijk} \left[\frac{d}{dt} \left(\frac{\partial x_1}{\partial X_i} \right) \right] \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) \\ &+ \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left[\frac{d}{dt} \left(\frac{\partial x_2}{\partial X_j} \right) \right] \left(\frac{\partial x_3}{\partial X_k} \right) \\ &+ \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left[\frac{d}{dt} \left(\frac{\partial x_3}{\partial X_k} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= \epsilon_{ijk} \left(\frac{\partial v_1}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) \\
&\quad + \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial v_2}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) \\
&\quad + \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial v_3}{\partial X_k} \right) \\
&= \epsilon_{ijk} \left(\frac{\partial v_1}{\partial x_m} \right) \left(\frac{\partial x_m}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) \\
&\quad + \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial v_2}{\partial x_m} \right) \left(\frac{\partial x_m}{\partial X_j} \right) \left(\frac{\partial x_3}{\partial X_k} \right) \\
&\quad + \epsilon_{ijk} \left(\frac{\partial x_1}{\partial X_i} \right) \left(\frac{\partial x_2}{\partial X_j} \right) \left(\frac{\partial v_3}{\partial x_m} \right) \left(\frac{\partial x_m}{\partial X_k} \right) \\
&= \left(\frac{\partial v_1}{\partial x_m} \right) J \delta_{1m} + \left(\frac{\partial v_2}{\partial x_m} \right) J \delta_{2m} + \left(\frac{\partial v_3}{\partial x_m} \right) J \delta_{3m} \\
&= \left(\frac{\partial v_i}{\partial x_i} \right) J
\end{aligned}$$

using Equation 58. Therefore

$$\frac{dJ}{dt} = J \partial_i v_i \quad . \quad 1)$$

Since

$$dx_1 dx_2 dx_3 = J dX_1 dX_2 dX_3 \quad ,$$

or

$$dV = J dV_0 \quad ,$$

where dV is an incremental volume in spatial coordinates and dV_0 is an incremental volume referred to the initial coordinate system at time t_0 , J is a measure of the expansion or contraction of the fluid and is called the dilatation. From Equation 1, it is seen that the divergence of the velocity is the relative rate of expansion of the volume following a particle. For a constant density fluid there is no expansion of the fluid so that J is a constant. Therefore $dJ/dt = 0$, or from Equation 1

$$\partial_i v_i = 0$$

for a constant density fluid.

The preceding paragraph developed a continuity equation for a constant density fluid, but in the more general case the following development is used. Let V^* be an arbitrary material volume, that is, a volume whose boundaries move with the fluid particles. Since there is no mass crossing the

boundaries, the mass of fluid within the volume is a constant, so that

$$\begin{aligned}
 0 &= \frac{d}{dt} \int_{V^*} \rho dV \\
 &= \frac{d}{dt} \int_{V_0^*} \rho J dV_0 \\
 &= \int_{V_0^*} \left(J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} \right) dV_0 \\
 &= \int_{V_0^*} \left(\frac{d\rho}{dt} + \rho \partial_i v_i \right) J dV_0 .
 \end{aligned}$$

Since the volume V^* and thus V_0^* is arbitrary, this results in the equation of continuity

$$\frac{d\rho}{dt} + \rho \partial_i v_i = 0 , \quad 2a)$$

or

$$\frac{\partial \rho}{\partial t} + v_i \partial_i \rho + \rho \partial_i v_i = 0 ,$$

which gives

$$\frac{\partial \rho}{\partial t} + \partial_i (\rho v_i) = 0 . \quad 2b)$$

These, of course, reduce to the previous result when ρ is a constant,

$$\partial_i v_i = 0 \quad . \quad 2c)$$

Let the stress tensor acting on a particle of fluid be T_{ij} and let the specific body force be F_i . Then the law of conservation of momentum applied to a material volume V^* with surface S^* and external normal n_i results in

$$\frac{d}{dt} \int_{V^*} \rho v_i dV = \int_{S^*} T_{ji} n_j dS + \int_{V^*} \rho F_i dV \quad .$$

Upon applying Equation 59,

$$\frac{d}{dt} \int_{V^*} \rho v_i dV = \int_{V^*} \partial_j T_{ji} dV + \int_{V^*} \rho F_i dV \quad ,$$

$$\frac{d}{dt} \int_{V_0^*} \rho v_i J dV_0 = \int_{V_0^*} [\partial_j T_{ji} + \rho F_i] J dV_0 \quad ,$$

$$\int_{V_0^*} \left[\rho J \frac{dv_i}{dt} + v_i \left(J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} \right) \right] dV_0 = \int_{V_0^*} [\partial_j T_{ji} + \rho F_i] J dV_0 \quad .$$

From Equations 1 and 2a,

$$J \frac{d\rho}{dt} + \rho \frac{dJ}{dt} = J \left[\frac{d\rho}{dt} + \rho \partial_i v_i \right] = 0 \quad .$$

Therefore

$$\int_{V_0^*} \left[\rho \frac{dv_i}{dt} - \partial_j T_{ji} - \rho F_i \right] J dV_0 = 0 .$$

Since V_0^* is an arbitrary volume, this results in the general equation of motion for a fluid,

$$\rho \frac{dv_i}{dt} = \rho F_i + \partial_j T_{ji} , \quad 3a)$$

or

$$\rho \left[\frac{\partial v_i}{\partial t} + v_j \partial_j v_i \right] = \rho F_i + \partial_j T_{ji} . \quad 3b)$$

Consider, now, a point P separated from a point 0 by the vector dx_i . The velocity of the fluid at P is then

$$\begin{aligned} v_i &= v_i^0 + \frac{\partial v_i}{\partial x_j} dx_j + O(dx_i^2) \\ &= v_i^0 + \partial_j v_i dx_j + O(dx_i^2) . \end{aligned}$$

The tensor $\partial_j v_i$ may be split into its symmetric and anti-symmetric parts by defining the tensors D_{ji} and Ω_{ji} by

$$\partial_j v_i = \frac{1}{2}(\partial_j v_i + \partial_i v_j) + \frac{1}{2}(\partial_j v_i - \partial_i v_j) = D_{ji} + \Omega_{ji} .$$

Thus, to terms of order dx_i ,

$$v_i = v_i^0 + \Omega_{ji} dx_j + D_{ji} dx_j \quad . \quad 4)$$

An antisymmetric tensor of second order such as Ω_{ji} has only three independent terms. These terms can be associated with the three components of a first order tensor by defining ω_i ,

$$\begin{aligned} \omega_i &= \epsilon_{ijk} \Omega_{jk} \\ &= \epsilon_{ijk} \left(\frac{1}{2} \right) \left[\partial_j v_k - \partial_k v_j \right] \\ &= \frac{1}{2} \left[\epsilon_{ijk} \partial_j v_k - \epsilon_{ijk} \partial_k v_j \right] \\ &= \frac{1}{2} \left[\epsilon_{ijk} \partial_j v_k + \epsilon_{ikj} \partial_k v_j \right] \quad , \end{aligned}$$

or

$$\omega_i = \epsilon_{ijk} \partial_j v_k \quad .$$

In vector notation this is written

$$\vec{\omega} = \nabla \times \vec{v} = \text{curl } \vec{v} \quad .$$

Upon inverting this relationship using Equation 57,

$$\begin{aligned}
\frac{1}{2} \epsilon_{imn} \omega_i &= \frac{1}{2} \epsilon_{imn} \epsilon_{ijk} \Omega_{jk} \\
&= \frac{1}{2} [\delta_{mj} \delta_{nk} - \delta_{mk} \delta_{nj}] \Omega_{jk} \\
&= \frac{1}{2} [\Omega_{mn} - \Omega_{nm}] ,
\end{aligned}$$

or

$$\Omega_{mn} = \frac{1}{2} \epsilon_{imn} \omega_i .$$

Therefore

$$\begin{aligned}
\Omega_{ji} dx_j &= \frac{1}{2} \epsilon_{kji} \omega_k dx_j \\
&= \frac{1}{2} \epsilon_{ikj} \omega_k dx_j \\
&= \left(\frac{1}{2} \vec{\omega} \right) \times d\vec{x} ,
\end{aligned}$$

in vector notation.

Therefore the second term of Equation 4 can be interpreted as a local rotation of the fluid about the point 0 with an angular velocity of $\frac{1}{2}\omega_i$. Because of this, the vector ω_i is called the vorticity vector or simply the vorticity, and Ω_{ij} is called the vorticity tensor.

The distance between the points 0 and P, ds , is given by

$$ds = \sqrt{dx_i dx_i} .$$

The material derivative of this distance with respect to time will then be the rate of change of the distance between two particles of fluid.

$$\begin{aligned}
 \frac{d}{dt}(ds) &= \frac{1}{2ds} \left[2dx_i \frac{d}{dt}(dx_i) \right] \\
 &= \frac{1}{ds} dx_i dv_i \\
 &= \frac{1}{ds} dx_i \partial_j (v_i) dx_j \\
 &= \frac{1}{ds} \left[\frac{1}{2} \partial_j v_i dx_i dx_j + \frac{1}{2} \partial_j v_i dx_i dx_j \right] \\
 &= \frac{1}{ds} \left[\frac{1}{2} (\partial_i v_j dx_j dx_i + \partial_j v_i dx_i dx_j) \right] \\
 &= \frac{1}{ds} D_{ij} dx_i dx_j
 \end{aligned}$$

A locally rigid body movement of the fluid at 0 results if the distance between particles is constant, that is, if

$$\frac{d}{dt}(ds) = 0 .$$

Thus a sufficient condition that the instantaneous motion of the fluid at a point is one which a rigid body could undergo is that $D_{ij} = 0$. This is also necessary since the point P and thus dx_i is arbitrary. D_{ij} is therefore called the rate of deformation tensor as it determines the amount of local deformation of the fluid.

The local instantaneous motion of the fluid at a point is made up of the three parts shown in Equation 4. The first term is a translation, the second is a rotation, and the third is a deformation of the fluid. The first two are motions which a rigid body could undergo, but the third is not.

Stokes (42, p. 80) assumes the following principle as to the behavior of a fluid:

"The difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned."

Serrin (41) reduces this principle to four statements regarding T_{ij} and D_{ij} .

- (1) T_{ij} is a continuous function of D_{ij} and independent of all other kinematic quantities.
- (2) T_{ij} does not depend explicitly on x_i .
- (3) There is no preferred direction in space.
- (4) When $D_{ij} = 0$, T_{ij} reduces to $-p\delta_{ij}$.

These statements express the intuitive idea that if a portion of fluid is moving as a rigid body, the surface stresses within the fluid should not differ from the case of the fluid in equilibrium.

To the four statements of Serrin, a fifth is added to linearize the resulting equations and arrive at the

Navier-Stokes equations.

(5) The continuous function of statement (1) is linear in the terms of D_{ij} .

This is often called the Newtonian hypothesis.

Statement (5), along with statements (1) and (2), imply that T_{ij} may be written

$$T_{ij} = A_{ij} + B_{ijkl} D_{kl} ,$$

where A_{ij} and B_{ijkl} are tensors whose terms depend only upon the thermodynamic state of the fluid. Statement (4) implies that

$$A_{ij} = -p\delta_{ij} ,$$

while statement (3) implies that B_{ijkl} is an isotropic tensor. Jeffreys and Swirles (21, p. 87) show that the most general isotropic tensor of the fourth order is of the form

$$\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) ,$$

where λ , μ and γ depend only on the state of the fluid. Since D_{kl} is symmetric and the last term above is antisymmetric with respect to k and l , the double contraction of these tensors will be zero from Equation 58. Therefore T_{ij} can be written

$$\begin{aligned} T_{ij} &= -p\delta_{ij} + \lambda\delta_{ij}\delta_{kl}D_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})D_{kl} \\ &= -p\delta_{ij} + \lambda D_{kk}\delta_{ij} + 2\mu D_{ij} . \end{aligned} \tag{6}$$

By using the constitutive Equation 6, Equation 3 becomes

$$\rho \frac{dv_i}{dt} = \rho F_i + \partial_j \left[-p \delta_{ij} + \lambda D_{kk} \delta_{ij} + 2\mu D_{ij} \right] .$$

This results in the Navier-Stokes equations

$$\rho \frac{dv_i}{dt} = \rho F_i - \partial_i p + \partial_i (\lambda D_{kk}) + 2\partial_j (\mu D_{ij}) , \quad 7)$$

where μ is the viscosity and λ is the bulk viscosity of the fluid.

For the case of constant density, ρ ,

$$\begin{aligned} D_{kk} &= \partial_k v_k \\ &= 0 , \end{aligned}$$

from Equation 2c. Also assuming a constant viscosity, μ , and that the specific body forces can be written as the negative gradient of a potential, g , Equation 7 can be written

$$\begin{aligned} \frac{dv_i}{dt} &= -\partial_i \left(g + \frac{p}{\rho} \right) + 2\frac{\mu}{\rho} \partial_j D_{ij} \\ &= -\partial_i \left(g + \frac{p}{\rho} \right) + \nu \partial_j \left[\partial_j v_i + \partial_i v_j \right] . \end{aligned}$$

The last term is zero from the continuity Equation 2c so that

$$\frac{dv_i}{dt} = -\partial_i \left(g + \frac{p}{\rho} \right) + \nu \partial_j \partial_j v_i , \quad 8a)$$

or

$$\frac{\partial v_i}{\partial t} + v_j \partial_j v_i = - \partial_i \left(g + \frac{p}{\rho} \right) + \nu \partial_j \partial_j v_i, \quad 8b)$$

where ν is the kinematic viscosity defined by

$$\nu = \frac{\mu}{\rho} .$$

Equation 8 is the same equation given by Stokes (42, p. 93). Rouse and Ince (37, pp. 193 ff.) state that essentially the same equation was derived by Navier in 1822 and Poisson in 1829. Both men derived the equation by considering the forces between molecules. Saint-Venant in 1843 and Stokes in 1845 derived Equation 8 on the basis of continuum mechanics. The method used here is primarily from Serrin (41) and Aris (3), but is much like that of Stokes in his paper "On the theories of the internal friction of fluids in motion, and the equilibrium and motion of elastic solids" (42, pp. 75 ff.), updated by tensor notation. Stokes also discusses the theories of his predecessors in this paper.

The validity of the Navier-Stokes equations has been studied both experimentally and theoretically. From a theoretical study, Coleman and Noll (6) report that the Navier-Stokes equations are a complete first-order correction to the theory of perfect fluids in the limit of slow motions. On the question of speed, Truesdell (48) states that for water at one atmosphere pressure, the rate of deformation of the fluid must

be on the order of $7 \times 10^5 \text{ sec}^{-1}$ before the nonlinear terms need be retained. For gases at low speeds and most common liquids, the validity of the assumption of constant density and viscosity has been experimentally verified where there are no large thermal gradients. This is discussed in Dryden, Murnaghan and Bateman (12, pp. 89 ff.). For viscous liquids, in particular, the viscosity is dependent upon the temperature and large variations in temperature must be avoided. McAdams (28, p. 229) and Kays (25, pp. 102 ff.) both discuss nonisothermal flow.

Milne-Thomson (29) notes that in the flow of a continuous medium practically all experimental evidence points to the fact that on solid boundaries, the fluid adheres to the boundary. The boundary conditions used with the Navier-Stokes equations are, therefore, that the fluid velocity is the velocity of the boundary at all solid boundaries.

It is often convenient to write the Navier-Stokes equations in terms of the vorticity, ω_i . By operating on Equation 8b with $\epsilon_{ijk} \partial_j ()_k$,

$$\begin{aligned} & \frac{\partial (\epsilon_{ijk} \partial_j v_k)}{\partial t} + v_m \partial_m (\epsilon_{ijk} \partial_j v_k) + \epsilon_{ijk} \partial_j v_m \partial_m v_k \\ & = - \epsilon_{ijk} \partial_j \partial_k \left(g + \frac{p}{\rho} \right) + \nu \epsilon_{ijk} \partial_j \partial_m \partial_m v_k \quad . \end{aligned} \quad 9)$$

Using the continuity Equation 2c and the relation

$$\epsilon_{ijk} \partial_j v_m \partial_k v_m = 0 ,$$

derived from Equation 58, the third term of Equation 9 can be written

$$\begin{aligned} \epsilon_{ijk} \partial_j v_m \partial_m v_k &= \epsilon_{ijk} \left[\partial_j v_m \partial_m v_k - \partial_j v_m \partial_k v_m \right] \\ &= \epsilon_{ijk} (\delta_{pm} \delta_{qk} - \delta_{pk} \delta_{qm}) \partial_j v_m \partial_p v_q \\ &= \epsilon_{ijk} \epsilon_{npq} \epsilon_{nmk} \partial_j v_m \partial_p v_q \\ &= \epsilon_{npq} (\delta_{in} \delta_{jm} - \delta_{im} \delta_{jn}) \partial_j v_m \partial_p v_q \\ &= \epsilon_{ipq} \partial_j v_j \partial_p v_q - \epsilon_{j pq} \partial_j v_i \partial_p v_q \\ &= - \epsilon_{j pq} \partial_j v_i \partial_p v_q . \end{aligned} \tag{10}$$

Since

$$\omega_i = \epsilon_{ijk} \partial_j v_k , \tag{5}$$

using Equations 58, 2c, and 10, Equation 9 becomes

$$\frac{\partial \omega_i}{\partial t} + v_j \partial_j \omega_i = \omega_j \partial_j v_i + v \partial_j \partial_j \omega_i . \tag{11}$$

FLOW PAST SPHERES AND CYLINDERS

It is instructive to look at one of the classical problems of fluid mechanics, the flow of a viscous fluid past a sphere or infinite circular cylinder. This is one of the earliest and most studied problems in viscous hydrodynamics, and illustrates the transition from the exact solutions formerly used to the numerical solutions which are becoming more popular.

The first solution was given by Stokes in his paper of 1850, "On the effect of the internal friction of fluids on the motion of pendulums" (43, pp. 1 ff.). He made the assumption that the velocities are so small that it is justifiable to neglect the terms in the Navier-Stokes equations involving the square of the velocity. This is equivalent to assuming that the viscous forces completely dominate the inertial forces at all positions of the region studied. For the case of steady motion, this assumption reduces Equation 8 to

$$0 = -\partial_i \left(g + \frac{p}{\rho} \right) + \nu \partial_j \partial_j v_i \quad . \quad 12)$$

This linearization of the Navier-Stokes equations is often called the equation for creeping flow.

Stokes solved Equation 12, together with the continuity Equation 2c, for flow around a sphere with the boundary conditions of zero velocity on the surface of the sphere and a constant velocity, U , in the x -direction at an infinite

distance from the sphere.

In spherical coordinates, Stokes obtained

$$\psi = \frac{Ua^2}{4} \left[-2\left(\frac{r}{a}\right)^2 + 3\left(\frac{r}{a}\right) - \left(\frac{a}{r}\right) \right] \sin^2\theta ,$$

for the stream function. This results in velocities in the r-direction and in the θ -direction given by

$$v_r = -U \left[1 - \frac{3}{2}\left(\frac{a}{r}\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3 \right] \cos\theta ,$$

$$v_\theta = U \left[1 - \frac{3}{4}\left(\frac{a}{r}\right) - \frac{1}{4}\left(\frac{a}{r}\right)^3 \right] \sin\theta ,$$

where a is the radius of the sphere and U is the free stream velocity.

By replacing θ by $\Pi - \theta$ in the equation for the stream function, it can be seen that the stream function upstream of the sphere has the same form as the stream function downstream of the sphere, that is, Stokes' solution does not predict a wake downstream of the sphere.

By integrating the pressure over the surface, Stokes obtained the equation for the drag on a sphere known as Stokes' Law,

$$D = 6\pi\mu aU . \quad (13)$$

Schlichting (39, p. 16) plots this equation along with experimental results, and Dryden, et al. (12, p. 9) reports that

it holds up to Reynolds numbers of 0.5.

When Stokes applied Equation 12 to flow perpendicular to an infinite circular cylinder, he found that the resulting solution could not satisfy all the necessary boundary conditions. Langlois (27, p. 142) notes that the best solution possible is one with a logarithmic singularity at infinity. Stokes reasoned that the difficulty was due to physical, rather than mathematical, phenomena. He argued that as a cylinder is moved in a fluid, the quantity of fluid carried with the cylinder continually increases so that a steady state is never reached, while for a sphere the quantity of fluid does eventually reach a maximum. The viscous forces tending to carry more fluid with the sphere are eventually cancelled out by the frictional forces of the fluid at a distance trying to strip the fluid from the sphere. This is known as Stokes' Paradox.

Oseen in 1910, as reported by Lamb (26, p. 609) and Langlois (27, p. 139), resolved Stokes' Paradox by criticizing the assumption that viscous forces dominate inertia forces at all points in the region. He pointed out that at regions far from the sphere, the reverse is actually the case. Oseen suggested writing the velocity as

$$v_i = U_i + u_i ,$$

where

$$U_1 = U ,$$

$$U_2 = U_3 = 0 ,$$

and then neglecting terms of second order in u_i rather than v_i as was done by Stokes. For steady motion, Equation 8 then reduces to

$$U_j \partial_j u_i = - \partial_i \left(g + \frac{p}{\rho} \right) + \nu \partial_j \partial_j u_i , \quad (14)$$

which is again linear. Oseen rationalized that Equation 14 is a much better approximation to Equation 8 at regions far from the sphere, and at regions near the sphere the viscous terms become dominant so that the left hand side of Equation 14 is negligible. Langlois (27, p. 144) shows that Oseen's solution for the stream function is the same as Stokes' for regions close to the sphere, differing only in terms of the order of Rr/a , where R is the Reynolds number,

$$R = \frac{Ua}{\nu} .$$

The drag on the sphere to one higher order of approximation than Equation 13 is given by Goldstein (16) as

$$D = 6\pi\mu a U \left[1 + \frac{3}{8}R + O(R^2) \right] .$$

The experimental results given by Schlichting fall between this equation and Equation 13.

Lamb (26, p. 614) has solved Equation 14 for the case of

a circular cylinder. As was previously noted, this solution for the creeping flow equation of Stokes is not possible, but when the inertia effects are partially taken into account, a solution is possible.

No significant mathematical solutions were found for these two problems until 1957 when Proudman and Pearson (35) combined the equations of Stokes, Equation 12 and of Oseen, Equation 14. They reasoned that Stokes' equation was valid near the sphere and Oseen's equation was valid far from the sphere, so that a combination of the two equations would result in a more accurate solution. Thus, Proudman and Pearson use two expansions; an expansion near the sphere which satisfies Equation 12 and no-slip boundary conditions on the sphere, and a second expansion far from the sphere which satisfies Equation 14 and the boundary condition of a constant velocity at an infinite distance from the sphere. These two expansions are determined in such a way that their terms match at an intermediate distance from the sphere. Proudman and Pearson give the drag on the sphere as

$$D = 6\pi\mu aU \left[1 + \frac{3}{8}R + \frac{9}{40}R^2 \ln R + O(R^2) \right] ,$$

which is seen to match Goldstein's solution through the first two terms.

They also utilize this technique for the flow past a circular cylinder.

This brief history of a classical problem shows that even

for a hydrodynamic problem involving a rather simple geometry the mathematical solution is not simple. Rather stringent assumptions were made and the resulting equations were still difficult to solve. The approximations used limited the range of Reynolds number for which the solutions were valid. Of the solutions presented here, none are applicable for Reynolds numbers greater than five.

Prandtl's boundary layer theory has also been used to determine the flow field around cylinders and spheres as reported by Schlichting (39). The results are accurate for large Reynolds numbers up to the point of separation of the fluid from the body.

In general, no analytical, as opposed to numeric, solution exists for the flow around spheres and cylinders at intermediate Reynolds numbers. In 1933 Thom (45) developed a numerical technique which he used to solve for the steady state velocity field around a circular cylinder at Reynolds numbers of 10 and 20. In the solution at a Reynolds number of 20, two weak symmetrical vortices resulted downstream of the cylinder. These have been determined experimentally as reported in Dryden, et al. (12, p. 11). At Reynolds numbers of 35 to 45 according to Birkhoff (4, p. 80), and Happel and Brenner (17, p. 40) the two vortices attached to cylinders become unsteady and are alternately shed from the cylinder into the wake.

Due to this phenomenon, a number of numerical studies

have been attempted for a circular cylinder with a Reynolds number of approximately 40. The most successful of these appears to be Kawaguti (24) for $R = 40$, and Apelt (2) for $R = 40$ and 44. The data of these solutions compare favorably with each other and with experimental results.

Jenson (22) has solved the problem for flow past a sphere at Reynolds numbers varying from 5 to 40, and notes that separation, or vorticity formation, occurs at $R = 17$.

In all of these results no indication of shedding of vortices is noted.

Allen and Southwell (1) attempted the numerical study of the flow past circular cylinders at Reynolds numbers of 10, 100, and 1000. Their data do not correlate with experimental data nor with other numerical studies as was noted by Kawaguti (23) and Apelt (2).

Payne (31) has numerically solved the problem of a suddenly accelerated circular cylinder at Reynolds numbers of 40 and 100. He noticed no shedding of the vortices at either Reynolds number.

Fromm and Harlow in a number of papers (14, 15, 18, 19) have discussed a method of solving the unsteady flow around a rectangular cylinder. The results compare, at least qualitatively, with experimental vortex street measurements for cylinders. The boundary conditions used for their earlier results are rather artificial and would be difficult to duplicate experimentally. The 1964 paper (19) utilizes

different boundary conditions at the inlet and exit of the channel and probably gives a closer approximation to experimental results.

The papers of Fromm and Harlow clearly show the development of the vortices and their shedding from the downstream edge of the cylinder. Asymmetric boundary conditions were introduced upstream of the cylinder to aid in the development of the vortex shedding. The 1964 paper also determines the heat transfer from the cylinder.

The NASA publication of Trulio, Carr, Niles, and Rentfrow (49) reports the study of the compressible flow around circular cylinders at a variety of Reynolds numbers. The authors also study the formation and shedding of vortices as did Fromm and Harlow.

FLOW IN A CHANNEL INLET

This study is primarily concerned with the time dependent, laminar flow of a viscous fluid in the inlet section between two suddenly accelerated parallel plates.

The first solution of this problem, for the case of steady motion, was that of Schlichting (40) in 1934. He made the usual boundary layer assumptions of negligible second partial derivatives in the direction of flow and of a negligible first derivative of the pressure perpendicular to the flow. For boundary conditions, Schlichting assumed a constant velocity at the inlet of the channel and a parabolic distribution an infinite distance downstream, along with zero velocity at the walls. Two series solutions were used; the upstream solution being a perturbation of the constant inlet velocity, and the downstream solution being a perturbation of the parabolic downstream velocity distribution. By matching the two series at an intermediate point, Schlichting determined the velocity field in the inlet region.

Collins and Schowalter (7) have determined more accurate numerical data for the Schlichting solution by extending the series to a greater number of terms.

Numerical solutions of the boundary layer equations have been done by a number of people. Two, in particular, have published results for inlet regions. Hornbeck (20) solved the boundary layer equations for the entrance region of a circular

pipe. Bodoia and Osterle (5) solved the boundary layer equations for the inlet region of a channel and compared their results to those of Schlichting. The comparison was not as good as was expected, due to the matching technique used by Schlichting, according to Bodoia and Osterle. Their data compare much better with the data of Collins and Schowalter, as might be expected.

Dix (8) solved the Navier-Stokes equations for the steady state flow of a conducting fluid between non-conducting parallel plates in the presence of a transverse magnetic field.

Wang and Longwell (51) published a steady state solution of the Navier-Stokes equations for the inlet region of a channel. They reported the results of two cases. The first is the classic case of a constant velocity at the inlet of the channel, while the second is an attempt to make the inlet boundary conditions more closely approximate experimental results. They assumed a constant velocity profile some distance upstream of the inlet and then calculated what the velocity profile at the inlet should be.

With the use of digital computers, it appears that there is a tendency of replacement of analytical methods of solving complex differential equations by numerical methods. Analytical methods have required restrictive conditions on the equations of hydrodynamics to allow them to be solved. Numerical methods allow the solution of the complete set of

Navier-Stokes equations, at least in two dimensions. Some methods are available for even the solution of the unsteady Navier-Stokes equations.

It was proposed, therefore, to attempt the solution of the unsteady Navier-Stokes equations for the case of a suddenly accelerated channel by numerical methods. Thom (44), Thom and Apelt (46, 47), and Russell (38) discuss techniques of solving the two dimensional steady state Navier-Stokes equations by numerical means. The method used in this dissertation is an amalgamation of these steady state methods and Fromm's unsteady state method, together with newer techniques of solving the resulting difference equations. The numerical technique used to solve the difference equations was a combination of a successive block overrelaxation method for solving elliptic equations and an alternating direction implicit method for solving parabolic equations. This technique results in fewer calculations and in more stable equations, so that less computer time is needed to achieve steady state convergence.

NUMERICAL METHOD

The equations to be solved by numerical techniques are Equations 2c, 5, and 11. For two semi-infinite parallel plates, the flow is two dimensional. Therefore, Equation 11 can be simplified by noting that,

$$v_3 = 0 ,$$

and

$$\partial_3 () = 0 .$$

These relations with Equation 5 establish that

$$\omega_1 = \omega_2 = 0 .$$

Let

$$\bar{\omega} = \omega_3 .$$

Equation 11 reduces to

$$\frac{\partial \bar{\omega}}{\partial t} + v_j \partial_j \bar{\omega} = v \partial_j \partial_j \bar{\omega} , \quad j = 1, 2. \quad 15)$$

Let the stream function, ψ , be defined by

$$v_i = \epsilon_{3ij} \partial_j \psi . \quad 16)$$

Operating on this equation with $\partial_i ()$,

$$\begin{aligned}\partial_i v_i &= \epsilon_{3ij} \partial_i \partial_j \psi \\ &= 0, \quad i, j = 1, 2\end{aligned}$$

from Equation 58. Thus the introduction of the stream function, ψ , defined by Equation 16 means that the continuity equation 2c is identically satisfied as long as the second derivatives of the stream function are continuous.

Also

$$\begin{aligned}\bar{\omega} &= \omega_3 \\ &= \epsilon_{3ij} \partial_i v_j \\ &= \epsilon_{3ij} \partial_i (\epsilon_{3jk} \partial_k \psi) \\ &= (\delta_{3k} \delta_{i3} - \delta_{33} \delta_{ik}) \partial_i \partial_k \psi\end{aligned}$$

$$\bar{\omega} = -\partial_i \partial_i \psi, \quad i = 1, 2. \quad (17)$$

Equation 16 used in Equation 15 results in

$$\frac{\partial \bar{\omega}}{\partial t} + \epsilon_{3jk} \partial_j \psi \partial_k \bar{\omega} = \nu \partial_j \partial_j \bar{\omega}, \quad j, k = 1, 2. \quad (18)$$

Equations 17 and 18 are two partial differential equations to be solved for the vorticity field, $\bar{\omega}$, and the stream function field, ψ . In vector notation they become

$$\frac{\partial \bar{\omega}}{\partial t} + \frac{\partial (\bar{\omega}, \psi)}{\partial (x_1, x_2)} = \nu \nabla_1^2 \bar{\omega}, \quad (19)$$

and

$$\nabla_1^2 \psi = -\bar{\omega} \quad 20)$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} .$$

The velocities as given by Equation 16 are

$$v_1 = \frac{\partial \psi}{\partial x_2} \quad 21a)$$

$$v_2 = - \frac{\partial \psi}{\partial x_1} . \quad 21b)$$

If U is a characteristic velocity of a set of solutions of Equations 17 and 18 with given boundary conditions, d is a characteristic length, and ν is the kinematic viscosity, let

$$u = \frac{v_1}{U} ,$$

$$v = \frac{v_2}{U} ,$$

$$x = \frac{x_1}{d} ,$$

and

$$y = \frac{x_2}{d} .$$

Then using Equations 21

$$\begin{aligned}
 u &= \frac{v_1}{U} \\
 &= \frac{1}{Ud} \frac{\partial \psi}{\partial \left(\frac{x_2}{d}\right)} \\
 &= \frac{\partial \phi}{\partial y} , \qquad \qquad \qquad 22a)
 \end{aligned}$$

where

$$\phi = \frac{\psi}{Ud} .$$

Similarly

$$v = - \frac{\partial \phi}{\partial x} . \qquad \qquad \qquad 22b)$$

Transforming Equation 19 and 20 to this nondimensional form results in

$$\frac{\partial \omega}{\partial \tau} + R \frac{\partial (\omega, \phi)}{\partial (x, y)} = \nabla^2 \omega , \qquad \qquad \qquad 23)$$

and

$$\nabla^2 \phi = - \omega , \qquad \qquad \qquad 24)$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} ,$$

$$\omega = \frac{\bar{\omega}d}{U} ,$$

$$\tau = \frac{vt}{d^2} ,$$

and

$$R = \frac{Ud}{\nu}$$

R is, of course, the Reynolds number. In the pages to follow, ϕ , the nondimensional stream function, will be called simply the stream function. Similarly ω will be called the vorticity, and τ , the time.

Initial and Boundary Conditions

Two different sets of initial and boundary conditions were used to solve Equations 23 and 24.

The first set, case I, was the classical problem of a constant velocity across the inlet of the channel after the plates were accelerated at time $\tau = 0$.

If d is the distance between the parallel plates, the flow will be symmetric about the line.

$$y = \frac{1}{2} .$$

Using this condition, the initial conditions for a suddenly accelerated channel are

$$\left. \begin{array}{l} u = 1 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} 0 \leq x \leq L \\ 0 \leq y \leq \frac{1}{2} \\ \tau = 0 \end{array} \right. \quad 25)$$

where L is a large distance downstream of the inlet. The boundary conditions are

$$\left. \begin{array}{l} u = 1 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = 0 \\ 0 \leq y \leq \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 26a)$$

$$\left. \begin{array}{l} \frac{\partial u}{\partial x} = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = L \\ 0 \leq y \leq \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 26b)$$

$$\left. \begin{array}{l} u = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} 0 \leq x \leq L \\ y = 0 \\ \tau > 0 \end{array} \right. \quad 26c)$$

and

$$\left. \begin{array}{l} \frac{\partial u}{\partial y} = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} 0 \leq x \leq L \\ y = \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 26d)$$

From Equation 5, using the nondimensional variables,

$$\begin{aligned}\omega &= \frac{d}{U} \left[\epsilon_{3ij} \partial_i v_j \right] \\ &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} .\end{aligned}\tag{27}$$

At the inlet, from Equations 26a and 27,

$$\omega(0, y) = \frac{\partial v}{\partial x} .$$

It is assumed that ω is continuous at the inlet and that the velocity of the fluid before it enters the channel has a constant velocity of U in the x -direction with no velocity component in the y -direction. Therefore,

$$\left. \frac{\partial v}{\partial x} \right|_{x=0^-} = 0 ,$$

or

$$\omega(0, y) = 0 .$$

With this comment, the initial and boundary conditions can be easily written in terms of the stream function and the vorticity. If c is an arbitrary constant,

$$\left. \begin{aligned}\phi &= y + c \\ \omega &= 0\end{aligned} \right\} , \quad \left\{ \begin{aligned}0 &\leq x \leq L \\ 0 &\leq y \leq \frac{1}{2} \\ \tau &= 0 ,\end{aligned} \right. \tag{28}$$

$$\left. \begin{array}{l} \phi = y + c \\ \omega = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} x = 0 \\ 0 < y < \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 29a)$$

$$\left. \begin{array}{l} \frac{\partial \phi}{\partial x} = 0 \\ \frac{\partial \omega}{\partial x} = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} x = L \\ 0 < y < \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 29b)$$

$$\left. \begin{array}{l} \phi = c \\ \frac{\partial \phi}{\partial y} = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} 0 < x < L \\ y = 0 \\ \tau > 0 \end{array} \right. \quad 29c)$$

$$\left. \begin{array}{l} \phi = \frac{1}{2} + c \\ \omega = 0 \end{array} \right\}, \quad \left\{ \begin{array}{l} 0 < x < L \\ y = \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 29d)$$

The second set of initial and boundary conditions to be used, case II, is the same as that used by Wang and Longwell (51) for the steady state. The channel studied is imagined as two interior plates in a cascade of similar plates. The velocity at a large distance upstream of the plates, L' , is a constant value U parallel to the plates after the cascade is accelerated at time $\tau = 0$. With a coordinate system attached to the leading edge of a plate, the initial and boundary conditions are

$$\left. \begin{array}{l} u = 1 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' \leq x \leq L \\ 0 \leq y \leq \frac{1}{2} \\ \tau = 0 \end{array} \right.$$

$$\left. \begin{array}{l} u = 1 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = -L' \\ 0 \leq y \leq \frac{1}{2} \\ \tau > 0 \end{array} \right.$$

$$\left. \begin{array}{l} \frac{\partial u}{\partial x} = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = L \\ 0 \leq y \leq \frac{1}{2} \\ \tau > 0 \end{array} \right.$$

$$\left. \begin{array}{l} \frac{\partial u}{\partial y} = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' \leq x < 0 \\ y = 0 \\ \tau > 0 \end{array} \right.$$

$$\left. \begin{array}{l} u = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} 0 \leq x \leq L \\ y = 0 \\ \tau > 0 \end{array} \right.$$

and

$$\left. \begin{array}{l} \frac{\partial u}{\partial y} = 0 \\ v = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' \leq x \leq L \\ y = \frac{1}{2} \\ \tau > 0 \end{array} \right.$$

In terms of the vorticity and the stream function, the

boundary conditions for case II are

$$\left. \begin{array}{l} \phi = y + c \\ \omega = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' \leq x \leq L \\ 0 \leq y \leq \frac{1}{2} \\ \tau = 0 \end{array} \right. \quad 30)$$

$$\left. \begin{array}{l} \phi = y + c \\ \omega = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = -L' \\ 0 < y < \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 31a)$$

$$\left. \begin{array}{l} \frac{\partial \phi}{\partial x} = 0 \\ \frac{\partial \omega}{\partial x} = 0 \end{array} \right\} , \left\{ \begin{array}{l} x = L \\ 0 < y < \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 31b)$$

$$\left. \begin{array}{l} \phi = c \\ \omega = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' < x < 0 \\ y = 0 \\ \tau > 0 \end{array} \right. \quad 31c)$$

$$\left. \begin{array}{l} \phi = c \\ \frac{\partial \phi}{\partial y} = 0 \end{array} \right\} , \left\{ \begin{array}{l} 0 < x < L \\ y = 0 \\ \tau > 0 \end{array} \right. \quad 31d)$$

$$\left. \begin{array}{l} \phi = \frac{1}{2} + c \\ \omega = 0 \end{array} \right\} , \left\{ \begin{array}{l} -L' < x < L \\ y = \frac{1}{2} \\ \tau > 0 \end{array} \right. \quad 31e)$$

Difference Equations

If a function $f(z)$ is expanded about the point z_0 in a Taylor series

$$f(z_0 + h) = f(z_0) + hf'(z_0) + \frac{h^2}{2}f''(z_0) + \frac{h^3}{6}f'''(z_0) + O(h^4) .$$

Similarly,

$$f(z_0 - g) = f(z_0) - gf'(z_0) + \frac{g^2}{2}f''(z_0) - \frac{g^3}{6}f'''(z_0) + O(h^4) .$$

Adding these two equations and setting g equal to h results in

$$f''(z_0) = \frac{f(z_0 + h) - 2f(z_0) + f(z_0 - h)}{h^2} + O(h^2) .$$

Also, upon subtracting the equations and rearranging,

$$f'(z_0) = \frac{f(z_0+h) - f(z_0-g)}{h + g} + \frac{(h-g)}{2}f''(z_0) + O(h^2 + g^2) .$$

If g is equal to h in this equation,

$$f'(z_0) = \frac{f(z_0 + h) - f(z_0 - h)}{2h} + O(h^2) .$$

With these results Equation 23 can be written

$$\begin{aligned}
& \frac{1}{\Delta\tau} \left[\omega(x, y, \tau + \Delta\tau) - \omega(x, y, \tau) \right] + O(\Delta\tau) \\
& - \frac{R}{4h^2} \left\{ \left[\omega(x, y+h, \tau) - \omega(x, y-h, \tau) \right] \left[\phi(x+h, y, \tau) - \phi(x-h, y, \tau) \right] \right. \\
& \left. - \left[\omega(x+h, y, \tau) - \omega(x-h, y, \tau) \right] \left[\phi(x, y+h, \tau) - \phi(x, y-h, \tau) \right] \right\} + O(h^2) \\
& = \frac{1}{h^2} \left[\omega(x+h, y, \tau) + \omega(x-h, y, \tau) + \omega(x, y+h, \tau) \right. \\
& \quad \left. + \omega(x, y-h, \tau) - 4\omega(x, y, \tau) \right] + O(h^2) \quad . \quad 32)
\end{aligned}$$

and Equation 24 can be written

$$\begin{aligned}
& \frac{1}{h^2} \left[\phi(x+h, y, \tau) + \phi(x-h, y, \tau) + \phi(x, y+h, \tau) \right. \\
& \quad \left. + \phi(x, y-h, \tau) - 4\phi(x, y, \tau) \right] + O(h^2) \\
& = -\omega(x, y, \tau) \quad . \quad 33)
\end{aligned}$$

Let

$$\tau_n = n\Delta\tau \quad ,$$

$$x_i = i\Delta x = ih \quad ,$$

and

$$y_j = j\Delta y = jh \quad .$$

Also use the notation

$$\omega_{i,j}^n = \omega(x_i, y_j, \tau_n) \quad ,$$

and

$$\phi_{i,j}^n = \phi(x_i, y_j, \tau_n) \quad .$$

Then Equations 23 and 24 are approximated by neglecting the higher order terms in Equations 32 and 33. This results in a numerical method of order $(h^2 + \Delta\tau)$. Douglas (9) has reported a method utilizing an equation very similar to Equation 32 which is of order $(h^2 + \Delta\tau^2)$. This method was tried but the difference equations with the boundary conditions used were not stable for large $\Delta\tau$. Thus the extra accuracy gained by the method was not utilized due to the small $\Delta\tau$ necessary for stability. Equation 32 was modified somewhat to allow for derivation of the alternating direction method used in this study.

Equation 33 is approximated by

$$\phi_{i+1,j}^n + \phi_{i-1,j}^n + \phi_{i,j+1}^n + \phi_{i,j-1}^n - 4\phi_{i,j}^n = -h^2\omega_{i,j}^n \quad , \quad 34)$$

and Equation 32 is approximated by

$$\begin{aligned}
& \frac{\omega_{i,j}^{n+\theta} - \omega_{i,j}^n}{\theta \Delta \tau} - \frac{R}{4h^2} \left[(\omega_{i,j+1}^{n+\theta'} - \omega_{i,j-1}^{n+\theta'}) (\phi_{i+1,j}^{n+\theta'} - \phi_{i-1,j}^{n+\theta'}) \right. \\
& \quad \left. - (\omega_{i+1,j}^{n+\theta'} - \omega_{i-1,j}^{n+\theta'}) (\phi_{i,j+1}^{n+\theta'} - \phi_{i,j-1}^{n+\theta'}) \right] \\
& = \frac{1}{h^2} \left[\omega_{i+1,j}^{n+\theta''} + \omega_{i-1,j}^{n+\theta''} - 2\omega_{i,j}^{n+\theta''} + \omega_{i,j+1}^{n+\theta'''} + \omega_{i,j-1}^{n+\theta'''} - 2\omega_{i,j}^{n+\theta'''} \right] .
\end{aligned}$$

35a)

Define the operators

$$\Delta_x(f^n) = f_{i+1,j}^n - f_{i-1,j}^n ,$$

$$\Delta_y(f^n) = f_{i,j+1}^n - f_{i,j-1}^n ,$$

$$\Delta_x^2(f^n) = f_{i+1,j}^n - 2f_{i,j}^n + f_{i-1,j}^n ,$$

and

$$\Delta_y^2(f^n) = f_{i,j+1}^n - 2f_{i,j}^n + f_{i,j-1}^n .$$

Then Equation 35a can be written

$$\begin{aligned} \frac{\omega_{i,j}^{n+\theta} - \omega_{i,j}^n}{\theta \Delta \tau} - \frac{R}{4h^2} \left[\Delta_Y(\omega^{n+\theta'}) \Delta_X(\phi^{n+\theta'}) - \Delta_X(\omega^{n+\theta'}) \Delta_Y(\phi^{n+\theta'}) \right] \\ = \frac{1}{h^2} \left[\Delta_X^2(\omega^{n+\theta''}) + \Delta_Y^2(\omega^{n+\theta''''}) \right] . \end{aligned} \quad 35b)$$

Successive Block Overrelaxation

Let it be assumed that Equation 34 is to be solved in the rectangular region,

$$0 \leq i \leq M + 1 ,$$

$$0 \leq j \leq N + 1 ,$$

with $\omega_{i,j}^n$ given throughout the region and $\phi_{i,j}^n$ given on the boundary. Further, let the $(N \times 1)$ vectors ϕ_i , ϕ_i^0 , and Ω_i , and the $(N \times N)$ matrix T be defined by

$$\phi_i = \begin{bmatrix} \phi_{i,1}^n \\ \phi_{i,2}^n \\ \vdots \\ \phi_{i,N}^n \end{bmatrix} ,$$

$$\phi_i^0 = \begin{bmatrix} \phi_{i,0}^n \\ 0 \\ \cdot \\ \cdot \\ 0 \\ \phi_{i,N+1}^n \end{bmatrix},$$

$$\Omega_i = h^2 \begin{bmatrix} \omega_{i,1}^n \\ \omega_{i,2}^n \\ \cdot \\ \cdot \\ \cdot \\ \omega_{i,N}^n \end{bmatrix},$$

and

$$T = \begin{bmatrix} 4 & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & 4 & -1 & & & & \\ 0 & -1 & 4 & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \\ 0 & & \cdot & \cdot & \cdot & & 4 \end{bmatrix},$$

where $1 \leq i \leq M$. Then Equation 34 can be written

$$T\phi_i - \phi_{i+1} - \phi_{i-1} = \Omega_i + \phi_i^0, \quad 1 \leq i \leq M. \quad (36)$$

Equation 36 can be solved in a number of ways, the best of them outlined in Varga (50) and in Forsythe and Wasow (13). The method used was the successive block or line overrelaxation (SBOR) iterative method described in both Varga (p. 194) and Forsythe and Wasow (p. 266). This method was chosen as it has a better asymptotic rate of convergence than the more common successive point overrelaxation. Also, due to the simple form of T, it does not require much additional computation. SBOR can be written as

$$T\hat{\phi}_i^{[m+1]} = \Omega_i + \phi_i^0 + \phi_{i+1}^{[m]} + \phi_{i-1}^{[m+1]} , \quad 37a)$$

$$\phi_i^{[m+1]} = \phi_i^{[m]} + r \left[\hat{\phi}_i^{[m+1]} - \phi_i^{[m]} \right] , \quad 37b)$$

$$1 \leq i \leq M ,$$

where the superscript [m] is the iteration parameter, and r is the relaxation factor.

The tridiagonal matrix T of Equation 37a is easily inverted to solve for $\hat{\phi}_i^{[m+1]}$ by Gaussian elimination as described in the following algorithm. Let Equation 37a be written in the form

$$A X = K , \quad 38a)$$

where X and K are (N×1) vectors and A is an (N×N) matrix

$$A = \begin{bmatrix} a & -1 & 0 & \cdot & \cdot & \cdot & 0 \\ -1 & a & -1 & & & & \\ 0 & -1 & a & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & \cdot \\ \cdot & & & & & \cdot & \\ 0 & & \cdot & \cdot & \cdot & & a \end{bmatrix} .$$

Define the components of the vector f by

$$f_1 = \frac{1}{a} ,$$

$$f_j = \frac{1}{a - f_{j-1}} , \quad 2 \leq j \leq N . \quad 38b)$$

Then X may be determined from the following algorithm,

$$g_1 = f_1 K_1 ,$$

$$g_j = f_j [K_j + g_{j-1}] , \quad 2 \leq j \leq N , \quad 38c)$$

and

$$x_N = g_N ,$$

$$x_j = g_j + f_j x_{j+1} , \quad 1 \leq j \leq N-1 . \quad 38d)$$

With $a = 4$, this algorithm can be used to solve Equation 37a.

The value of r was chosen such that the asymptotic rate of convergence was a maximum. Varga (50) and Forsythe and Wasow (13) show that this value of r is given by

$$r = \frac{2}{1 + \sqrt{1 - \sigma^2}}, \quad 39)$$

where σ is the spectral radius of the block Jacobi matrix or the matrix of the method of simultaneous displacement by lines. This value of r minimizes the spectral radius of the SBOR matrix associated with Equation 37.

The block Jacobi matrix is the matrix of the method described by

$$T\phi_i^{[m+1]} = \phi_{i+1}^{[m]} + \phi_{i-1}^{[m]} + (\Omega_i + \phi_i^0), \quad 1 \leq i \leq M,$$

or

$$\begin{bmatrix} T & 0 & 0 & \dots & 0 \\ 0 & T & 0 & & \\ 0 & 0 & T & & \\ \vdots & & & \ddots & \\ 0 & & & & T \end{bmatrix} \begin{bmatrix} \phi_1^{[m+1]} \\ \phi_2^{[m+1]} \\ \vdots \\ \phi_M^{[m+1]} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ I & 0 & I & & \\ 0 & I & 0 & & \\ \vdots & & & \ddots & \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} \phi_1^{[m]} \\ \vdots \\ \phi_M^{[m]} \end{bmatrix}$$

$$+ \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \vdots \\ \Omega_M \end{bmatrix} + \begin{bmatrix} \phi_1^0 \\ \phi_2^0 \\ \vdots \\ \phi_M^0 \end{bmatrix} + \begin{bmatrix} \phi_0 \\ 0 \\ \vdots \\ 0 \\ \phi_{M+1} \end{bmatrix},$$

where I is the $(N \times N)$ identity matrix. The block Jacobi matrix, B , is then

$$B = \begin{bmatrix} T & 0 & 0 & \dots & 0 \\ 0 & T & 0 & & \\ 0 & 0 & T & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & & & & T \end{bmatrix}^{-1} \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ I & 0 & I & & \\ 0 & I & 0 & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} T^{-1} & 0 & 0 & \dots & 0 \\ 0 & T^{-1} & 0 & & \\ 0 & 0 & T^{-1} & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & & & & T^{-1} \end{bmatrix} \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ I & 0 & I & & \\ 0 & I & 0 & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & T^{-1} & 0 & \dots & 0 \\ T^{-1} & 0 & T^{-1} & & \\ 0 & T^{-1} & 0 & & \\ \vdots & & & \ddots & \\ \vdots & & & & \ddots \\ 0 & & & & 0 \end{bmatrix}$$

Let X_i be an $(N \times 1)$ vector with components $X_{i,j}$. Then

the eigenvalues of B are given by the solutions, $\lambda_{m,n}$, of the equation

$$B \begin{bmatrix} X_{1.} \\ X_{2.} \\ \cdot \\ \cdot \\ \cdot \\ X_{M.} \end{bmatrix} = \lambda_{m,n} \begin{bmatrix} X_{1.} \\ X_{2.} \\ \cdot \\ \cdot \\ \cdot \\ X_{M.} \end{bmatrix}$$

When B is replaced by its equivalent and the equation put in difference notation, it becomes

$$T^{-1}X_{i-1.} + T^{-1}X_{i+1.} = \lambda_{m,n}X_{i.} \quad , \quad 1 \leq i \leq M$$

$$X_{0.} = X_{M+1.} = 0 \quad .$$

Premultiplying by T results in

$$X_{i-1.} + X_{i+1.} = \lambda_{m,n}TX_{i.}$$

or

$$X_{i-1,j} + X_{i+1,j} = \lambda_{m,n} [4X_{i,j} - X_{i,j+1} - X_{i,j-1}] \quad . \quad 40a)$$

Therefore the solution of the eigenvalue problem for B is equivalent to the solution of the difference Equation 40a with

the boundary conditions

$$\begin{aligned}
 X_{0,j} = X_{i,0} = X_{M+1,j} = X_{i,N+1} = 0 \quad , \\
 1 \leq i \leq M \\
 1 \leq j \leq N \quad . \quad 40b)
 \end{aligned}$$

Since the boundary conditions imply that the solution must reduce to zero at all boundaries, try a solution of the form

$$X_{i,j} = \sin pi \sin qj \quad .$$

When this is placed in Equation 40a,

$$\begin{aligned}
 & \sin p(i+1) \sin qj + \sin p(i-1) \sin qj \\
 + \lambda & \left[\sin pi \sin q(j+1) + \sin pi \sin q(j-1) - 4\sin pi \sin qj \right] \\
 & = 0 \quad .
 \end{aligned}$$

On applying the identity,

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \quad ,$$

this becomes

$$\begin{aligned}
 2\sin pi \sin qj \cos p + 2\lambda_{m,n} \sin pi \sin qj \cos q \\
 - 4\lambda_{m,n} \sin pi \sin qj = 0 \quad .
 \end{aligned}$$

Solving for $\lambda_{m,n}$ gives

$$\lambda_{m,n} = \frac{\cos p}{2 - \cos q} .$$

The first two boundary conditions of 40b are already satisfied due to the form chosen for $X_{i,j}$. To satisfy the third and fourth boundary conditions it is necessary that

$$\sin p(M + 1) = 0 ,$$

$$\sin q(N + 1) = 0 .$$

This implies

$$p = \frac{m\pi}{M + 1} , \quad 1 \leq m \leq M ,$$

$$q = \frac{n\pi}{N + 1} , \quad 1 \leq n \leq N .$$

Therefore

$$\lambda_{m,n} = \frac{\cos \frac{m\pi}{M+1}}{2 - \cos \frac{n\pi}{N+1}} , \quad \begin{cases} 1 \leq m \leq M , \\ 1 \leq n \leq N . \end{cases}$$

This determines the MN eigenvalues of the matrix, B . Since the spectral radius, σ , is the maximum of the absolute values of the eigenvalues,

$$\begin{aligned}
\sigma &= \max_{m,n} |\lambda_{m,n}| \\
&= \frac{\cos \frac{\pi}{M+1}}{2 - \cos \frac{\pi}{N+1}} \\
&= \frac{1 - \frac{1}{2} \left(\frac{\pi}{M+1} \right)^2 + \frac{1}{24} \left(\frac{\pi}{M+1} \right)^4 + \dots}{1 + \frac{1}{2} \left(\frac{\pi}{N+1} \right)^2 - \frac{1}{24} \left(\frac{\pi}{N+1} \right)^4 + \dots} \\
&= \left[1 - \frac{1}{2} \left(\frac{\pi}{M+1} \right)^2 + \frac{1}{24} \left(\frac{\pi}{M+1} \right)^4 - \dots \right] \\
&\quad \times \left[1 - \frac{1}{2} \left(\frac{\pi}{N+1} \right)^2 + \frac{7}{24} \left(\frac{\pi}{N+1} \right)^4 + \dots \right] \\
&= 1 - \frac{\pi^2}{2} \left[\left(\frac{1}{M+1} \right)^2 + \left(\frac{1}{N+1} \right)^2 \right] + O \left\{ \left[\frac{\pi}{M+1} \right]^4 + \left[\frac{\pi}{N+1} \right]^4 \right\} ,
\end{aligned}$$

or

$$\sigma \approx 1 - \frac{\pi^2}{2} \left[\left(\frac{1}{M+1} \right)^2 + \left(\frac{1}{N+1} \right)^2 \right] . \quad (41)$$

Since both M and N will be greater than 10, the approximation is within about 1 percent.

Alternating Direction Implicit Method

The solution of the numerical approximation of the parabolic partial differential Equation 23 can also be accomplished in a number of ways. Douglas (10) has described a number of ways to solve parabolic equations.

The method used was an alternating direction implicit (ADI) method described by Douglas (9). This is a further modification of the method first used by Peaceman and Rachford (32) and modified by Douglas and Rachford (11).

The ADI method results upon applying Equation 35 twice per time step with two sets of values for θ , θ' , θ'' , and θ''' .

The first half-step is taken with

$$\theta = \frac{1}{2} ,$$

$$\theta' = 0 ,$$

$$\theta'' = \frac{1}{2} ,$$

and

$$\theta''' = 0 .$$

That is,

$$\begin{aligned} \frac{\omega_{i,j}^{n+1/2} - \omega_{i,j}^n}{\Delta\tau/2} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ = \frac{1}{h^2} \Delta_X^2(\omega^{n+1/2}) + \frac{1}{h^2} \Delta_Y^2(\omega^n) . \end{aligned} \quad (42)$$

The second half-step is taken with

$$\theta = \frac{1}{2} ,$$

$$\theta' = -\frac{1}{2} ,$$

$$\theta'' = 0 ,$$

and

$$\theta''' = \frac{1}{2} ,$$

and with n replaced by $n+\frac{1}{2}$.

$$\begin{aligned} & \frac{\omega^{n+1} - \omega^{n+1/2}}{\Delta\tau/2} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ & = \frac{1}{h^2} \Delta_X^2(\omega^{n+1/2}) + \frac{1}{h^2} \Delta_Y^2(\omega^{n+1}) . \end{aligned} \quad 43)$$

Subtracting Equation 43 from 42 results in

$$2\omega_{i,j}^{n+1/2} = \omega_{i,j}^{n+1} + \omega_{i,j}^n - \frac{\Delta\tau}{2h^2} \Delta_Y^2(\omega^{n+1} - \omega^n) .$$

When this equation is used in either Equation 42 or 43, the resulting equation is that for the total time step,

$$\begin{aligned} & \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^n}{\Delta\tau} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ & = \frac{1}{2h^2} \left[\Delta_X^2 + \Delta_Y^2 \right] \left[\omega^{n+1} + \omega^n \right] - \frac{\Delta\tau}{4h^4} \Delta_X^2 \Delta_Y^2 (\omega^{n+1} - \omega^n) . \end{aligned} \quad 44)$$

Douglas (9) notes that the final term is of order $(\Delta\tau)^2$.

When Equations 42 and 43 are written in matrix notation rather than finite difference notation, they both require the inversion of a tridiagonal matrix. This is easily accomplished by the algorithm defined by Equations 38.

However the method described by Douglas results in simpler equations to solve and gives the same final Equation 44. Let $\omega_{i,j}^*$ be defined by

$$\begin{aligned} \frac{\omega_{i,j}^* - \omega_{i,j}^n}{\Delta\tau} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ = \frac{1}{2h^2} \Delta_X^2(\omega^* + \omega^n) + \frac{1}{h^2} \Delta_Y^2(\omega^n) \quad , \end{aligned} \quad (45)$$

and let $\omega_{i,j}^{n+1}$ be defined by

$$\begin{aligned} \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^n}{\Delta\tau} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ = \frac{1}{2h^2} \Delta_X^2(\omega^* + \omega^n) + \frac{1}{2h^2} \Delta_Y^2(\omega^{n+1} + \omega^n) \quad . \end{aligned} \quad (46)$$

Subtracting Equation 46 from 45 gives

$$\omega^* = \omega^{n+1} - \frac{\Delta\tau}{2h^2} \Delta_Y^2(\omega^{n+1} - \omega^n) \quad . \quad (47)$$

Replacing this equation for ω^* in either Equation 45 or 46 results in Equation 44, so that the two methods are equivalent

for the total time step.

Let

$$\alpha = \frac{h^2}{\Delta\tau} .$$

Then Equations 45 and 47 can be put in the form

$$\begin{aligned} (\Delta_X^2 - 2\alpha)\omega^* &= - (\Delta_X^2 + 2\Delta_Y^2 + 2\alpha)\omega^n \\ &+ \frac{R}{2} \left[\Delta_Y(\omega^n)\Delta_X(\phi^n) - \Delta_X(\omega^n)\Delta_Y(\phi^n) \right] , \end{aligned} \quad 48a)$$

$$(\Delta_Y^2 - 2\alpha)\omega^{n+1} = \Delta_Y^2(\omega^n) - 2\alpha\omega^* . \quad 48b)$$

These equations are both tridiagonal in form and can be solved with Equations 38 for the values of the vorticity at time (n+1) given the vorticity and the stream function at time n, and the boundary values of the vorticity at time (n+1).

Difference Initial and Boundary Conditions

The initial and boundary conditions to be used with Equations 37 and 48 are derived from the initial and boundary conditions for the partial differential equations which the difference equations approximate.

Expanding the stream function,

$$\phi_{i,j}^n = \phi_{i,j-1}^n + h \left(\frac{\partial \phi}{\partial y} \right)_{i,j-1}^n + \frac{h^2}{2} \left(\frac{\partial^2 \phi}{\partial y^2} \right)_{i,j-1}^n + \frac{h^3}{6} \left(\frac{\partial^3 \phi}{\partial y^3} \right)_{i,j-1}^n + \dots .$$

From Equation 24,

$$-\omega = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} ,$$

so that

$$\begin{aligned} \phi_{i,j}^n &= \phi_{i,j-1}^n + h \left(\frac{\partial \phi}{\partial y} \right)_{i,j-1}^n \\ &\quad - \frac{h^2}{2} \left[\omega_{i,j-1}^n + \left(\frac{\partial^2 \phi}{\partial x^2} \right)_{i,j-1}^n \right] + \frac{h^3}{6} \left(\frac{\partial^3 \phi}{\partial y^3} \right)_{i,j-1}^n + \dots . \end{aligned}$$

Along the wall of the channel, $j = 0$,

$$\left(\frac{\partial \phi}{\partial y} \right)_{i,0}^n = \left(\frac{\partial \phi}{\partial x} \right)_{i,0}^n = 0 , \quad \begin{array}{l} 0 < x < L \\ n > 0 \end{array} ,$$

from Equation 29c or 31d. Therefore

$$\phi_{i,1}^n = \phi_{i,0}^n - \frac{h^2}{2} \omega_{i,0}^n + O(h^3) .$$

Solving for the vorticity and neglecting higher order terms,

$$\omega_{i,0}^n = \frac{2(\phi_{i,0}^n - \phi_{i,1}^n)}{h^2} , \quad \begin{cases} 0 < i \leq M \\ n > 0 \end{cases} .$$

With this relation, the initial and boundary conditions for the finite difference approximations can be determined from Equations 28, 29, 30, and 31.

For case I, with $c = 0$, $M+1 = L/h$, and $N+1 = l/2h$,

$$\left. \begin{array}{l} \phi_{i,j}^0 = jh \\ \omega_{i,j}^0 = 0 \end{array} \right\}, \left\{ \begin{array}{l} 0 \leq i \leq M+1 \\ 0 \leq j \leq N+1 \end{array} \right., \quad 49)$$

$$\left. \begin{array}{l} \phi_{0,j}^n = jh \\ \omega_{0,j}^n = 0 \end{array} \right\}, \left\{ \begin{array}{l} 0 < j < N+1 \\ n > 0 \end{array} \right., \quad 50a)$$

$$\left. \begin{array}{l} \phi_{M+1,j}^n = \phi_{M,j}^n \\ \omega_{M+1,j}^n = \omega_{M,j}^n \end{array} \right\}, \left\{ \begin{array}{l} 0 < j < N+1 \\ n > 0 \end{array} \right., \quad 50b)$$

$$\left. \begin{array}{l} \phi_{i,0}^n = 0 \\ \omega_{i,0}^n = -\frac{2\phi_{i,1}^n}{h^2} \end{array} \right\}, \left\{ \begin{array}{l} 0 < i < M+1 \\ n > 0 \end{array} \right., \quad 50c)$$

$$\left. \begin{array}{l} \phi_{i,N+1}^n = (N+1)h \\ \omega_{i,N+1}^n = 0 \end{array} \right\}, \left\{ \begin{array}{l} 0 < i < M+1 \\ n > 0 \end{array} \right. . \quad 50d)$$

For case II with $M' = L'/h$, and $M+1 = (L+L')/h$,

$$\left. \begin{array}{l} \phi_{i,j}^0 = jh \\ \omega_{i,j}^0 = 0 \end{array} \right\}, \left\{ \begin{array}{l} 0 \leq i \leq M+1 \\ 0 \leq j \leq N+1 \end{array} \right., \quad 51)$$

$$\left. \begin{aligned} \phi_{0,j}^n &= jh \\ \omega_{0,j}^n &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 < j < N + 1 \\ n > 0 \end{aligned} \right. \quad 52a)$$

$$\left. \begin{aligned} \phi_{M+1,j}^n &= \phi_{M,j}^n \\ \omega_{M+1,j}^n &= \omega_{M,j}^n \end{aligned} \right\} , \left\{ \begin{aligned} 0 < j < N + 1 \\ n > 0 \end{aligned} \right. \quad 52b)$$

$$\left. \begin{aligned} \phi_{i,0}^n &= 0 \\ \omega_{i,0}^n &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 < i \leq M' \\ n > 0 \end{aligned} \right. \quad 52c)$$

$$\left. \begin{aligned} \phi_{i,0}^n &= 0 \\ \omega_{i,0}^n &= -\frac{2\phi_{i,1}^n}{h^2} \end{aligned} \right\} , \left\{ \begin{aligned} M' < i < M + 1 \\ n > 0 \end{aligned} \right. \quad 52d)$$

$$\left. \begin{aligned} \phi_{i,N+1}^n &= (N+1)h \\ \omega_{i,N+1}^n &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 < i < M + 1 \\ n > 0 \end{aligned} \right. \quad 52e)$$

Numerical Algorithm

The method of solution of Equations 37 and 48 with boundary conditions 49 and 50 or 51 and 52 is as follows.

Assume values are known for $\phi_{i,j}^n$ and $\omega_{i,j}^n$, $0 \leq i \leq M+1$, $0 \leq j \leq N+1$, for a given value of n . Equation 48 could be solved for $\omega_{i,j}^{n+1}$ if the boundary values of the vorticity were known, but Equations 49c and 51d show that these boundary values are

a function of $\phi_{i,j}^{n+1}$ which is in turn a function of $\omega_{i,j}^{n+1}$. Hence an iterative technique must be used.

As a first approximation, the boundary values of the vorticity at time n are used for the boundary values at time $(n+1)$. Therefore Equation 48 can be solved for an approximation of the vorticity field at time $(n+1)$. The solution is only approximate due to the approximate boundary values used. Equation 37 is then iteratively solved for the stream function field corresponding to the approximate vorticity field. At each iteration, the downstream boundary values of the stream function are changed to correspond with Equation 50b or 52b. This new stream function is then used in either Equation 50c or 52d to obtain a better approximation to the boundary values of the vorticity on the channel wall, and the downstream boundary values are changed as indicated by Equation 50b or 52b. Equation 48 is then solved with these better boundary values and the process repeated until it converges.

This scheme is modified somewhat in practice so that convergence of the iterative method is achieved. Thom and Apelt (46, 47) have reported, and it has been verified by others, that changing the boundary values of the vorticity at a solid boundary by the total amount indicated by the new stream function can result in nonconvergence. Therefore, it is common practice to change the boundary values by only a certain fraction of the indicated change. That is, for the $(k+1)$ th iterate,

$$\omega_{i,0}^{n+1,[k+1]} = \omega_{i,0}^{n+1,[k]} + \beta \left[-\frac{2\phi_{i,1}^{n+1,[k]}}{h^2} - \omega_{i,0}^{n+1,[k]} \right] ,$$

or

$$\begin{aligned} & 0 < i < M + 1 , \\ & M' < i < M + 1 . \end{aligned} \quad 53)$$

In this study a value of β of 0.5 was used following Thom's example. Other values may result in faster convergence, but this type of optimization was not attempted.

The process is more easily visualized by means of a flow chart. Let i , j , and n indicate the same variables as previously. Let (k) be the counter for the number of iterations of the vorticity, or the outer iteration loop, and let (m) be the counter for the iterations of the stream function, or the inner iteration loop. $\omega_{i,j}^n$ and $\phi_{i,j}^n$ without the superscript iteration indicators are the values of the vorticity and stream function when convergence has been achieved. Upon entering the flow chart, the values of $\omega_{i,j}^n$ and $\phi_{i,j}^n$ are known for $0 \leq i \leq M+1$ and $0 \leq j \leq N+1$. The boundary values which remain constant will not be referred to in the flow chart. The chart is presented in Figure 1.

Once the stream function field at a given time plane has been determined, the velocity components are easily found from finite difference approximations to Equation 22,

$$u_{i,j}^n = \frac{1}{2h} \left[\phi_{i,j+1}^n - \phi_{i,j-1}^n \right] ,$$

$$v_{i,j}^n = \frac{-1}{2h} [\phi_{i+1,j}^n - \phi_{i-1,j}^n] ,$$

$$\begin{cases} 1 \leq i \leq M \\ 1 \leq j \leq N \\ n \geq 0 \end{cases} . \quad 54)$$

On the boundaries, the velocities are given by the boundary conditions except for

$$u_{i,N+1}^n = \frac{1}{h} [\phi_{i,N+1}^n - \phi_{i,N}^n] , \quad \begin{cases} 1 \leq i \leq M + 1 \\ n \geq 0 \end{cases} , \quad 55a)$$

$$u_{M+1,j}^n = \frac{1}{2h} [\phi_{M+1,j+1}^n - \phi_{M+1,j-1}^n] , \quad \begin{cases} 1 \leq j \leq N \\ n \geq 0 \end{cases} , \quad 55b)$$

and for case II

$$u_{i,0}^n = \frac{\phi_{i,1}^n}{h} , \quad \begin{cases} 1 \leq i \leq M' \\ n > 0 \end{cases} . \quad 55c)$$

The tests for convergence of the iterative procedures are that

$$\max_{i,j} \left| \phi_{i,j}^{n,[k],[m+1]} - \phi_{i,j}^{n,[k],[m]} \right| \leq \epsilon_s ,$$

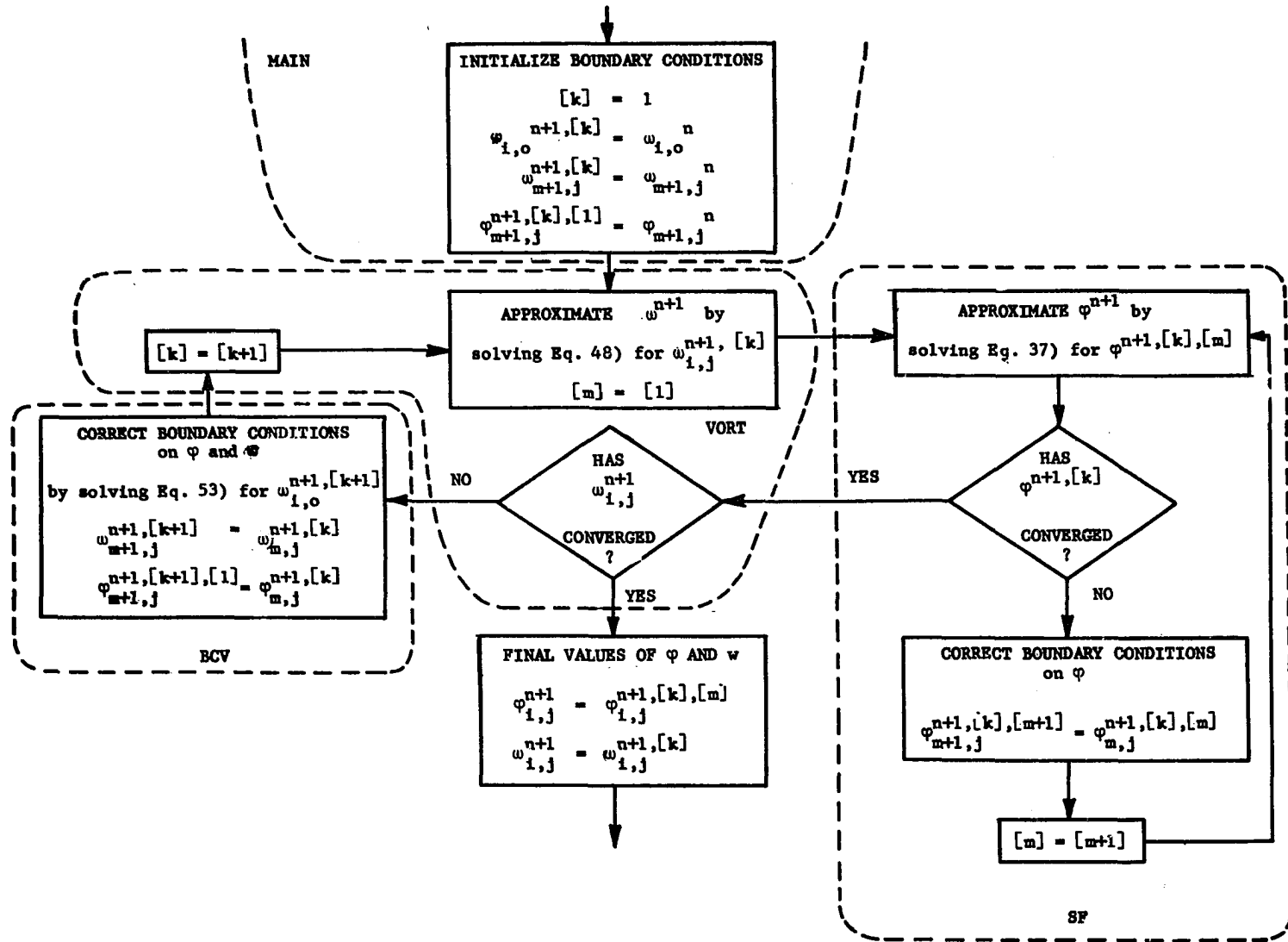
and

$$\max_{i,j \in \text{boundary}} \left| \omega_{i,j}^{n,[k+1]} - \omega_{i,j}^{n,[k]} \right| \leq \epsilon_v .$$

Since the maximum value of the stream function was known a priori, it was not thought necessary to place a limit on the relative change of the stream function, but only on the absolute change. Only the changing boundary values of the vorticity were examined for convergence criteria since these were the critical values which varied from one iteration to the next.

The SBOR and the ADI methods were chosen to solve the difference equations since they are two of the more powerful methods. Theoretically the ADI method is stable for all values of $\Delta\tau$, but it was found necessary to limit $\Delta\tau$ in this case, probably due to the boundary conditions used. However, a large enough $\Delta\tau$ could be used to achieve steady state in less than 200 time planes with less than 20 minutes of calculation time on an IBM 360/50. Although not directly comparable, Fromm and Harlow and Trulio, et al., required thousands of time planes and hours of IBM 7094 time.

Figure 1. Flow chart



DISCUSSION AND RESULTS

Equations 39 and 41 were used to determine the relaxation factor for the method of successive block overrelaxation. These equations are correct if the boundary conditions on the stream function remain constant between iterations. However, boundary condition 50b or 52b was used in a manner such that the stream function vector, $\phi_{M+1}^{[m]}$, varied between iterations. Therefore, rather than the block Jacobi matrix previously defined, \hat{B} , the block Jacobi matrix for this method becomes

$$\hat{B} = \begin{bmatrix} T & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & T & 0 & & & & & \\ 0 & 0 & T & & & & \cdot & \cdot \\ \cdot & & & \cdot & & & \cdot & \cdot \\ \cdot & & & & \cdot & & \cdot & \cdot \\ \cdot & & & & & \cdot & & \cdot \\ 0 & \cdot & \cdot & \cdot & & T & 0 & \\ 0 & \cdot & \cdot & \cdot & & 0 & I & \end{bmatrix}^{-1} \begin{bmatrix} 0 & I & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ I & 0 & I & & & & & & \\ 0 & I & 0 & \cdot & & & \cdot & & \\ \cdot & & & \cdot & & & \cdot & & \\ \cdot & & & & \cdot & & \cdot & & \\ \cdot & & & & & \cdot & & & \\ 0 & & & & & & 0 & I & 0 \\ 0 & \cdot & \cdot & \cdot & & I & 0 & 0 & \\ 0 & & & & & & 0 & I & 0 \end{bmatrix}$$

$$= \begin{bmatrix} & & & & & & & & 0 \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & 0 \\ & & & & & & & & 0 \\ \hline 0 & \cdot & \cdot & \cdot & 0 & I & & & 0 \end{bmatrix} \cdot$$

The spectral radius of \hat{B} , $\sigma(\hat{B})$, can be found by solving the difference Equation 40a with the boundary conditions

$$\left. \begin{aligned} X_{0,j} &= X_{i,0} = X_{i,N+1} = 0 \\ X_{M+1,j} &= X_{M,j} \end{aligned} \right\} \begin{cases} 1 \leq i \leq M \\ 1 \leq j \leq N \end{cases} .$$

Using the same technique as was used to develop Equation 41,

$$X_{i,j} = \sin pi \sin qj ,$$

$$\lambda_{m,n} = \frac{\cos p}{2 - \cos q}$$

$$\approx 1 - \frac{1}{2}[p^2 + q^2] .$$

As previously

$$q = \frac{n\pi}{N+1} , \quad 1 \leq n \leq N .$$

However, the fourth boundary condition results in a different value of p . This boundary condition gives

$$X_{M+1,j} = \sin p(M+1) \sin qj$$

$$= X_{M,j}$$

$$= \sin p M \sin qj$$

or

$$\sin p(M+1) = \sin p(M)$$

A glance at a graph of the sine function shown in Figure 2 demonstrates that this relation can be satisfied around the points $(2m-1)\pi/2$, $1 \leq m \leq M$, if

$$p(M + \frac{1}{2}) = \frac{1}{2}(2m - 1)\pi ,$$

$$p = \frac{(2m - 1)\pi}{2M + 1} .$$

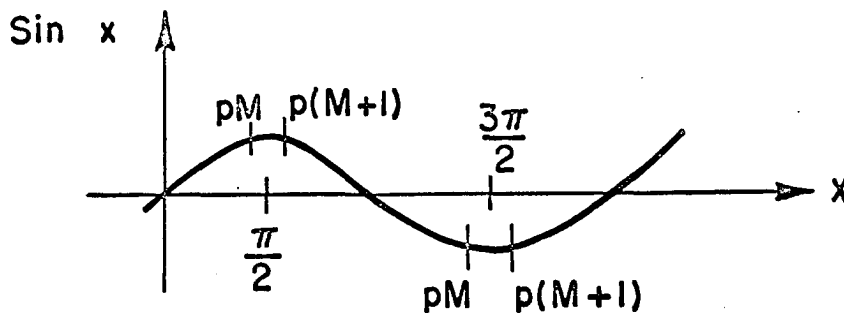


Figure 2. Sine function

Since the matrix \hat{B} has been augmented by N rows and columns, instead of the MN eigenvalues found for B , \hat{B} has $(M+1)N$ eigenvalues. The extra N eigenvalues are all zero and do not increase the spectral radius of the matrix. Therefore the spectral radius of \hat{B} is given by

$$\sigma(\hat{B}) \approx 1 - \frac{\pi^2}{2} \left[\left(\frac{1}{2M+1} \right)^2 + \left(\frac{1}{N+1} \right)^2 \right] .$$

For the problem reported in this dissertation $N+1$ was 10, and $M+1$ was greater than or equal to 60. With these values, the difference between the spectral radius of \hat{B} and of B is about one part in a thousand. Thus the use of Equation 44 is justified.

The directed graph of \hat{B} is not strongly connected so that \hat{B} itself is not irreducible although B was. Therefore, Equation 39 cannot be rigorously applied as the irreducibility of B was used to derive it in Varga (50) and Forsythe and Wasow (13). However, both Varga and Forsythe and Wasow conjecture that this equation is a good estimate of r even though it cannot be theoretically justified at the present time.

A Reynolds number of 50 was assumed in all the results presented. This value is the result of a compromise. Experimental results usually appear with larger values of the Reynolds number. However, as the Reynolds number increases, the fluid requires a longer length of channel to achieve one dimensional flow. This results in a larger number of mesh points to calculate and more calculation to achieve convergence. The numerical equations also appear to become less stable at larger Reynolds numbers so that more care must be taken in the manner in which boundary conditions are incorporated into the solution and in the size of the time and distance increments.

The parameters for the convergence criteria and the time and distance increments were chosen in a trial and error fashion. Reasonable values were first picked and then these values were decreased until the solution did not change significantly with the decrease. The meaning of a nonsignificant change was, generally, that the digit in the second decimal place of the velocity, u or v , did not change. The

final values used to calculate the results presented were

$$\epsilon_s = 0.0001 ,$$

$$\epsilon_v = 0.001 ,$$

$$h = 0.05 ,$$

$$\Delta\tau = 0.001 ,$$

$$\alpha = h^2/\Delta\tau = 2.5 .$$

The value of α was found to affect the stability of the solution. With $\alpha = 1.25$, the solution was almost, but not quite, stable. The low order digits of the velocity field for $\tau > 0.10$ were found to vary in a random fashion. As α was increased, however, the variation damped out. For a value of $\alpha = 0.5$ the solution diverged.

Since the velocity fields asymptotically approach the steady state solution, it was thought that a small value of $\Delta\tau$ could be used initially and then increased as the change of the velocity field between time planes became small. It was found, however, that any value of $\alpha < 1.25$ was unstable so that larger time increments could not be used, even when steady state was nearly reached.

From considerations of the change of the solution with a change in the values of the convergence parameters and the time and distance increments, it was estimated that the velocities presented are within ± 0.01 of the true value except

very near the channel entrance where the error may be as much as ± 0.03 for case I at the early time planes.

This indicates that the use of two values of h might have been beneficial, a smaller h near the entrance with a larger h downstream. A second possibility along this line which was considered was the use of a transformation so that the value of h would vary continuously from the entrance to the exit of the channel. Wang and Longwell (51) used the transformation

$$\eta = 1 - \frac{1}{1 + cx} .$$

This idea was not used since the simplicity of the difference matrix T depends on a constant value of h . It was decided that an increase in accuracy could be achieved more cheaply by decreasing h all over the net than by using a transformation which increased the complexity.

Case II does not result in as sudden a change in the derivative of the velocity as the fluid enters the channel as does case I, and so the accuracy is not greatly affected by the channel entrance.

Figures 3 and 4 present the steady state dimensionless velocity in the x -direction for cases I and II. The arrows at the right side indicate the parabolic profile to which the solution should converge. It can be seen that solution near the middle of the channel has not quite achieved this parabolic profile, but is converging towards it. If the value of M were increased the solution would, of course, more closely

approximate the parabolic profile. However, it was found that an increase in M did not alter significantly the velocities calculated with $M = 60$ for case I and $M = 75$ with $M' = 20$ for case II.

Figure 3 compares very well with a similar figure given in Bodoia and Osterle (5). To the accuracy of the curve presented in Bodoia and Osterle, the comparison is exact.

The steady-state velocity distributions calculated compare very favorably with other results available for comparison. For example, Figure 5 presents the velocity in the x -direction for case I. Values of the velocity according to Schlichting (40) are also given. It can be seen that after the fluid has traveled about half a channel width downstream, the comparison between Schlichting's results and those presented here is very good. The variance near the entrance of the channel is probably due to two causes. In the first place, Schlichting's results are based upon a boundary layer solution which neglects the second derivative of the velocities in the x -direction. Although this is a valid assumption downstream of the entrance, it becomes less valid near the entrance. Secondly, Schlichting's results are not accurate. Collins and Schowalter (7), as was previously noted, found that the number of terms of the series solution used by Schlichting was not sufficient to accurately compute the velocities.

Unfortunately Collins and Schowalter present very little

data. They do mention, however, that their results very closely match the numerical results of Bodoia and Osterle (5). In their paper, Bodoia and Osterle present a curve comparing the center-line velocity from their results with the center-line velocity of Schlichting. This curve indicates the Schlichting's dimensionless velocities are as much as 0.04 higher than the velocities of Bodoia and Osterle, and hence of Collins and Schowalter. This is approximately what is shown in Figure 5.

Figure 6 presents a similar graph for case II.

Wang and Longwell (51) present the results of the steady state solution of the Navier-Stokes equations for both case I and case II. Their results are presented for a Reynolds number of 300 so a direct comparison of the results is not possible, but a general comparison gives very good correlation.

Wang and Longwell compare their data with Schlichting and find that Schlichting's velocities are as much as 0.03 higher than theirs for case I as was noted by Bodoia and Osterle, and also noted in Figure 5.

A point of inflection in the profiles of u versus y was noted by Wang and Longwell which was not found in Schlichting's results, nor in the results of Bodoia and Osterle. This phenomenon is also found in the results presented here for case I, but as can be seen from Figure 5 it is not as evident as in Wang and Longwell. Wang and Longwell also report a point of inflection for case II which the results presented

here do not show. When tests were being run to determine the value of h and $\Delta\tau$ to be used in this investigation, it was found that large values of h or $\Delta\tau$ resulted in very distinct points of inflection for case I. As the values were decreased, the peaks at the edge of the channel became less distinct until the height of the peaks for the results presented here is less than the predicted error. It is, therefore, thought that the points of inflection reported by Wang and Longwell, and shown in Figure 5 to a lesser extent, are due to the effect of the sudden change of the derivative of the velocity at the entrance of the channel on the numerical method.

Figures 7 and 8 present the velocity profiles for cases I and II at time $\tau = 0.02$. As would be expected, the profiles are flatter than for the fully developed case as the effect of the walls has not been completely carried throughout the channel.

The transverse or y-velocity, v , is shown in Figures 9 and 10. The value of v is larger for case I due to the sudden effect of the channel wall, rather than a more gradual effect experienced in case II.

The entrance or development length compares very favorably with values reported by Schlichting, Bodoia and Osterle, and Wang and Longwell. Arbitrarily defining entrance length as the distance from the inlet at which the center-line velocity reaches 98 percent of its asymptotic value, the results can be expressed in the form

$$x_e = lR, \quad 56)$$

where x_e is the entrance length divided by the channel width and R is the Reynolds number based on the channel width. The following values for l have been reported.

Table 1. Entrance length

Reference	l
Schlichting (40)	0.034
Bodoia and Osterle (5)	0.034
Wang and Longwell (51)	
Case I	0.0335
Case II	0.034
Pearson	
Case I	0.034
Case II	0.036

All of these comparisons with existing data are for the steady state case. There was no data available to directly compare the time dependent solution. A one dimensional, time dependent solution of the constant pressure gradient problem is given in Rouse (36, pp. 221 ff.). The boundary conditions for this problem and the problems presented here are different; Rouse using a constant pressure gradient, and the solutions presented here having a constant flow rate. However, a qualitative comparison shows similar time constants and similar

shaped curves as shown in Figure 11.

Figures 11 and 12 show the downstream velocities as a function of time for both cases I and II. As would be expected, the effect of the entrance boundary conditions is damped out so that there is no significant difference between the two cases for x greater than 1.7 or 1.8. Figure 12 shows that steady state has been achieved for any time $\tau \geq 0.14$.

The numerical data from which these graphs were drawn is presented in the Appendix.

The method of solution used here is applicable to a wider variety of problems than those presented. Using this technique, the laminar velocity field for many two-dimensional rectangular regions can be determined if only the velocities on the boundaries of the regions are known. If, as is the case in the problem presented here, the velocities on some portion of the boundary are not known, but the derivatives are, the problem can probably still be solved, but the determination of the relaxation parameter becomes more difficult.

It is also felt that the solution of the Navier-Stokes equations for regions other than rectangular could be accomplished with some thought as to possible transformations of the region and as to the meaning of the vorticity at sharp projections into the region. Thom and Apelt (46) have discussed these problems for the steady state case. As was previously noted, the use of transformations results in a more complex set of equations to be solved. Whether the extra complexity is worth while will depend upon the problem to be solved.

Figure 3. u as a function of x and y for case I, steady state

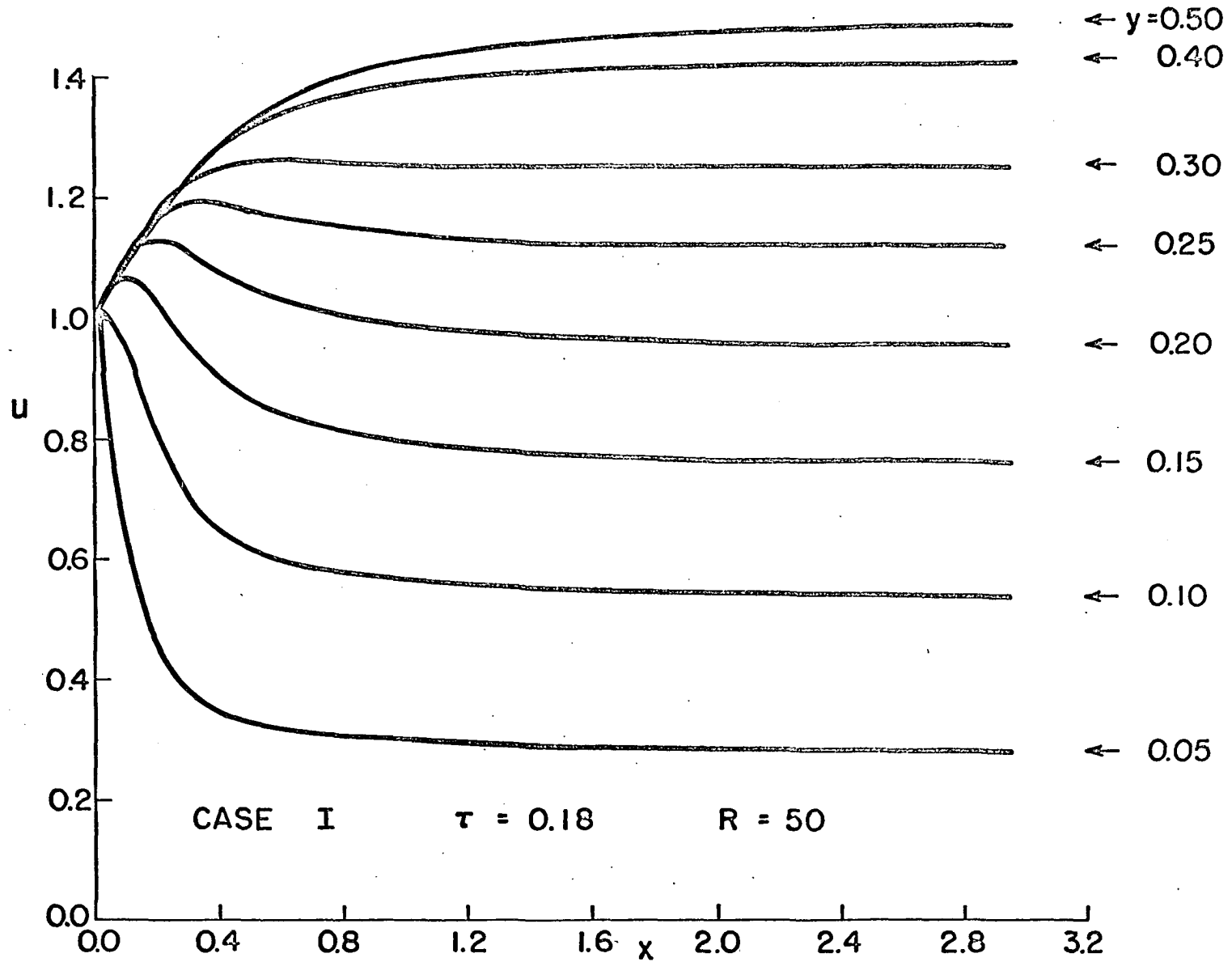


Figure 4. u as a function of x and y for case II, steady state

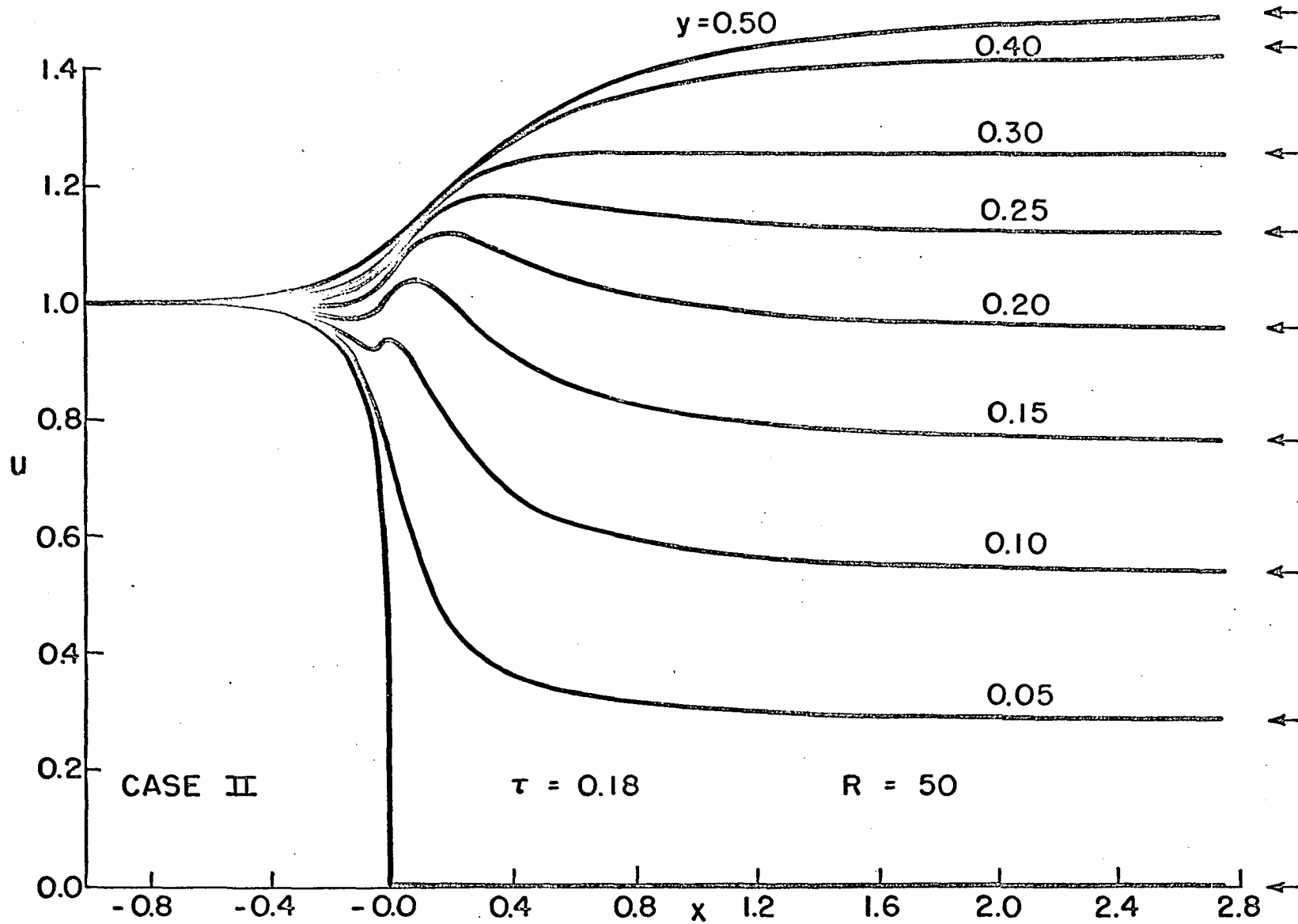
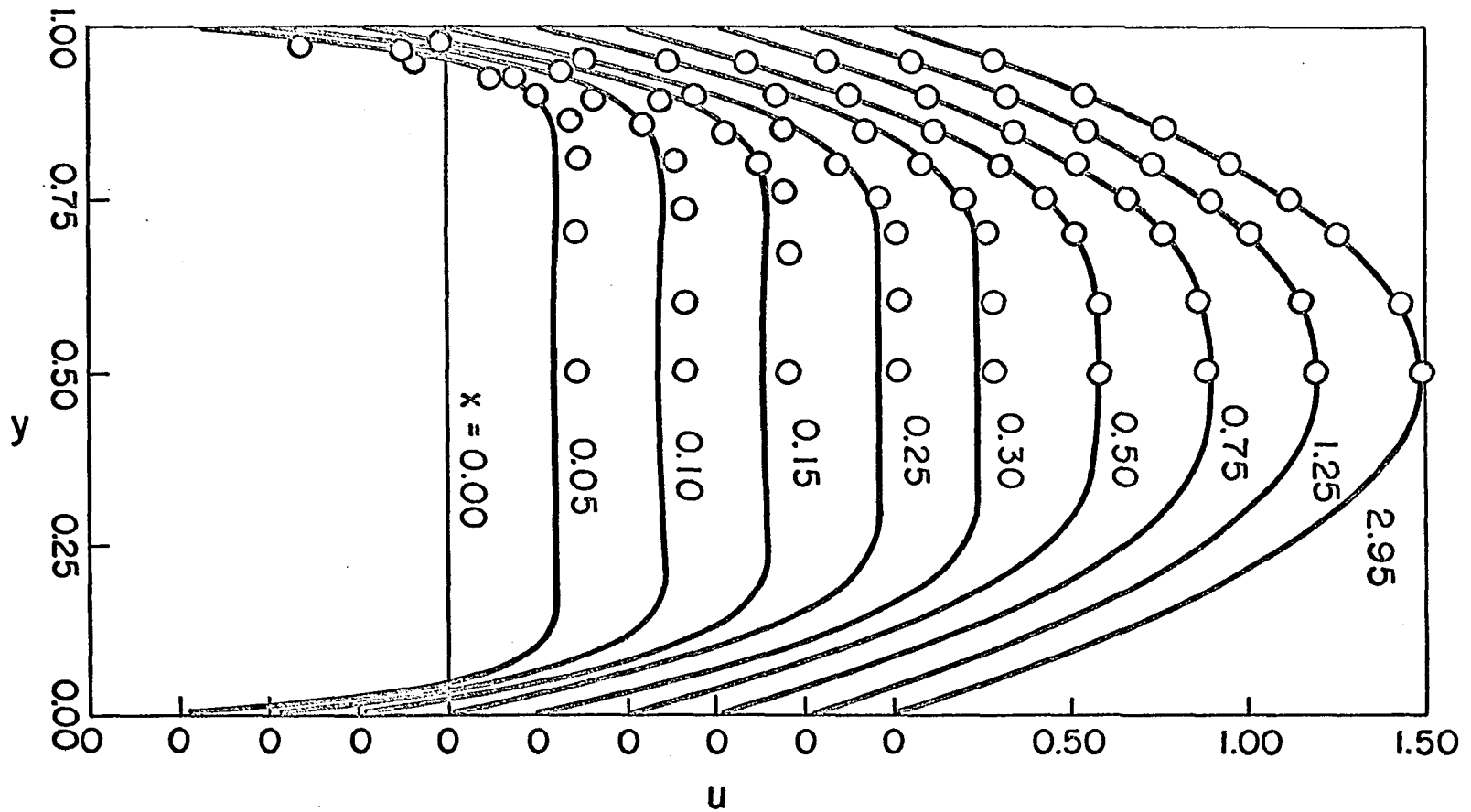


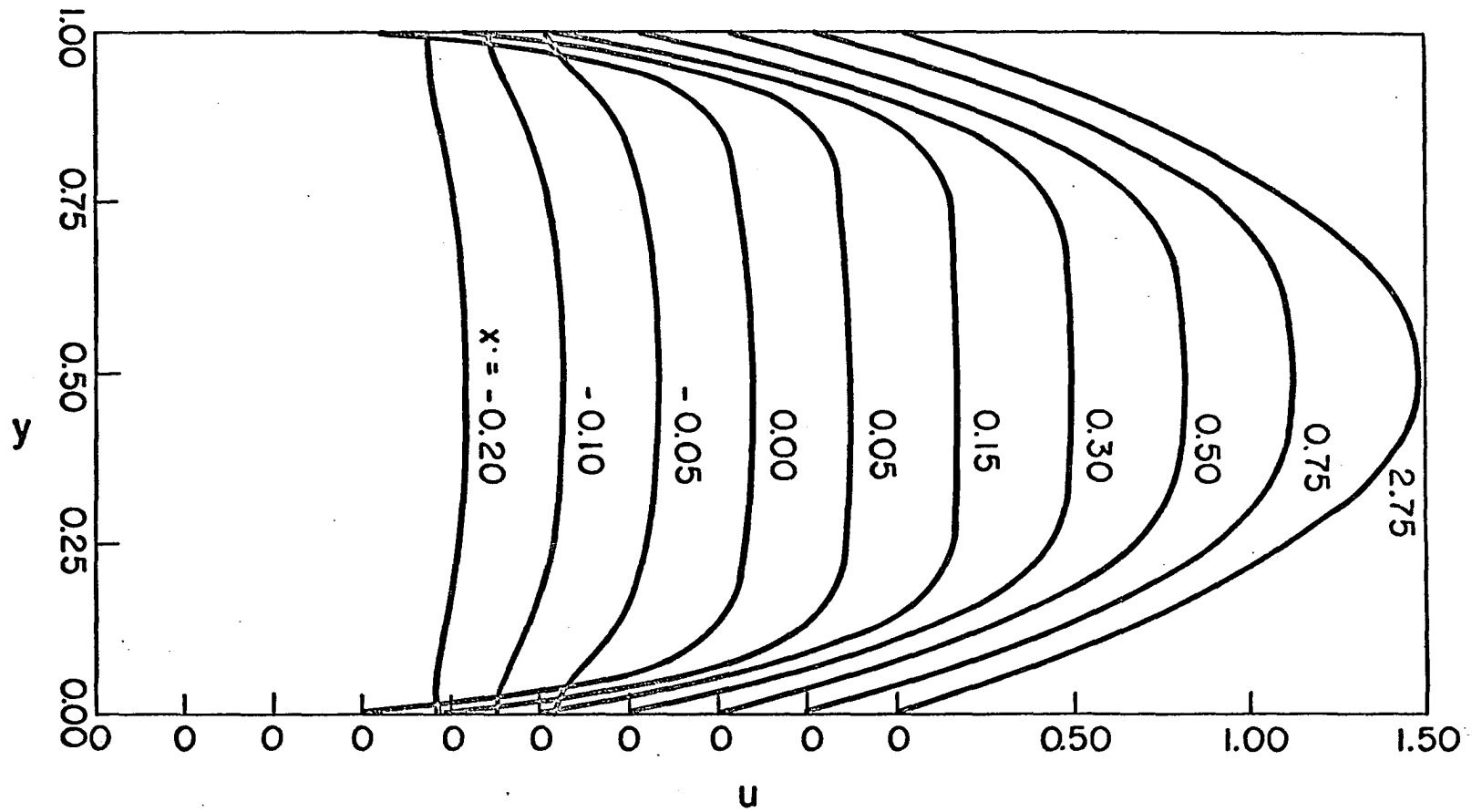
Figure 5. u as a function of y and x for case I, steady state

O - Schlichting's data (40)



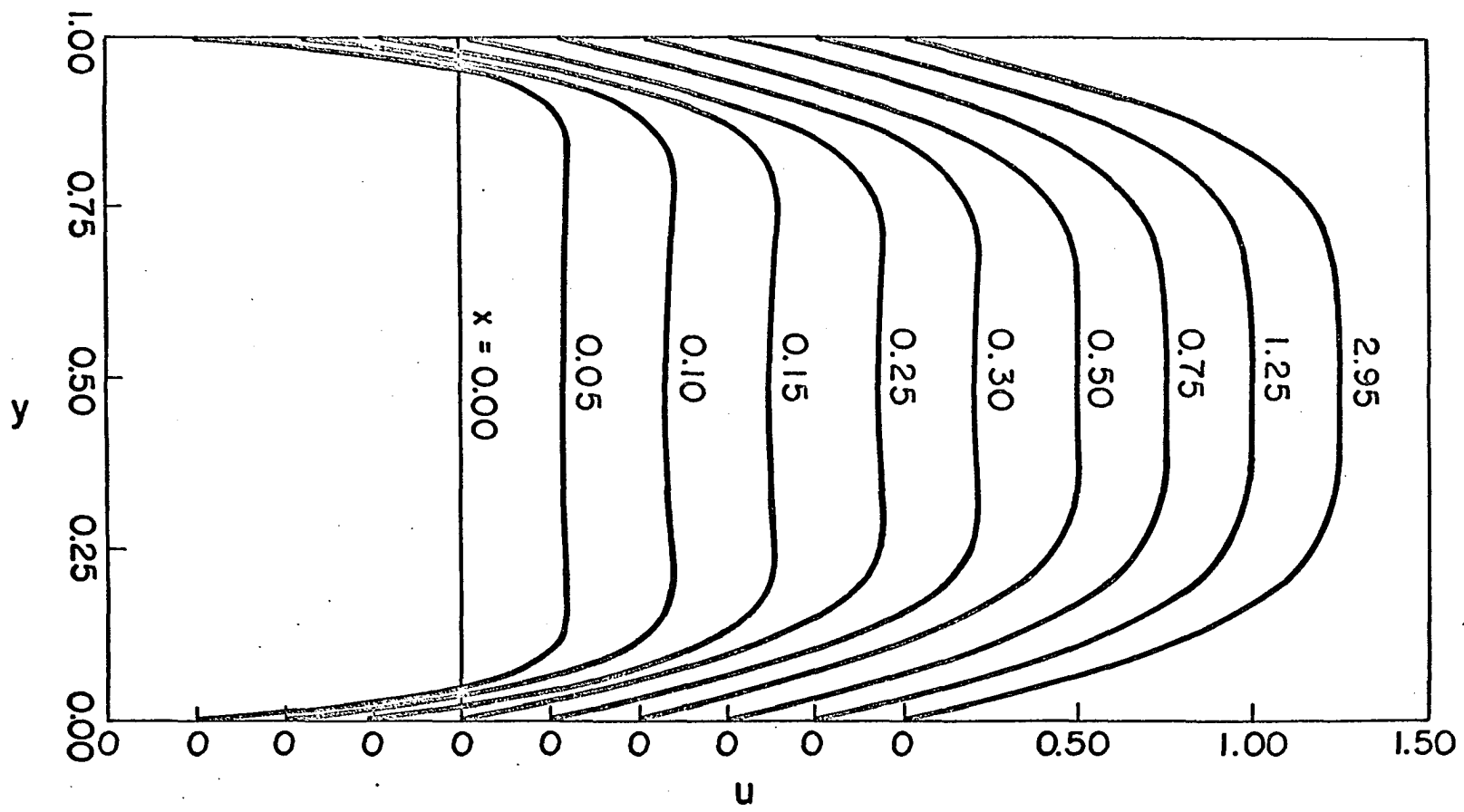
VELOCITY PROFILE $\tau = 0.18$
 Case I Reynolds Number = 50.

Figure 6. u as a function of y and x for case II, steady state



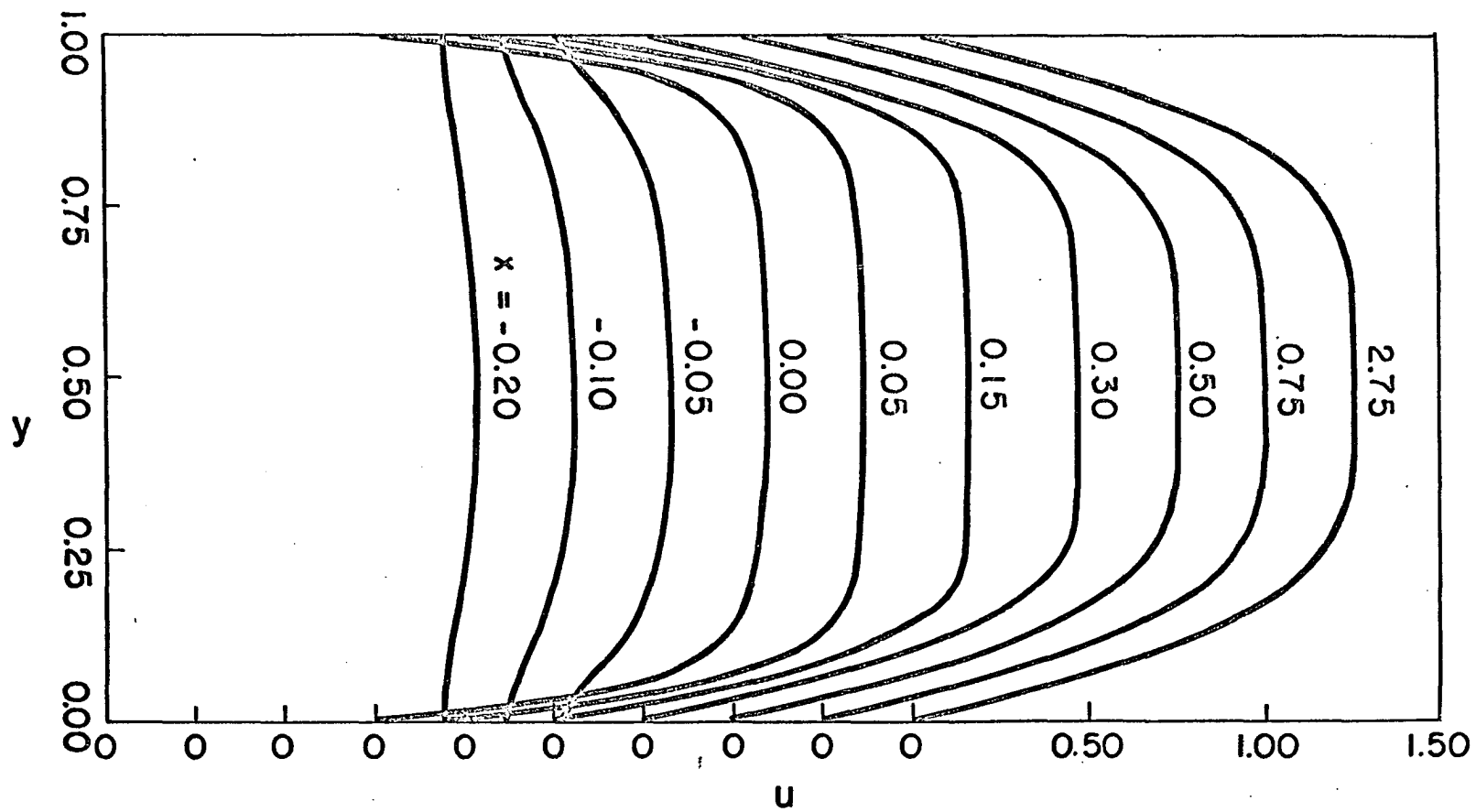
VELOCITY PROFILE $\tau = 0.18$
Case II Reynolds Number = 50.

Figure 7. u as a function of y and x for case I, $\tau = 0.02$



VELOCITY PROFILE $\tau = 0.02$
Case I Reynolds Number = 50

Figure 8. u as a function of y and x for case II, $\tau = 0.02$



VELOCITY PROFILE $\tau = 0.02$
Case II Reynolds Number = 50.

Figure 9. v as a function of y and x for case I, steady state

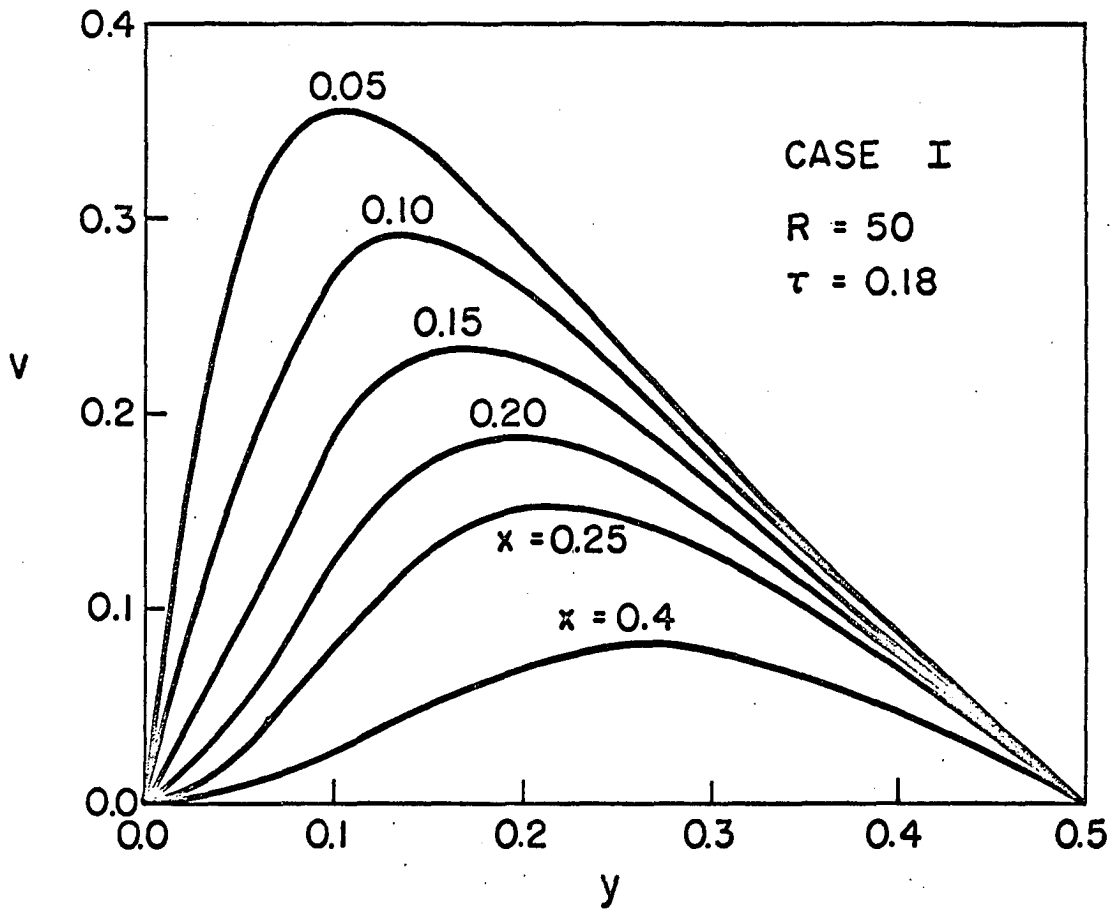


Figure 10. v as a function of y and x for case II, steady state

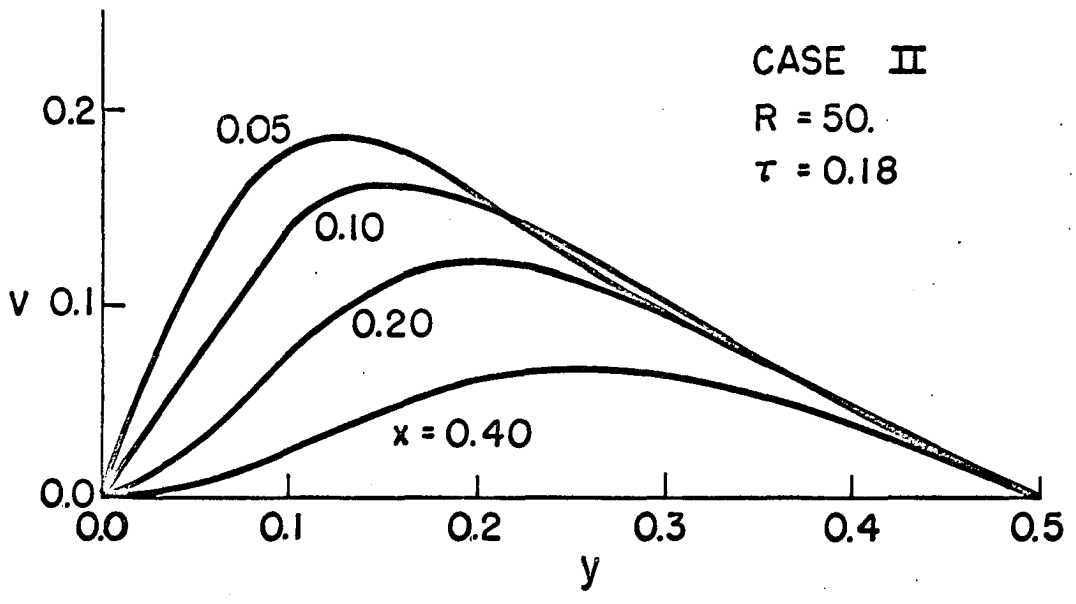
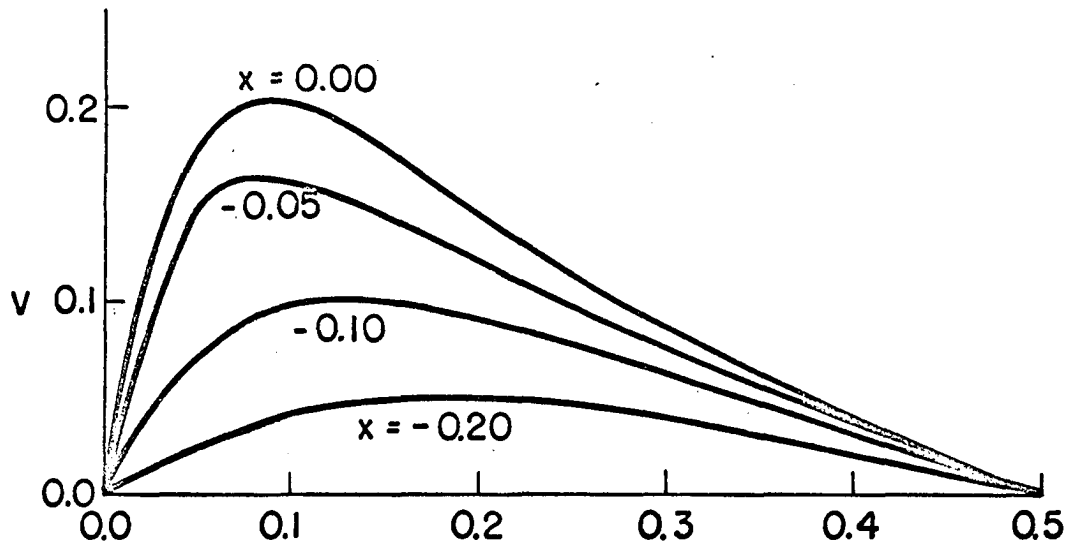


Figure 11. u as a function of y and τ for $x = 2.75$

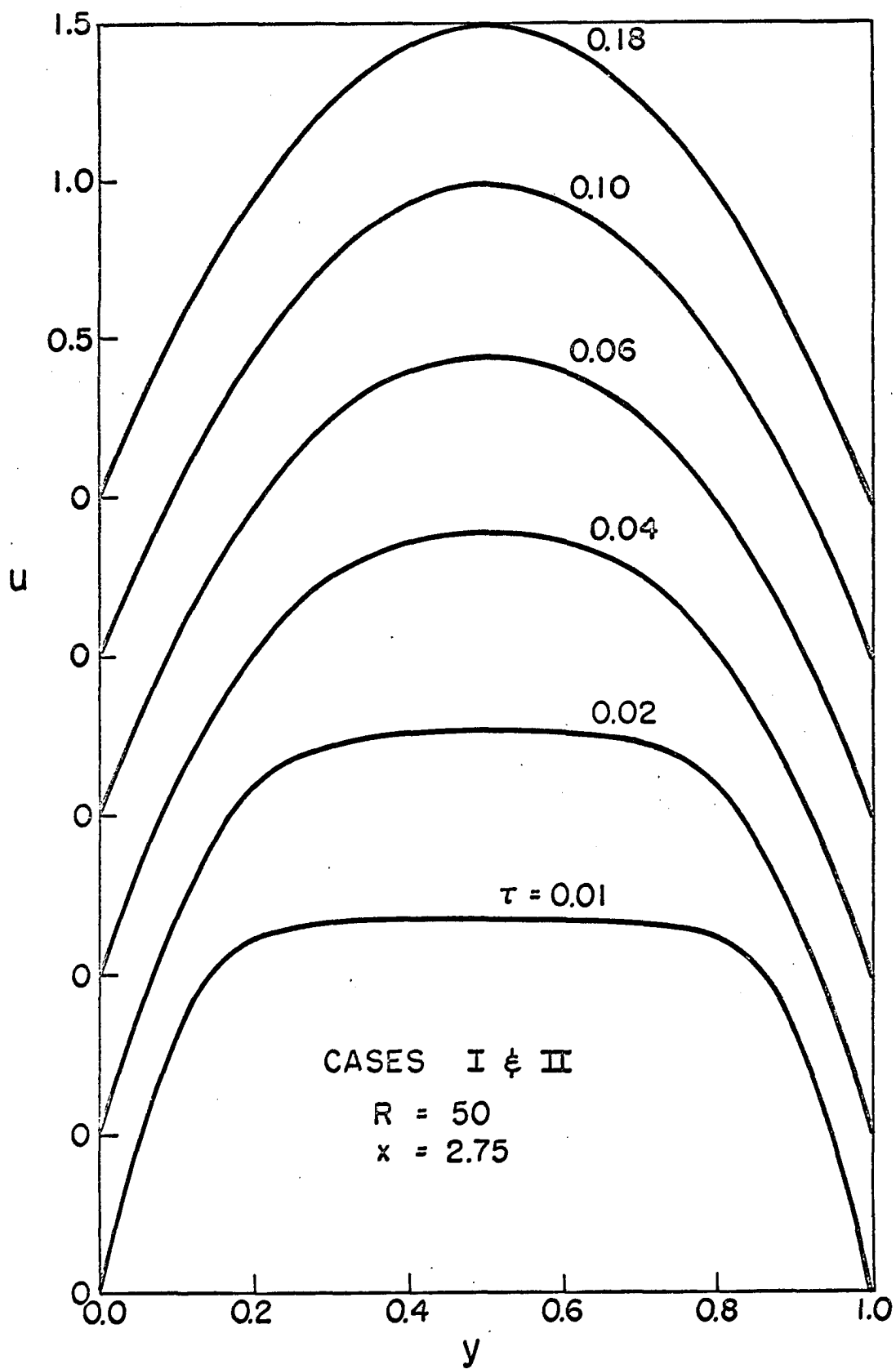
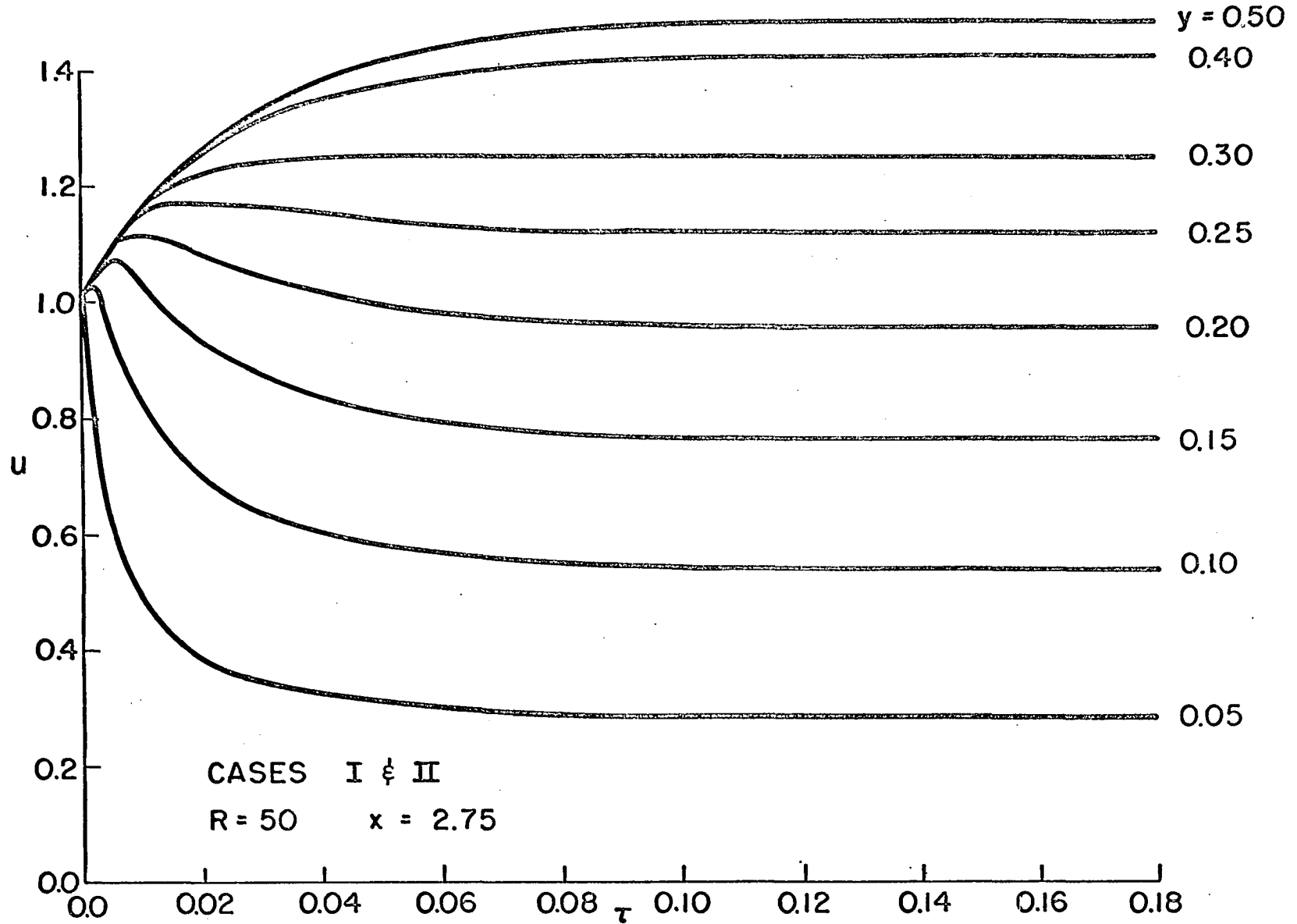


Figure 12. u as a function of τ and y for $x = 2.75$



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APPENDIX

Tensor Notation

Since the geometry of the inlet of parallel plates is most naturally described by a Cartesian coordinate system, Cartesian tensor analysis was used.

The equations of hydrodynamics were derived in three dimensions so that all tensor subscripts ranged over the values of 1, 2 and 3 unless specifically noted otherwise.

A convenient notational convention usually employed with tensor analysis is the summation convention. If a particular subscript appears twice in any term of an expression, that term actually stands for the sum of three terms obtained by giving the subscript the values 1, 2 and 3. For example

$$A_{ii\dots} = A_{11\dots} + A_{22\dots} + A_{33\dots} ,$$

or

$$A_{ij\dots} B_{jk\dots} = A_{i1\dots} B_{1k\dots} + A_{i2\dots} B_{2k\dots} \\ + A_{i3\dots} B_{3k\dots} , \quad i,k = 1,2,3.$$

When two subscripts of a tensor are made the same so that the summation convention is brought into use, the operation is called contraction with respect to the two subscripts.

Two useful tensors are the Kronecker delta, δ_{ij} , and the epsilon tensor, ϵ_{ijk} . The Kronecker delta tensor is defined by

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

and the epsilon tensor is defined by

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } ijk \text{ form an even} \\ & \text{permutation of } 123, \\ 0, & \text{if any two of } ijk \\ & \text{are alike,} \\ -1, & \text{if } ijk \text{ form an odd} \\ & \text{permutation of } 123. \end{cases}$$

It is easy, but tedious, to show that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \quad (57)$$

by taking all possible values of $j, k, m,$ and n . This relation is proved in Jeffreys and Swirles (21, p. 73).

A tensor is called symmetric with respect to two subscripts if

$$A_{ij\dots} = A_{ji\dots},$$

and antisymmetric if

$$A_{ij\dots} = -A_{ji\dots}.$$

A useful result of tensor analysis is that the double contraction of any symmetric tensor with an antisymmetric tensor results in a null tensor. That is, if

$$A_{ij\dots} = A_{ji\dots},$$

and

$$B_{ij\dots} = -B_{ji\dots} ,$$

then

$$\begin{aligned} A_{ij\dots} B_{ij\dots} &= \frac{1}{2}(A_{ij\dots} B_{ij\dots} + A_{ij\dots} B_{ij\dots}) \\ &= \frac{1}{2}(A_{ij\dots} B_{ij\dots} + A_{ji\dots} B_{ij\dots}) \\ &= \frac{1}{2}(A_{ij\dots} B_{ij\dots} - A_{ji\dots} B_{ji\dots}) \\ &= \frac{1}{2}(A_{ij\dots} B_{ij\dots} - A_{ij\dots} B_{ij\dots}) \\ &= 0 \end{aligned} \tag{58}$$

In particular

$$\epsilon_{ijk} A_{jk\dots} = 0 .$$

The operator $\partial_i ()$ is defined by

$$\partial_i A_{jk\dots} = \frac{\partial A_{jk\dots}}{\partial x_i}$$

This operator is the same as the del operator in vector notation.

Occasion will also be found for use of Gauss' theorem, or the divergence theorem. This theorem is proved in many texts such as Prager (33, p. 28) and Jeffreys and Swirles (21, p. 193). In tensor notation as given by Prager it takes the form

$$\int_V \partial_i A_{jk} \dots dV = \int_S n_i A_{jk} \dots dS \quad , \quad 59)$$

where V is a volume, bounded by the surface S consisting of a finite number of parts with the continuously turning, external normal, n_i . This equation can be put in a variety of forms by selective contraction or by choosing various forms for the tensor $A_{jk} \dots$.

Cartesian tensor notation is related to vector notation by results such as

$$f_i g_i = \vec{f} \cdot \vec{g} \quad ,$$

$$\epsilon_{ijk} f_j g_k = \vec{f} \times \vec{g} \quad .$$

$$\partial_i \theta = \nabla \theta = \text{grad } \theta \quad ,$$

$$\partial_i f_i = \nabla \cdot \vec{f} = \text{div } \vec{f} \quad ,$$

and

$$\epsilon_{ijk} \partial_j f_k = \nabla \times \vec{f} = \text{curl } \vec{f} \quad .$$

Program Listings

The flow chart presented in Figure 1 gives the essential details of the method of solution so that a flow chart of the main program and each subprogram will not be presented. However, a complete listing of each routine for case I is given in Figures 13 to 20.

Two changes from the notation given in body of the dissertation should be noted. The x and y directions are reversed so that x is the direction perpendicular to the flow and y is the direction parallel to the flow. The second change is that the program is not written using the nondimensional variables. The time, as used in the program, is actually vt as given in the previous description.

MAIN routine

MAIN functions as the executive routine for the program, besides initializing the boundary conditions of the vorticity each time VORT is entered as shown in Figure 1. The two matrices, VORTA and VORTB, as defined in MAIN are alternately the new and old vorticity matrices. STRFN is the stream function matrix at the present time plane. The other variables are housekeeping variables to control such things as output, change of time increment and completion of the program.

MAIN is composed of two separate sections which are identical in logical structure. The first section consists of the lines from statement 1 through statement 6+1. This

section is used to calculate the new vorticity, VORTB, from the old vorticity, VORTA. The second section, statement 7 through statement 12+2, is used to calculate the new vorticity, VORTA, from the old vorticity, VORTB.

Statements 1 through 3 and statements 7 through 9 initialize the boundary conditions of the vorticity as shown in Figure 1. Other statements update parameters and call routines as required.

Subprogram INIT

INIT is the initialization routine. It is used to input parameters, output a heading, initialize parameters, and to input the initial conditions for the vorticity and the stream function.

The important variables are described as follows:

VORT(I,J)	Initial vorticity matrix
STRFN(I,J)	Stream function matrix
M	Number of nodes in the x-direction
N	Number of nodes in the y-direction
H	Distance increment
DELT	Time increment
VIS	Kinematic viscosity
VEL	Inlet velocity
RELAX	Relaxation parameter

Statements 1-1 through 2 are used to calculate the

vectors f of Equation 38b for the SBOR and ADI methods. Statements 2+5 through 3 set the initial conditions for the vorticity and the stream function.

Subroutine PRNT

PRNT is the routine used to output data. Each time PRNT is called the matrices STRFN and VORT are output onto a peripheral tape. Also at specified intervals the values of u and v , the velocity fields, are calculated from Equations 54 and 55 and output on the printer. The subroutine as listed is for case I. For case II some of the boundary values of the velocities must be changed.

Subroutine VORT

VORT is the subroutine which calculates the new value of the vorticity, VORTB, from the old, VORTA. This is done by the ADI method of Equation 48. Figure 1 presents a simplified flow chart of VORT. As can be seen from the flow chart, VORT calls subroutine SF to calculate the stream function matrix associated with VORTB and calls the subroutine BCV to calculate the corrected boundary values of the vorticity from this stream function matrix.

The temporary storage matrix AMAT is used for two purposes. The interior locations are used to store the right hand side of Equation 48a. This is calculated by statement 1. Also the first column and the first row of AMAT are used to store the vector g of Equation 38c for the solution of

Equations 48a and 48b. Equation 48a is solved by statements 2 through 4 using the algorithm of Equation 38. The value of ω^* is temporarily stored in the matrix VORTB. Then Equation 48b is solved by statements 4+1 through 6. These statements put the values of ω^{n+1} into the matrix VORTB.

Subroutine SF

Subroutine SF is called by VORT to calculate the new stream function matrix, STRFN, associated with the vorticity matrix, VORT. It does this by means of the SBOR technique described by Equations 37. A simplified flow chart is shown in Figure 1.

TMAT in STRFN is used in the same manner as was AMAT in subroutine VORT. Statements 1 through 2 place the right hand side of Equation 37a into the interior locations of TMAT. Then the first column of TMAT is used to store the vector g of Equation 38c. Vector g is calculated by statements 3+3 through 4. Statements 4+1 through 5+2 calculate $(\hat{\phi}_i^{[m+1]} - \phi_i^{[m]})$ and store this vector in the first column of TMAT. The maximum value is then determined. Statement 6 calculates $\phi_i^{[m+1]}$ from Equation 37b, and the downstream boundary conditions are changed according to Equation 50b or 52b by statement 7. Statement 7+1 then checks the maximum change of the stream function between iterations to determine whether or not another iteration is required.

Subroutine BCV

Subroutine BCV calculates the corrected boundary values of the vorticity from the stream function. It also checks the difference between the old and new boundary values to determine whether the vorticity field has converged. The listing of BCV is for case I. For case II, the boundary values are calculated only along the actual wall of the channel.

Subroutine DTCHAN

Subroutine DTCHAN is the routine used to change the value of the time increment, DELT, or to change the number of time planes calculated before the data is output. DT(I) is a vector of eight time increments which may be used, NT(I) is the number of times the time increment DT(I) is to be used, and MTAP(I) is the number of time planes to be calculated before the data is output onto tape. When DT(I) is zero, this routine completes the program.

Figure 13. MAIN routine

```

COMMON CONA, CONB, CONC, DELT, FRS, FRV, FRSMAX, FRVMAX, H, HSO,
1 INOBCV, M, M1, M2, M3, MAXITS, MAXITV, MAXPRN, MAXTAP, N, N1, N2,
2 N3, NITS, NITV, NPRN, NTAP, NTCHAN, NTIME, RELAX, RH, RHSO, TIME,
3 VEL, VIS, DT(8), MTAP(8), NT(8), STREN(15,100), VECSE(15),
4 VECV(100)
DIMENSION VORTA(15,100), VORTB(15,100)
PAUSE
REWIND 10
CALL INIT (VORTA)
1 DO 2 I = 1, M
  VORTB(I,1) = VORTA(I,1)
2 VORTB(I,N) = VORTA(I,N)
  DO 3 J = 2, N1
  VORTB(I,J) = VORTA(I,J)
3 VORTB(M,J) = VORTA(M,J)
  CALL VORT(VORTA, VORTB)
  TIME = TIME + DELT
  NTIME = NTIME - 1
  NTAP = NTAP - 1
  IF (NTAP) 13,4,5
4 NTAP = MAXTAP
  CALL PRNT(VORTB)
5 IF (NTIME) 13,6,7
6 CALL DTCHAN
  NTAP = MAXTAP
7 DO 8 I = 1, M
  VORTA(I,1) = VORTB(I,1)
8 VORTA(I,N) = VORTB(I,N)
  DO 9 J = 2, N1
  VORTA(I,J) = VORTB(I,J)
9 VORTA(M,J) = VORTB(M,J)
  CALL VORT(VORTB, VORTA)
  TIME = TIME + DELT
  NTIME = NTIME - 1
  NTAP = NTAP - 1
  IF (NTAP) 13,10,11
10 NTAP = MAXTAP
  CALL PRNT(VORTA)
11 IF (NTIME) 13,12,1
12 CALL DTCHAN
  NTAP = MAXTAP
  GO TO 1
13 CALL RUN(10)
  WRITE(15,101)
  STOP 1
101 FORMAT(' REMOVE AND SAVE TAPE #527')
END

```

Figure 14. Subroutine INIT

```

SUBROUTINE INIT (VCRT)
COMMON CCNA,CCNB,CCNC,DELT,ERS,ERV,FRSMAX,ERVMAX,H,HSQ,
1 INDBCV,M,M1,M2,M3,MAXITS,MAXITV,MAXPRN,MAXTAP,N,N1,N2,
2 N3,NITS,NITV,NPRN,NTAP,NTCHAN,NTIME,RELAX,RH,RHSQ,TIME,
3 VEL,VIS,DT(8),MTAP(8),NT(8),STRFN(15,100),VECSF(15),
4 VECV(100)
DIMENSION VCRT(15,100)
READ(1,100) M,N,MAXITV,MAXITS,MAXPRN
READ(1,101) H,VIS,VEL,ERVMAX,ERSMAX
READ(1,102) DT
READ(1,103) NT
READ(1,103) MTAP
READ(1,104)
READ(1,105)
WRITE(3,104)
WRITE(3,105)
WRITE(10) M,N,MAXITV,MAXITS,H,VIS,VEL,ERVMAX,ERSMAX
NTCHAN = 1
RELAX = 2./((1. + SQRT(1. - (1. - 0.5*3.1416**2.*
1 (1./M**2. + 1./N**2.))**2.))
NTIME = NT(1)
DELT = DT(1)
MAXTAP = MTAP(1)
WRITE(3,106) M,N,H,VIS,VEL,ERVMAX,ERSMAX,RELAX,MAXPRN
WRITE(3,107) (DT(I),I=1,8),(NT(I),I=1,8),(MTAP(I),I=1,8)
HSQ = H*H
RH = 0.5/H
RHSQ = 1./HSQ
M1 = M - 1
M2 = M1 - 1
M3 = M2 - 1
N1 = N - 1
N2 = N1 - 1
N3 = N2 - 1
CCNA = 4.*HSQ/DELT
CCNB = 6. - CCNA
CCNC = 2. + CCNA
VECSF(2) = 0.25

```

Figure 14. Subroutine INIT (continued)


```

DO 1 I = 3,M1
1  VECSE(I) = 1./(4.-VECSF(I-1))
  VECV(2) = 1./CONC
DO 2 J = 3,M1
2  VECV(J) = 1./(CONC-VECV(J-1))
  VIS = 2.*VIS
  TIME = 0.
  NTAP = MAXTAP
  NPRN = MAXPPN
  DELSF = -VFL*H
  TEMP = -DELSF
DO 3 I = 1,M
  TEMP = TEMP + DELSF
DO 3 J = 1,N
  STRFN(I,J) = TEMP
3  VORT(I,J) = 0.
  RETURN
100 FORMAT (I3)
101 FORMAT(F10.0)
102 FORMAT (8F10.0)
103 FORMAT (8I10)
104 FORMAT(' HEADING ' )
105 FORMAT(' HEADING ' //)
106 FORMAT(' DIFFERENCE EQUATION SOLUTION OF NAVIER-STOKES' /
1 5X, 'EQUATIONS FOR A RECTANGULAR REGION' /5X, 'WITH', I3,
2 ' BY', I3, ' POINTS' // ' DISTANCE INCREMENT', T35, F12.5 /
3 ' VISCOSITY', T35, E12.5 // ' INLET VELOCITY', T35, E12.5 /
4 ' ERROR CHECKS' /6X, 'VORTICITY', T35, E12.5 /6X,
5 ' STREAM FUNCTION', T35, E12.5 // ' RELAXATION PARAMETER',
6 T35, E12.5 // ' PRINT FACTOR', T35, I8)
107 FORMAT(/' TIME INCREMENTS' /3X, 8F7.2 //
1 ' NUMBER OF TIMES INCREMENT USED' /3X, 8I7 //
2 ' TAPED EVERY NTH TIME' /3X, 8I7)
  END

```

Figure 15. Subroutine PRNT

```

SUBROUTINE PRNT(VORT)
COMMON CCNA,CCNB,CCNC,DELT,FRS,ERV,ERSMAX,ERVMAX,H,H5Q,
1 INDBCV,M,M1,M2,M3,MAXITS,MAXITV,MAXPRN,MAXTAP,N,N1,N2,
2 N3,NITS,NITV,NPRN,NTAP,NTCHAN,NTIME,RELAX,RH,RHSQ,TIME,
3 VLL,VIS,DT(8),MIAP(8),NT(8),STRFN(15,100),VECSF(15),
4 VECV(100)
DIMENSION VEC(15),VORT(15,100)
WRITE(10) TIME, ((VORT(I,J), I = 1,M), J = 1,N)
WRITE(10) TIME, ((STRFN(I,J), I = 1,M), J = 1,N)
NPRN = NPRN - 1
IF(NPRN) 7,1,6
1 NPRN = MAXPRN
NITV = MAXITV - NITV
WRITE(3,100) TIME,NITV,ERV,ERS
DO 3 J = 1,N
DO 2 I = 2,M1
2 VEC(I) = RH*(STRFN(I-1,J) - STRFN(I+1,J))
VEC(I) = 0.
VEC(M) = (STRFN(M1,J) - STRFN(M,J))/H
3 WRITE(3,101) J, (VEC(I), I = 1,M)
WRITE(3,102)
J = 1
V = 0.
WRITE(3,101) J, (V, I = 1,M)
DO 5 J = 2,N1
DO 4 I = 1,M
4 VEC(I) = RH*(STRFN(I,J+1) - STRFN(I,J-1))
5 WRITE(3,101) J, (VEC(I), I = 1,M)
WRITE(3,101) N, (V, I = 1,M)
6 RETURN
7 CALL KUN(10)
WRITE(15,103)
STOP 2
100 FORMAT(' TIME',T40,F12.3// ' ITERATIONS'/6X,' VORTICITY',
1 T40,I12/// ' ERROR'/6X,' VORTICITY',T40,F12.5/
2 6X,' STREAM FUNCTION',T40,F12.5/' Y VELOCITY')
101 FORMAT(I3,2X,I1F10.4)
102 FORMAT(' Y VELOCITY')
103 FORMAT(' REMOVE AND SAVE TAPE #527')
END

```

Figure 16. Subroutine VORT

```

SUBROUTINE VORT(VORTA,VORTB)
COMMON CONA,CONB,CONC,DFLT,ERS,ERV,ERSMAX,ERVMAX,H,HSQ,
1 INDBCV,M,M1,M2,M3,MAXITS,MAXITV,MAXPRN,MAXTAP,N,N1,N2,
2 N3,NITS,NITV,NPRN,NTAP,NTCHAN,NTIME,RELAX,RH,RHSQ,TIME,
3 VEL,VIS,DT(8),MTAP(8),NT(8),STRFN(15,100),VECSF(15),
4 VECV(100)
DIMENSION VORTA(15,100),VORTB(15,100),AMAT(15,100)
NITV = MAXITV
DO 1 J = 2,N1
DO 1 I = 2,M1
1 AMAT(I,J) = VORTA(I+1,J) - CONB*VORTA(I,J) + VORTA(I-1,J)
1 + 2.*(VORTA(I,J+1) + VORTA(I,J-1)) - ((STRFN(I,J+1)
2 - STRFN(I,J-1))*(VORTA(I+1,J) - VORTA(I-1,J)) -
3 (STRFN(I+1,J) - STRFN(I-1,J))*(VORTA(I,J+1) -
4 VORTA(I,J-1)))/VIS
2 DO 4 J = 2,N1
AMAT(2,1) = (AMAT(2,J) + VORTB(1,J))*VECV(2)
DO 3 I = 3,M1
3 AMAT(I,1) = (AMAT(I,J) + AMAT(I-1,1))*VECV(I)
DO 4 I = 1,M2
K = M - I
4 VORTB(K,J) = AMAT(K,1) + VECV(K)*VORTB(K+1,J)
DO 6 I = 2,M1
AMAT(1,2) = (-VORTA(I,3) + 2.*VORTA(I,2) - VORTA(I,1)
1 + CONA*VORTB(I,2) + VORTB(I,1))*VECV(2)
DO 5 J = 3,N1
5 AMAT(1,J) = VECV(J)*(-VORTA(I,J+1) + VORTA(I,J)*2. -
1 VORTA(I,J-1) + CONA*VORTB(I,J) + AMAT(1,J-1))
DO 6 J = 1,N2
K = N - J
6 VORTB(I,K) = AMAT(I,K) + VECV(K)*VORTB(I,K+1)
CALL SF(VORTB)
CALL BCV(VORTB)
IF(INDBCV) 9,8,7
7 NITV = NITV - 1
IF(NITV) 9,9,2
8 RETURN
9 CALL RUN(10)
WRITE(15,100)
STOP 3
100 FORMAT(' REMOVE AND SAVE TAPE #527')
END

```

Figure 17. Subroutine SF

```

SUBROUTINE SF(VORT)
COMMON CQNA,CCNB,CCNC,DELT,ERS,ERV,ERSMAX,ERVMAX,H,HSQ,
1 INDRCV,M,M1,M2,M3,MAXITS,MAXITV,MAXPRN,MAXITAP,N,N1,N2,
2 NB,NITS,NITV,NPRN,NTAP,NTCHAN,NTIME,RELAX,RH,RHSQ,TIME,
3 VFL,VIS,DT(8),MTAP(8),NT(8),STRFN(15,100),VECSF(15),
4 VECV(100)
DIMENSION VORT(15,100),TMAT(15,100)
NITS = MAXITS
DO 2 J = 2,N1
DO 1 I = 2,M1
1 TMAT(I,J) = HSC*VORT(I,J)
TMAT(2,J) = TMAT(2,J) + STRFN(1,J)
2 TMAT(M1,J) = TMAT(M1,J) + STRFN(M,J)
3 ERS = 0.
NITS = NITS - 1
DO 6 J = 2,N1
TMAT(2,1) = VECSF(2)*(TMAT(2,J) + STRFN(2,J-1) +
1 STRFN(2,J+1))
DO 4 I = 3,M1
4 TMAT(I,1) = VECSF(I)*(TMAT(I,J) + STRFN(I,J-1) +
1 STRFN(I,J+1) + TMAT(I-1,1))
DO 5 I = 1,M3
K = M - I
TMAT(K-1,1) = TMAT(K-1,1) + VECSF(K-1)*TMAT(K,1)
TMAT(K,1) = TMAT(K,1) - STRFN(K,J)
5 ERS = AMAX1(ERS,ABS(TMAT(K,1)))
TMAT(2,1) = TMAT(2,1) - STRFN(2,J)
ERS = AMAX1(ERS,ABS(TMAT(2,1)))
DO 6 I = 2,M1
6 STRFN(I,J) = RELAX*TMAT(I,1) + STRFN(I,J)
DO 7 I = 2,M1
7 STRFN(I,N) = STRFN(I,N-1)
IF(ERSMAX - ERS) 8,9,9
8 IF(NITS) 10,10,3
9 RETURN
10 CALL RUN(10)
WRITE(15,100)
STOP 4
100 FORMAT(' REMOVE AND SAVE TAPE #527')
END

```

Figure 18. Subroutine BCV


```

SUBROUTINE BCV(VORT)
COMMON CCNA,CCNB,CCNC,DELT,ERS,ERV,ERSMAX,ERVMAX,H,HSG,
1 INDBC, M, M1, M2, M3, MAXITS, MAXITV, MAXPRN, MAXTAP, N, N1, N2,
2 N3, NITS, NITV, NPRN, NTAP, NCHAN, NTIME, RELAX, RH, RHSV, TIME,
3 VEL, VIS, DT(8), MTAP(8), NT(8), STRFN(15,100), VECSE(15),
4 VECV(100)
DIMENSION VORT(15,100)
INDBC = 0
ERV = 0.
DO 1 J = 2, N1
TEMP = -2.*RHSV*STRFN(2, J)
ERV = AMAX1(ERV, ABS(TEMP-VORT(1, J)))
1 VORT(1, J) = 0.5*(VORT(1, J) + TEMP)
DO 2 I = 2, M1
ERV = AMAX1(ERV, ABS(VORT(I, N)-VORT(I, N-1)))
2 VORT(I, N) = VORT(I, N-1)
IF(ERVMAX-ERV) 3,4,4
3 INDBC = 1
4 RETURN
END

```

Figure 19. Subroutine DTCHAN

```

SUBROUTINE OTCHAN
COMMON CCNA,CCNB,CCNC,DELT,FRS,FRV,ERSMAX,ERVMAX,H,HSD,
1 INDBCV,M,M1,M2,M3,MAXITS,MAXITV,MAXPRN,MAXTAP,N,N1,N2,
2 N3,NITS,NITV,NPRN,NTAP,NTCHAN,NTIME,RELAX,RH,RHSQ,TIME,
3 VEL,VIS,DT(8),MTAP(8),NT(8),STREN(15,100),VECSF(15),
4 VECV(100)
NTCHAN = NTCHAN + 1
NTIME = NT(NTCHAN)
IF(NTIME) 4,3,1
1 DELT = DT(NTCHAN)
MAXTAP = MTAP(NTCHAN)
CCNA = 4.*HSD/DFLT
CCNB = 6. - CCNA
CCNC = 2. + CCNA
VECV(2) = 1./CCNC
DO 2 J = 3,N1
2 VECV(J) = 1./((CCNC - VECV(J-1)))
RETURN
3 CALL RUN(10)
WRITE(15,100)
STOP
4 CALL RUN(10)
WRITE(15,100)
STOP 5
100 FORMAT(' REMOVE AND SAVE TAPE #527')
END

```

Computer Output

The following set of figures presents sample velocity fields for both case I and case II as they were output from the computer. Each figure lists the velocity in either the x-direction or the y-direction, u or v, as a function of the x- and y-coordinates at a given time plane. The velocity fields are for a Reynolds number of 50. All variables are in the nondimensional form

$$u = v_1/U ,$$

$$v = v_2/U ,$$

$$x = x_1/d ,$$

$$y = x_2/d ,$$

and

$$\tau = vt/d^2 .$$

As can be seen from the figures, the solution has converged to a steady state solution at a time of $\tau = 0.14$.

Figure 20. Computer output

VELOCITY IN X-DIRECTION

TIME = 0.01

REYNOLDS NUMBER = 50.0

CASE I

	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X												
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.821	1.020	1.060	1.060	1.051	1.043	1.037	1.033	1.031	1.030	1.030
0.10	0.000	0.679	0.991	1.092	1.108	1.097	1.083	1.072	1.064	1.060	1.058	1.058
0.15	0.000	0.585	0.943	1.096	1.136	1.131	1.116	1.102	1.092	1.086	1.084	1.084
0.20	0.000	0.530	0.900	1.085	1.148	1.152	1.139	1.125	1.114	1.108	1.106	1.106
0.25	0.000	0.501	0.869	1.070	1.148	1.162	1.154	1.141	1.131	1.125	1.123	1.123
0.30	0.000	0.488	0.851	1.056	1.143	1.165	1.161	1.152	1.144	1.139	1.137	1.137
0.35	0.000	0.483	0.841	1.046	1.136	1.163	1.165	1.159	1.153	1.149	1.147	1.147
0.40	0.000	0.482	0.836	1.039	1.131	1.161	1.166	1.163	1.158	1.155	1.154	1.154
0.45	0.000	0.482	0.833	1.034	1.126	1.159	1.166	1.165	1.162	1.160	1.159	1.159
0.50	0.000	0.482	0.831	1.032	1.123	1.157	1.166	1.166	1.165	1.163	1.163	1.163
0.60	0.000	0.483	0.831	1.029	1.120	1.155	1.165	1.167	1.167	1.166	1.166	1.166
0.70	0.000	0.484	0.830	1.028	1.119	1.154	1.165	1.167	1.168	1.168	1.168	1.168
0.80	0.000	0.484	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
0.90	0.000	0.484	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.00	0.000	0.484	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.10	0.000	0.484	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.20	0.000	0.484	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.30	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.40	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.50	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
1.75	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
2.00	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
2.25	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
2.50	0.000	0.485	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
2.75	0.000	0.485	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168
2.95	0.000	0.485	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168	1.168

Figure 21. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.01	REYNOLDS NUMBER = 50.0								CASE i
	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X													
0.00		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
0.05		0.000	0.275	0.321	0.284	0.229	0.177	0.132	0.093	0.060	0.029	0.000	
0.10		0.000	0.157	0.236	0.235	0.200	0.158	0.120	0.086	0.055	0.027	0.000	
0.15		0.000	0.079	0.150	0.170	0.156	0.130	0.101	0.074	0.048	0.024	0.000	
0.20		0.000	0.036	0.084	0.109	0.110	0.097	0.079	0.060	0.040	0.020	0.000	
0.25		0.000	0.014	0.041	0.063	0.071	0.068	0.058	0.046	0.031	0.016	0.000	
0.30		0.000	0.004	0.018	0.033	0.042	0.044	0.041	0.033	0.023	0.012	0.000	
0.35		0.000	0.000	0.006	0.016	0.024	0.028	0.027	0.023	0.017	0.009	0.000	
0.40		0.000	-0.001	0.001	0.007	0.013	0.017	0.018	0.016	0.011	0.006	0.000	
0.45		0.000	-0.001	-0.000	0.003	0.007	0.010	0.011	0.010	0.008	0.004	0.000	
0.50		0.000	-0.001	-0.001	0.001	0.004	0.006	0.007	0.007	0.005	0.003	0.000	
0.60		0.000	-0.001	-0.001	-0.000	0.001	0.002	0.003	0.003	0.002	0.001	0.000	
0.70		0.000	-0.000	-0.000	-0.000	0.000	0.001	0.001	0.001	0.001	0.000	0.000	
0.80		0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	
0.90		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	
1.00		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	
1.10		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
1.20		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
1.30		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
1.40		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	
1.50		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	
1.75		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	
2.00		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
2.25		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
2.50		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	
2.75		0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	
2.95		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	

Figure 22. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.02	REYNOLDS NUMBER = 50.0							CASE I
	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X												
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.806	1.006	1.052	1.058	1.055	1.050	1.046	1.043	1.041	1.041	1.041
0.10	0.000	0.651	0.960	1.074	1.104	1.104	1.097	1.090	1.084	1.081	1.080	1.080
0.15	0.000	0.542	0.893	1.066	1.130	1.143	1.139	1.131	1.123	1.119	1.117	1.117
0.20	0.000	0.470	0.828	1.040	1.138	1.170	1.173	1.166	1.158	1.153	1.151	1.151
0.25	0.000	0.425	0.776	1.010	1.135	1.186	1.199	1.195	1.189	1.183	1.182	1.182
0.30	0.000	0.397	0.738	0.982	1.126	1.194	1.217	1.218	1.214	1.209	1.207	1.207
0.35	0.000	0.381	0.712	0.960	1.115	1.196	1.228	1.235	1.233	1.229	1.228	1.228
0.40	0.000	0.372	0.697	0.944	1.105	1.194	1.234	1.246	1.246	1.244	1.243	1.243
0.45	0.000	0.369	0.689	0.933	1.097	1.191	1.236	1.252	1.255	1.255	1.254	1.254
0.50	0.000	0.370	0.686	0.928	1.091	1.187	1.235	1.255	1.260	1.261	1.261	1.261
0.60	0.000	0.375	0.689	0.925	1.085	1.180	1.231	1.254	1.263	1.266	1.266	1.266
0.70	0.000	0.380	0.694	0.928	1.084	1.177	1.227	1.251	1.261	1.265	1.266	1.266
0.80	0.000	0.383	0.698	0.930	1.084	1.176	1.225	1.248	1.259	1.263	1.264	1.264
0.90	0.000	0.385	0.700	0.932	1.085	1.175	1.224	1.247	1.257	1.261	1.262	1.262
1.00	0.000	0.385	0.701	0.933	1.085	1.175	1.223	1.246	1.256	1.260	1.261	1.261
1.10	0.000	0.386	0.702	0.933	1.086	1.175	1.223	1.246	1.256	1.260	1.261	1.261
1.20	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.246	1.255	1.259	1.260	1.260
1.30	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
1.40	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
1.50	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
1.75	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
2.00	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
2.25	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
2.50	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
2.75	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260
2.95	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260	1.260

Figure 23. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.02	REYNOLDS NUMBER = 50.0						CASE 1
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.286	0.349	0.325	0.275	0.222	0.171	0.124	0.081	0.040	0.000
0.10	0.000	0.166	0.264	0.280	0.251	0.208	0.163	0.119	0.078	0.038	0.000
0.15	0.000	0.087	0.180	0.220	0.214	0.185	0.148	0.110	0.072	0.036	0.000
0.20	0.000	0.046	0.117	0.162	0.172	0.157	0.129	0.097	0.065	0.032	0.000
0.25	0.000	0.024	0.073	0.114	0.131	0.126	0.108	0.083	0.056	0.028	0.000
0.30	0.000	0.013	0.044	0.076	0.095	0.096	0.085	0.067	0.046	0.023	0.000
0.35	0.000	0.006	0.025	0.047	0.063	0.068	0.063	0.051	0.035	0.018	0.000
0.40	0.000	0.002	0.011	0.025	0.038	0.044	0.043	0.036	0.025	0.013	0.000
0.45	0.000	-0.001	0.003	0.010	0.018	0.024	0.026	0.023	0.017	0.009	0.000
0.50	0.000	-0.002	-0.003	0.000	0.005	0.010	0.013	0.012	0.010	0.005	0.000
0.60	0.000	-0.002	-0.005	-0.007	-0.006	-0.004	-0.001	0.001	0.001	0.001	0.000
0.70	0.000	-0.002	-0.004	-0.006	-0.007	-0.006	-0.005	-0.003	-0.002	-0.001	0.000
0.80	0.000	-0.001	-0.002	-0.004	-0.005	-0.004	-0.004	-0.003	-0.002	-0.001	0.000
0.90	0.000	-0.000	-0.001	-0.002	-0.003	-0.003	-0.002	-0.002	-0.001	-0.001	0.000
1.00	0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	0.000
1.10	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.000	-0.000	0.000
1.20	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
1.30	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
1.40	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
1.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
1.75	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000
2.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	0.000	0.000
2.25	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	-0.000	0.000
2.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 24. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.04	REYNOLDS NUMBER = 50.0						CASE 1
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.803	1.003	1.050	1.057	1.055	1.051	1.048	1.045	1.044	1.044
0.10	0.000	0.645	0.954	1.069	1.102	1.105	1.100	1.094	1.090	1.087	1.086
0.15	0.000	0.533	0.882	1.057	1.126	1.144	1.143	1.137	1.132	1.128	1.127
0.20	0.000	0.460	0.813	1.027	1.132	1.171	1.179	1.176	1.170	1.167	1.165
0.25	0.000	0.412	0.755	0.992	1.126	1.186	1.206	1.208	1.205	1.202	1.201
0.30	0.000	0.380	0.711	0.958	1.113	1.193	1.226	1.236	1.236	1.233	1.232
0.35	0.000	0.359	0.678	0.928	1.097	1.194	1.240	1.258	1.262	1.261	1.261
0.40	0.000	0.345	0.653	0.903	1.082	1.192	1.250	1.275	1.283	1.285	1.285
0.45	0.000	0.335	0.635	0.883	1.068	1.188	1.255	1.288	1.302	1.306	1.307
0.50	0.000	0.328	0.621	0.867	1.055	1.183	1.259	1.299	1.317	1.324	1.325
0.60	0.000	0.319	0.603	0.845	1.035	1.173	1.261	1.313	1.339	1.351	1.354
0.70	0.000	0.314	0.594	0.832	1.022	1.164	1.261	1.320	1.354	1.369	1.374
0.80	0.000	0.313	0.590	0.825	1.014	1.158	1.259	1.324	1.362	1.381	1.387
0.90	0.000	0.314	0.590	0.823	1.010	1.153	1.256	1.324	1.365	1.387	1.393
1.00	0.000	0.316	0.593	0.824	1.010	1.151	1.253	1.322	1.364	1.387	1.394
1.10	0.000	0.319	0.596	0.828	1.012	1.151	1.251	1.319	1.361	1.384	1.391
1.20	0.000	0.321	0.599	0.831	1.014	1.152	1.250	1.317	1.358	1.380	1.387
1.30	0.000	0.322	0.602	0.833	1.016	1.153	1.250	1.315	1.355	1.377	1.384
1.40	0.000	0.323	0.603	0.834	1.017	1.153	1.250	1.314	1.354	1.375	1.382
1.50	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.381
1.75	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.00	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.25	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.50	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.75	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.95	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380

Figure 25. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.04	REYNOLDS NUMBER = 50.0						CASE I	
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.288	0.355	0.334	0.286	0.232	0.181	0.133	0.087	0.043	0.000	0.000
0.10	0.000	0.167	0.270	0.289	0.263	0.220	0.174	0.128	0.084	0.042	0.000	0.000
0.15	0.000	0.088	0.185	0.229	0.227	0.199	0.161	0.120	0.080	0.040	0.000	0.000
0.20	0.000	0.047	0.122	0.173	0.187	0.174	0.145	0.110	0.074	0.037	0.000	0.000
0.25	0.000	0.025	0.080	0.127	0.149	0.146	0.127	0.099	0.067	0.034	0.000	0.000
0.30	0.000	0.015	0.053	0.092	0.116	0.121	0.108	0.086	0.059	0.030	0.000	0.000
0.35	0.000	0.009	0.035	0.067	0.090	0.098	0.091	0.074	0.052	0.026	0.000	0.000
0.40	0.000	0.006	0.024	0.049	0.069	0.078	0.076	0.063	0.045	0.023	0.000	0.000
0.45	0.000	0.004	0.017	0.036	0.053	0.063	0.062	0.053	0.038	0.020	0.000	0.000
0.50	0.000	0.003	0.012	0.027	0.041	0.050	0.051	0.044	0.032	0.017	0.000	0.000
0.60	0.000	0.001	0.006	0.015	0.024	0.031	0.033	0.030	0.023	0.012	0.000	0.000
0.70	0.000	0.001	0.003	0.007	0.013	0.018	0.020	0.019	0.015	0.008	0.000	0.000
0.80	0.000	-0.000	0.000	0.002	0.005	0.008	0.010	0.010	0.009	0.005	0.000	0.000
0.90	0.000	-0.001	-0.002	-0.002	-0.002	-0.000	0.002	0.003	0.003	0.002	0.000	0.000
1.00	0.000	-0.001	-0.003	-0.004	-0.005	-0.005	-0.004	-0.003	-0.002	-0.001	0.000	0.000
1.10	0.000	-0.001	-0.002	-0.004	-0.006	-0.007	-0.006	-0.005	-0.004	-0.002	0.000	0.000
1.20	0.000	-0.001	-0.002	-0.003	-0.005	-0.005	-0.006	-0.005	-0.004	-0.002	0.000	0.000
1.30	0.000	-0.000	-0.001	-0.002	-0.003	-0.003	-0.004	-0.003	-0.003	-0.001	0.000	0.000
1.40	0.000	-0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.002	-0.001	-0.001	0.000	0.000
1.50	0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.000	0.000	0.000
1.75	0.000	0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000
2.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.25	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000
2.75	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 26. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.06	REYNOLDS NUMBER = 50.0							CASE I
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.803	1.003	1.050	1.057	1.055	1.051	1.048	1.045	1.044	1.044	1.044
0.10	0.000	0.645	0.953	1.069	1.102	1.105	1.100	1.094	1.090	1.087	1.086	1.086
0.15	0.000	0.533	0.851	1.057	1.126	1.144	1.143	1.137	1.132	1.129	1.128	1.128
0.20	0.000	0.459	0.812	1.027	1.132	1.171	1.179	1.176	1.171	1.167	1.166	1.166
0.25	0.000	0.411	0.754	0.991	1.125	1.186	1.206	1.209	1.206	1.203	1.202	1.202
0.30	0.000	0.379	0.710	0.957	1.112	1.193	1.227	1.236	1.237	1.235	1.234	1.234
0.35	0.000	0.358	0.676	0.927	1.096	1.194	1.241	1.259	1.263	1.263	1.262	1.262
0.40	0.000	0.344	0.651	0.901	1.080	1.191	1.250	1.276	1.285	1.287	1.288	1.288
0.45	0.000	0.334	0.633	0.881	1.066	1.187	1.256	1.290	1.304	1.309	1.310	1.310
0.50	0.000	0.326	0.618	0.864	1.053	1.182	1.260	1.301	1.320	1.327	1.329	1.329
0.60	0.000	0.317	0.599	0.840	1.032	1.171	1.262	1.316	1.344	1.356	1.360	1.360
0.70	0.000	0.311	0.587	0.824	1.016	1.162	1.262	1.325	1.361	1.378	1.383	1.383
0.80	0.000	0.307	0.579	0.813	1.005	1.154	1.261	1.331	1.373	1.395	1.401	1.401
0.90	0.000	0.304	0.573	0.805	0.997	1.148	1.260	1.336	1.383	1.408	1.415	1.415
1.00	0.000	0.301	0.568	0.799	0.991	1.144	1.258	1.339	1.390	1.418	1.426	1.426
1.10	0.000	0.300	0.565	0.794	0.986	1.140	1.257	1.341	1.395	1.425	1.435	1.435
1.20	0.000	0.298	0.563	0.791	0.983	1.138	1.256	1.342	1.399	1.431	1.441	1.441
1.30	0.000	0.298	0.561	0.790	0.981	1.136	1.255	1.343	1.401	1.435	1.445	1.445
1.40	0.000	0.298	0.561	0.789	0.980	1.134	1.254	1.343	1.402	1.437	1.448	1.448
1.50	0.000	0.298	0.562	0.789	0.980	1.134	1.254	1.342	1.402	1.437	1.448	1.448
1.75	0.000	0.300	0.565	0.793	0.982	1.135	1.253	1.340	1.399	1.433	1.444	1.444
2.00	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441
2.25	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441
2.50	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441
2.75	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441
2.95	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441

Figure 27. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.06	REYNOLDS NUMBER = 50.0							CASE I
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.288	0.355	0.335	0.286	0.233	0.181	0.133	0.087	0.043	0.000	0.000
0.10	0.000	0.167	0.270	0.289	0.263	0.221	0.174	0.129	0.085	0.042	0.000	0.000
0.15	0.000	0.088	0.185	0.230	0.228	0.200	0.162	0.121	0.080	0.040	0.000	0.000
0.20	0.000	0.047	0.122	0.174	0.188	0.174	0.146	0.111	0.074	0.037	0.000	0.000
0.25	0.000	0.025	0.080	0.128	0.150	0.147	0.128	0.099	0.067	0.034	0.000	0.000
0.30	0.000	0.015	0.053	0.093	0.117	0.122	0.110	0.087	0.060	0.030	0.000	0.000
0.35	0.000	0.009	0.036	0.068	0.091	0.099	0.093	0.076	0.053	0.027	0.000	0.000
0.40	0.000	0.006	0.025	0.050	0.070	0.080	0.077	0.065	0.046	0.024	0.000	0.000
0.45	0.000	0.004	0.018	0.037	0.055	0.065	0.064	0.055	0.040	0.021	0.000	0.000
0.50	0.000	0.003	0.013	0.028	0.043	0.052	0.053	0.047	0.034	0.018	0.000	0.000
0.60	0.000	0.002	0.007	0.017	0.027	0.035	0.037	0.034	0.025	0.014	0.000	0.000
0.70	0.000	0.001	0.005	0.011	0.018	0.024	0.027	0.025	0.019	0.010	0.000	0.000
0.80	0.000	0.001	0.003	0.008	0.013	0.017	0.020	0.019	0.015	0.008	0.000	0.000
0.90	0.000	0.001	0.003	0.006	0.010	0.013	0.015	0.014	0.011	0.006	0.000	0.000
1.00	0.000	0.001	0.002	0.004	0.007	0.010	0.011	0.011	0.009	0.005	0.000	0.000
1.10	0.000	0.000	0.002	0.003	0.005	0.007	0.008	0.008	0.007	0.004	0.000	0.000
1.20	0.000	0.000	0.001	0.002	0.003	0.005	0.006	0.006	0.005	0.003	0.000	0.000
1.30	0.000	0.000	0.000	0.001	0.002	0.002	0.003	0.003	0.003	0.002	0.000	0.000
1.40	0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.001	0.001	0.001	0.001	0.000	0.000
1.50	0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.001	-0.001	-0.000	-0.000	0.000	0.000
1.75	0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.002	-0.002	-0.002	-0.001	0.000	0.000
2.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.000	-0.000	0.000	0.000
2.25	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.50	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.75	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 28. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.10	REYNOLDS NUMBER = 50.0						CASE I	
	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X												
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.803	1.003	1.050	1.057	1.055	1.051	1.048	1.048	1.045	1.044	1.044
0.10	0.000	0.645	0.953	1.069	1.102	1.105	1.100	1.094	1.094	1.090	1.087	1.086
0.15	0.000	0.533	0.881	1.057	1.126	1.144	1.143	1.137	1.137	1.132	1.129	1.128
0.20	0.000	0.459	0.812	1.027	1.132	1.171	1.179	1.176	1.176	1.171	1.167	1.166
0.25	0.000	0.411	0.754	0.991	1.125	1.186	1.206	1.209	1.209	1.206	1.203	1.202
0.30	0.000	0.379	0.710	0.957	1.112	1.193	1.227	1.236	1.236	1.237	1.235	1.234
0.35	0.000	0.358	0.676	0.927	1.096	1.194	1.241	1.259	1.259	1.263	1.263	1.262
0.40	0.000	0.344	0.651	0.901	1.080	1.191	1.250	1.276	1.276	1.285	1.287	1.288
0.45	0.000	0.334	0.632	0.881	1.066	1.187	1.256	1.290	1.290	1.304	1.309	1.310
0.50	0.000	0.326	0.618	0.864	1.053	1.182	1.260	1.301	1.301	1.320	1.327	1.329
0.60	0.000	0.317	0.599	0.840	1.031	1.171	1.262	1.316	1.316	1.344	1.357	1.360
0.70	0.000	0.311	0.587	0.824	1.016	1.161	1.262	1.326	1.326	1.361	1.379	1.384
0.80	0.000	0.307	0.578	0.812	1.005	1.154	1.261	1.332	1.332	1.374	1.396	1.402
0.90	0.000	0.303	0.572	0.804	0.996	1.148	1.260	1.336	1.336	1.384	1.409	1.417
1.00	0.000	0.301	0.567	0.797	0.990	1.143	1.258	1.339	1.339	1.391	1.419	1.428
1.10	0.000	0.299	0.563	0.792	0.984	1.139	1.257	1.342	1.342	1.397	1.428	1.438
1.20	0.000	0.297	0.560	0.788	0.980	1.136	1.257	1.344	1.344	1.402	1.435	1.446
1.30	0.000	0.295	0.557	0.784	0.977	1.134	1.256	1.346	1.346	1.407	1.441	1.452
1.40	0.000	0.294	0.554	0.781	0.974	1.131	1.256	1.348	1.348	1.410	1.446	1.458
1.50	0.000	0.292	0.552	0.778	0.971	1.130	1.255	1.349	1.349	1.413	1.451	1.463
1.75	0.000	0.290	0.548	0.773	0.966	1.126	1.255	1.352	1.352	1.419	1.459	1.472
2.00	0.000	0.288	0.545	0.770	0.963	1.124	1.254	1.354	1.354	1.423	1.464	1.478
2.25	0.000	0.287	0.543	0.768	0.961	1.123	1.254	1.355	1.355	1.426	1.468	1.482
2.50	0.000	0.287	0.542	0.767	0.960	1.122	1.254	1.355	1.355	1.426	1.469	1.483
2.75	0.000	0.287	0.543	0.767	0.961	1.122	1.254	1.355	1.355	1.426	1.469	1.483
2.95	0.000	0.287	0.543	0.768	0.961	1.123	1.254	1.354	1.354	1.426	1.468	1.483

Figure 29. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.10	REYNOLDS NUMBER = 50.0						CASE I
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.288	0.355	0.335	0.286	0.233	0.181	0.133	0.087	0.043	0.000
0.10	0.000	0.167	0.270	0.289	0.263	0.221	0.174	0.129	0.085	0.042	0.000
0.15	0.000	0.088	0.185	0.230	0.228	0.200	0.162	0.121	0.080	0.040	0.000
0.20	0.000	0.047	0.122	0.174	0.188	0.174	0.146	0.111	0.074	0.037	0.000
0.25	0.000	0.025	0.080	0.128	0.150	0.147	0.128	0.099	0.067	0.034	0.000
0.30	0.000	0.015	0.053	0.093	0.117	0.122	0.110	0.087	0.060	0.030	0.000
0.35	0.000	0.009	0.036	0.068	0.091	0.099	0.093	0.076	0.053	0.027	0.000
0.40	0.000	0.006	0.025	0.050	0.071	0.080	0.077	0.065	0.046	0.024	0.000
0.45	0.000	0.004	0.018	0.037	0.055	0.065	0.064	0.055	0.040	0.021	0.000
0.50	0.000	0.003	0.013	0.028	0.043	0.052	0.054	0.047	0.034	0.018	0.000
0.60	0.000	0.002	0.007	0.017	0.027	0.035	0.037	0.034	0.025	0.014	0.000
0.70	0.000	0.001	0.005	0.011	0.018	0.024	0.027	0.025	0.019	0.010	0.000
0.80	0.000	0.001	0.004	0.008	0.013	0.018	0.020	0.019	0.015	0.008	0.000
0.90	0.000	0.001	0.003	0.006	0.010	0.014	0.015	0.015	0.012	0.006	0.000
1.00	0.000	0.001	0.002	0.005	0.008	0.011	0.012	0.012	0.010	0.005	0.000
1.10	0.000	0.001	0.002	0.004	0.007	0.009	0.010	0.010	0.008	0.004	0.000
1.20	0.000	0.000	0.002	0.004	0.006	0.008	0.009	0.008	0.007	0.004	0.000
1.30	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.007	0.006	0.003	0.000
1.40	0.000	0.000	0.001	0.003	0.004	0.006	0.006	0.006	0.005	0.003	0.000
1.50	0.000	0.000	0.001	0.002	0.004	0.005	0.005	0.005	0.004	0.002	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.003	0.004	0.003	0.003	0.001	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.001	0.000
2.25	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000
2.50	0.000	0.000	0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.75	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 30. Computer output

VELOCITY IN X-DIRECTION		TIME = 0.14					REYNOLDS NUMBER = 50.0					CASE I
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X												
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	
0.05	0.000	0.803	1.003	1.050	1.057	1.055	1.051	1.048	1.045	1.044	1.044	
0.10	0.000	0.645	0.953	1.069	1.102	1.105	1.100	1.094	1.090	1.087	1.086	
0.15	0.000	0.533	0.881	1.057	1.126	1.144	1.143	1.137	1.132	1.129	1.128	
0.20	0.000	0.459	0.812	1.027	1.132	1.171	1.179	1.176	1.171	1.167	1.166	
0.25	0.000	0.411	0.754	0.991	1.125	1.186	1.206	1.209	1.206	1.203	1.202	
0.30	0.000	0.379	0.710	0.957	1.112	1.193	1.227	1.236	1.237	1.235	1.234	
0.35	0.000	0.358	0.676	0.927	1.096	1.194	1.241	1.259	1.263	1.263	1.262	
0.40	0.000	0.344	0.651	0.901	1.080	1.191	1.250	1.276	1.285	1.287	1.288	
0.45	0.000	0.334	0.632	0.881	1.066	1.187	1.256	1.290	1.304	1.309	1.310	
0.50	0.000	0.326	0.618	0.864	1.053	1.182	1.260	1.301	1.320	1.327	1.329	
0.60	0.000	0.317	0.599	0.840	1.031	1.171	1.262	1.316	1.344	1.357	1.360	
0.70	0.000	0.311	0.587	0.824	1.016	1.161	1.262	1.326	1.361	1.379	1.384	
0.80	0.000	0.307	0.578	0.812	1.005	1.154	1.261	1.332	1.374	1.396	1.402	
0.90	0.000	0.303	0.572	0.804	0.996	1.148	1.260	1.336	1.384	1.409	1.417	
1.00	0.000	0.301	0.567	0.797	0.990	1.143	1.258	1.339	1.391	1.419	1.428	
1.10	0.000	0.299	0.563	0.792	0.984	1.139	1.257	1.342	1.397	1.428	1.438	
1.20	0.000	0.297	0.560	0.788	0.980	1.136	1.257	1.344	1.402	1.435	1.446	
1.30	0.000	0.295	0.557	0.784	0.977	1.134	1.256	1.346	1.407	1.441	1.452	
1.40	0.000	0.294	0.554	0.781	0.974	1.131	1.256	1.348	1.410	1.446	1.458	
1.50	0.000	0.292	0.552	0.778	0.971	1.130	1.255	1.349	1.413	1.451	1.463	
1.75	0.000	0.290	0.548	0.773	0.966	1.126	1.255	1.352	1.419	1.459	1.472	
2.00	0.000	0.288	0.544	0.769	0.963	1.124	1.254	1.354	1.423	1.465	1.478	
2.25	0.000	0.287	0.542	0.767	0.960	1.123	1.254	1.355	1.426	1.469	1.483	
2.50	0.000	0.286	0.541	0.765	0.959	1.122	1.254	1.356	1.428	1.471	1.486	
2.75	0.000	0.285	0.540	0.764	0.958	1.121	1.254	1.357	1.430	1.473	1.488	
2.95	0.000	0.285	0.540	0.764	0.958	1.121	1.254	1.357	1.430	1.474	1.488	

Figure 31. Computer output

VELOCITY IN Y-DIRECTION				TIME =	REYNOLDS NUMBER =			50.0			CASE I
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.288	0.355	0.335	0.286	0.233	0.181	0.133	0.087	0.043	0.000
0.10	0.000	0.167	0.270	0.289	0.263	0.221	0.174	0.129	0.085	0.042	0.000
0.15	0.000	0.088	0.185	0.230	0.228	0.200	0.162	0.121	0.080	0.040	0.000
0.20	0.000	0.047	0.122	0.174	0.188	0.174	0.146	0.111	0.074	0.037	0.000
0.25	0.000	0.025	0.080	0.128	0.150	0.147	0.128	0.099	0.067	0.034	0.000
0.30	0.000	0.015	0.053	0.093	0.117	0.122	0.110	0.087	0.060	0.030	0.000
0.35	0.000	0.009	0.036	0.068	0.091	0.099	0.093	0.076	0.053	0.027	0.000
0.40	0.000	0.006	0.025	0.050	0.071	0.080	0.077	0.065	0.046	0.024	0.000
0.45	0.000	0.004	0.018	0.037	0.055	0.065	0.064	0.055	0.040	0.021	0.000
0.50	0.000	0.003	0.013	0.028	0.043	0.052	0.054	0.047	0.034	0.018	0.000
0.60	0.000	0.002	0.007	0.017	0.027	0.035	0.037	0.034	0.025	0.014	0.000
0.70	0.000	0.001	0.005	0.011	0.018	0.024	0.027	0.025	0.019	0.010	0.000
0.80	0.000	0.001	0.004	0.008	0.013	0.018	0.020	0.019	0.015	0.008	0.000
0.90	0.000	0.001	0.003	0.006	0.010	0.014	0.015	0.015	0.012	0.006	0.000
1.00	0.000	0.001	0.002	0.005	0.008	0.011	0.012	0.012	0.010	0.005	0.000
1.10	0.000	0.001	0.002	0.004	0.007	0.009	0.010	0.010	0.008	0.004	0.000
1.20	0.000	0.000	0.002	0.004	0.006	0.008	0.009	0.008	0.007	0.004	0.000
1.30	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.007	0.006	0.003	0.000
1.40	0.000	0.000	0.001	0.003	0.004	0.006	0.006	0.006	0.005	0.003	0.000
1.50	0.000	0.000	0.001	0.002	0.004	0.005	0.005	0.005	0.004	0.002	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.003	0.004	0.004	0.003	0.002	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.002	0.003	0.002	0.002	0.001	0.000
2.25	0.000	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.000
2.50	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 32. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.18		REYNOLDS NUMBER = 50.0					CASE I	
	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X												
0.00	0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	0.000	0.803	1.003	1.050	1.057	1.055	1.051	1.048	1.045	1.044	1.044	1.044
0.10	0.000	0.645	0.953	1.069	1.102	1.105	1.100	1.094	1.090	1.087	1.086	1.086
0.15	0.000	0.533	0.881	1.057	1.126	1.144	1.143	1.137	1.132	1.129	1.128	1.128
0.20	0.000	0.459	0.812	1.027	1.132	1.171	1.179	1.176	1.171	1.167	1.166	1.166
0.25	0.000	0.411	0.754	0.991	1.125	1.186	1.206	1.209	1.206	1.203	1.202	1.202
0.30	0.000	0.379	0.710	0.957	1.112	1.193	1.227	1.236	1.237	1.235	1.234	1.234
0.35	0.000	0.358	0.676	0.927	1.096	1.194	1.241	1.259	1.263	1.263	1.262	1.262
0.40	0.000	0.344	0.651	0.901	1.080	1.191	1.250	1.276	1.285	1.287	1.288	1.288
0.45	0.000	0.334	0.632	0.881	1.066	1.187	1.256	1.290	1.304	1.309	1.310	1.310
0.50	0.000	0.326	0.618	0.864	1.053	1.182	1.260	1.301	1.320	1.327	1.329	1.329
0.60	0.000	0.317	0.599	0.840	1.031	1.171	1.262	1.316	1.344	1.357	1.360	1.360
0.70	0.000	0.311	0.587	0.824	1.016	1.161	1.262	1.326	1.361	1.379	1.384	1.384
0.80	0.000	0.307	0.578	0.812	1.005	1.154	1.261	1.332	1.374	1.396	1.402	1.402
0.90	0.000	0.303	0.572	0.804	0.996	1.148	1.260	1.336	1.384	1.409	1.417	1.417
1.00	0.000	0.301	0.567	0.797	0.990	1.143	1.258	1.339	1.391	1.419	1.428	1.428
1.10	0.000	0.299	0.563	0.792	0.984	1.139	1.257	1.342	1.397	1.428	1.438	1.438
1.20	0.000	0.297	0.560	0.788	0.980	1.136	1.257	1.344	1.402	1.435	1.446	1.446
1.30	0.000	0.295	0.557	0.784	0.977	1.134	1.256	1.346	1.407	1.441	1.452	1.452
1.40	0.000	0.294	0.554	0.781	0.974	1.131	1.256	1.348	1.410	1.446	1.458	1.458
1.50	0.000	0.292	0.552	0.778	0.971	1.130	1.255	1.349	1.413	1.451	1.463	1.463
1.75	0.000	0.290	0.548	0.773	0.966	1.126	1.255	1.352	1.419	1.459	1.472	1.472
2.00	0.000	0.288	0.544	0.769	0.963	1.124	1.254	1.354	1.423	1.465	1.478	1.478
2.25	0.000	0.287	0.542	0.767	0.960	1.123	1.254	1.355	1.426	1.469	1.483	1.483
2.50	0.000	0.286	0.541	0.765	0.959	1.122	1.254	1.356	1.428	1.471	1.486	1.486
2.75	0.000	0.285	0.540	0.764	0.958	1.121	1.254	1.357	1.430	1.473	1.488	1.488
2.95	0.000	0.285	0.540	0.764	0.958	1.121	1.254	1.357	1.430	1.474	1.488	1.488

Figure 33. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.18	REYNOLDS NUMBER = 50.0							CASE I
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.288	0.355	0.335	0.286	0.233	0.181	0.133	0.087	0.043	0.000	0.000
0.10	0.000	0.167	0.270	0.289	0.263	0.221	0.174	0.129	0.085	0.042	0.000	0.000
0.15	0.000	0.088	0.185	0.230	0.228	0.200	0.162	0.121	0.080	0.040	0.000	0.000
0.20	0.000	0.047	0.122	0.174	0.188	0.174	0.146	0.111	0.074	0.037	0.000	0.000
0.25	0.000	0.025	0.080	0.128	0.150	0.147	0.128	0.099	0.067	0.034	0.000	0.000
0.30	0.000	0.015	0.053	0.093	0.117	0.122	0.110	0.087	0.060	0.030	0.000	0.000
0.35	0.000	0.009	0.036	0.068	0.091	0.099	0.093	0.076	0.053	0.027	0.000	0.000
0.40	0.000	0.006	0.025	0.050	0.071	0.080	0.077	0.065	0.046	0.024	0.000	0.000
0.45	0.000	0.004	0.018	0.037	0.055	0.065	0.064	0.055	0.040	0.021	0.000	0.000
0.50	0.000	0.003	0.013	0.028	0.043	0.052	0.054	0.047	0.034	0.018	0.000	0.000
0.60	0.000	0.002	0.007	0.017	0.027	0.035	0.037	0.034	0.025	0.014	0.000	0.000
0.70	0.000	0.001	0.005	0.011	0.018	0.024	0.027	0.025	0.019	0.010	0.000	0.000
0.80	0.000	0.001	0.004	0.008	0.013	0.018	0.020	0.019	0.015	0.008	0.000	0.000
0.90	0.000	0.001	0.003	0.006	0.010	0.014	0.015	0.015	0.012	0.006	0.000	0.000
1.00	0.000	0.001	0.002	0.005	0.008	0.011	0.012	0.012	0.010	0.005	0.000	0.000
1.10	0.000	0.001	0.002	0.004	0.007	0.009	0.010	0.010	0.008	0.004	0.000	0.000
1.20	0.000	0.000	0.002	0.004	0.006	0.008	0.009	0.008	0.007	0.004	0.000	0.000
1.30	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.007	0.006	0.003	0.000	0.000
1.40	0.000	0.000	0.001	0.003	0.004	0.006	0.006	0.006	0.005	0.003	0.000	0.000
1.50	0.000	0.000	0.001	0.002	0.004	0.005	0.005	0.005	0.004	0.002	0.000	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.003	0.004	0.004	0.003	0.002	0.000	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.002	0.003	0.002	0.002	0.001	0.000	0.000
2.25	0.000	0.000	0.000	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.000	0.000
2.50	0.000	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.001	0.001	0.001	0.000	0.000	0.000	0.000
2.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 34. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.01		REYNOLDS NUMBER = 50.0					CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50	0.994	0.994	0.995	0.996	0.998	1.000	1.002	1.004	1.005	1.006	1.006
-0.40	0.988	0.988	0.990	0.993	0.997	1.001	1.004	1.007	1.009	1.010	1.011
-0.30	0.975	0.977	0.982	0.988	0.996	1.003	1.009	1.013	1.017	1.019	1.019
-0.25	0.965	0.967	0.975	0.985	0.995	1.005	1.012	1.018	1.022	1.025	1.026
-0.20	0.948	0.953	0.965	0.981	0.996	1.008	1.018	1.025	1.030	1.032	1.033
-0.15	0.921	0.931	0.954	0.978	0.999	1.015	1.026	1.034	1.039	1.042	1.043
-0.10	0.875	0.897	0.940	0.979	1.006	1.025	1.038	1.046	1.051	1.053	1.054
-0.05	0.786	0.840	0.931	0.989	1.023	1.043	1.054	1.061	1.065	1.067	1.067
0.00	0.000	0.744	0.957	1.028	1.056	1.069	1.075	1.079	1.081	1.081	1.082
0.05	0.000	0.650	0.949	1.057	1.090	1.098	1.099	1.098	1.097	1.097	1.097
0.10	0.000	0.579	0.920	1.068	1.116	1.124	1.121	1.117	1.114	1.112	1.111
0.15	0.000	0.533	0.890	1.066	1.130	1.143	1.139	1.133	1.128	1.125	1.124
0.20	0.000	0.507	0.866	1.058	1.134	1.153	1.151	1.145	1.140	1.137	1.135
0.25	0.000	0.494	0.851	1.050	1.133	1.157	1.158	1.153	1.149	1.146	1.145
0.30	0.000	0.488	0.842	1.042	1.130	1.158	1.162	1.159	1.155	1.153	1.152
0.35	0.000	0.485	0.837	1.037	1.127	1.158	1.164	1.162	1.160	1.158	1.157
0.40	0.000	0.484	0.834	1.034	1.124	1.156	1.164	1.164	1.163	1.161	1.161
0.45	0.000	0.484	0.833	1.031	1.122	1.155	1.165	1.166	1.165	1.164	1.163
0.50	0.000	0.484	0.832	1.030	1.121	1.155	1.165	1.166	1.166	1.165	1.165
0.60	0.000	0.484	0.831	1.028	1.119	1.154	1.164	1.167	1.167	1.167	1.167
0.70	0.000	0.484	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168
0.80	0.000	0.484	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168
0.90	0.000	0.484	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168
1.00	0.000	0.484	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168
1.25	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168
1.50	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168
1.75	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168
2.00	0.000	0.485	0.831	1.027	1.118	1.153	1.164	1.167	1.168	1.168	1.168
2.75	0.000	0.485	0.831	1.028	1.118	1.153	1.164	1.167	1.168	1.168	1.168

Figure 35. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.01	REYNOLDS NUMBER = 50.0							CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X												
-0.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75	0.000	0.000	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
-0.50	0.000	0.002	0.004	0.005	0.006	0.006	0.006	0.005	0.004	0.002	0.000	0.000
-0.40	0.000	0.004	0.008	0.010	0.012	0.012	0.011	0.009	0.006	0.003	0.000	0.000
-0.30	0.000	0.009	0.016	0.021	0.023	0.022	0.019	0.015	0.011	0.005	0.000	0.000
-0.25	0.000	0.014	0.024	0.030	0.031	0.029	0.025	0.020	0.014	0.007	0.000	0.000
-0.20	0.000	0.022	0.036	0.043	0.043	0.039	0.033	0.025	0.017	0.009	0.000	0.000
-0.15	0.000	0.036	0.056	0.062	0.059	0.051	0.042	0.032	0.021	0.010	0.000	0.000
-0.10	0.000	0.068	0.091	0.090	0.079	0.066	0.052	0.038	0.025	0.012	0.000	0.000
-0.05	0.000	0.147	0.153	0.130	0.104	0.081	0.061	0.043	0.028	0.014	0.000	0.000
0.00	0.000	0.176	0.191	0.159	0.123	0.092	0.067	0.047	0.030	0.015	0.000	0.000
0.05	0.000	0.117	0.165	0.154	0.124	0.094	0.068	0.048	0.030	0.015	0.000	0.000
0.10	0.000	0.065	0.117	0.124	0.107	0.085	0.063	0.044	0.028	0.014	0.000	0.000
0.15	0.000	0.033	0.072	0.087	0.082	0.068	0.053	0.038	0.025	0.012	0.000	0.000
0.20	0.000	0.015	0.039	0.054	0.056	0.050	0.041	0.031	0.021	0.010	0.000	0.000
0.25	0.000	0.006	0.019	0.030	0.035	0.034	0.030	0.024	0.016	0.008	0.000	0.000
0.30	0.000	0.002	0.009	0.016	0.021	0.023	0.021	0.017	0.012	0.006	0.000	0.000
0.35	0.000	0.000	0.004	0.008	0.012	0.014	0.014	0.012	0.009	0.004	0.000	0.000
0.40	0.000	-0.000	0.001	0.004	0.007	0.009	0.009	0.008	0.006	0.003	0.000	0.000
0.45	0.000	-0.000	0.000	0.002	0.004	0.006	0.006	0.005	0.004	0.002	0.000	0.000
0.50	0.000	-0.000	0.000	0.001	0.002	0.003	0.004	0.004	0.003	0.001	0.000	0.000
0.60	0.000	-0.000	-0.000	0.000	0.001	0.001	0.002	0.001	0.001	0.001	0.000	0.000
0.70	0.000	-0.000	-0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.000
0.80	0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.90	0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
1.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000
1.25	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000
1.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000
1.75	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000
2.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 36. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.02	REYNOLDS NUMBER = 50.0						CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50	0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.006	1.007
-0.40	0.986	0.987	0.989	0.993	0.997	1.001	1.005	1.008	1.010	1.012	1.012
-0.30	0.973	0.974	0.980	0.987	0.995	1.003	1.009	1.015	1.019	1.021	1.022
-0.25	0.961	0.964	0.972	0.983	0.994	1.005	1.013	1.020	1.025	1.028	1.029
-0.20	0.943	0.948	0.962	0.978	0.994	1.008	1.020	1.028	1.034	1.037	1.038
-0.15	0.915	0.925	0.948	0.974	0.997	1.015	1.028	1.038	1.045	1.048	1.050
-0.10	0.866	0.888	0.932	0.973	1.004	1.026	1.041	1.052	1.058	1.062	1.063
-0.05	0.774	0.828	0.921	0.982	1.019	1.043	1.059	1.069	1.075	1.078	1.080
0.00	0.000	0.727	0.941	1.016	1.050	1.070	1.082	1.089	1.094	1.097	1.098
0.05	0.000	0.629	0.924	1.040	1.083	1.100	1.108	1.113	1.116	1.118	1.118
0.10	0.000	0.550	0.884	1.043	1.107	1.129	1.136	1.138	1.139	1.139	1.140
0.15	0.000	0.492	0.838	1.032	1.120	1.152	1.161	1.163	1.162	1.161	1.161
0.20	0.000	0.452	0.796	1.012	1.123	1.169	1.183	1.185	1.184	1.182	1.182
0.25	0.000	0.423	0.762	0.991	1.120	1.179	1.200	1.205	1.204	1.202	1.201
0.30	0.000	0.404	0.737	0.973	1.114	1.184	1.213	1.220	1.220	1.219	1.218
0.35	0.000	0.392	0.719	0.957	1.107	1.186	1.221	1.232	1.234	1.232	1.232
0.40	0.000	0.385	0.707	0.946	1.100	1.186	1.226	1.240	1.244	1.243	1.243
0.45	0.000	0.381	0.701	0.939	1.095	1.184	1.228	1.245	1.250	1.251	1.250
0.50	0.000	0.380	0.698	0.934	1.091	1.182	1.228	1.248	1.254	1.256	1.256
0.60	0.000	0.382	0.697	0.931	1.087	1.178	1.227	1.249	1.257	1.260	1.261
0.70	0.000	0.383	0.699	0.932	1.085	1.176	1.225	1.248	1.257	1.261	1.262
0.80	0.000	0.385	0.700	0.932	1.085	1.176	1.224	1.247	1.257	1.260	1.261
0.90	0.000	0.385	0.701	0.933	1.086	1.175	1.223	1.246	1.256	1.260	1.261
1.00	0.000	0.386	0.702	0.933	1.086	1.175	1.223	1.246	1.256	1.260	1.261
1.25	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260
1.50	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260
1.75	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260
2.00	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260
2.75	0.000	0.386	0.702	0.934	1.086	1.175	1.223	1.245	1.255	1.259	1.260

Figure 37. Computer output

VELOCITY IN Y-DIRECTION		TIME = 0.02										REYNOLDS NUMBER = 50.0										CASE II									
Y		0.00		0.05		0.10		0.15		0.20		0.25		0.30		0.35		0.40		0.45		0.50									
X																															
-0.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000								
-0.75	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000								
-0.50	0.000	0.002	0.004	0.006	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.007	0.006	0.004	0.004	0.004	0.004	0.004	0.004	0.002	0.000								
-0.40	0.000	0.005	0.009	0.012	0.013	0.013	0.013	0.012	0.010	0.007	0.004	0.000	0.000	0.000	0.010	0.007	0.004	0.004	0.004	0.004	0.004	0.004	0.000								
-0.30	0.000	0.010	0.018	0.023	0.025	0.025	0.022	0.018	0.012	0.006	0.000	0.000	0.000	0.000	0.018	0.012	0.006	0.006	0.006	0.006	0.006	0.006	0.000								
-0.25	0.000	0.015	0.026	0.033	0.035	0.033	0.029	0.023	0.016	0.008	0.000	0.000	0.000	0.000	0.023	0.016	0.008	0.008	0.008	0.008	0.008	0.008	0.000								
-0.20	0.000	0.023	0.039	0.047	0.048	0.044	0.038	0.030	0.020	0.010	0.000	0.000	0.000	0.000	0.030	0.020	0.010	0.010	0.010	0.010	0.010	0.010	0.000								
-0.15	0.000	0.039	0.061	0.068	0.066	0.059	0.049	0.037	0.025	0.013	0.000	0.000	0.000	0.000	0.037	0.025	0.013	0.013	0.013	0.013	0.013	0.013	0.000								
-0.10	0.000	0.071	0.097	0.098	0.089	0.075	0.061	0.045	0.030	0.015	0.000	0.000	0.000	0.000	0.045	0.030	0.015	0.015	0.015	0.015	0.015	0.015	0.000								
-0.05	0.000	0.149	0.160	0.141	0.117	0.094	0.073	0.053	0.035	0.017	0.000	0.000	0.000	0.000	0.053	0.035	0.017	0.017	0.017	0.017	0.017	0.017	0.000								
0.00	0.000	0.177	0.199	0.174	0.141	0.110	0.084	0.061	0.039	0.019	0.000	0.000	0.000	0.000	0.061	0.039	0.019	0.019	0.019	0.019	0.019	0.019	0.000								
0.05	0.000	0.119	0.177	0.176	0.150	0.120	0.091	0.065	0.042	0.021	0.000	0.000	0.000	0.000	0.065	0.042	0.021	0.021	0.021	0.021	0.021	0.021	0.000								
0.10	0.000	0.071	0.136	0.157	0.145	0.120	0.093	0.067	0.044	0.021	0.000	0.000	0.000	0.000	0.067	0.044	0.021	0.021	0.021	0.021	0.021	0.021	0.000								
0.15	0.000	0.042	0.098	0.129	0.130	0.113	0.090	0.066	0.043	0.021	0.000	0.000	0.000	0.000	0.066	0.043	0.021	0.021	0.021	0.021	0.021	0.021	0.000								
0.20	0.000	0.025	0.069	0.101	0.109	0.100	0.082	0.061	0.040	0.020	0.000	0.000	0.000	0.000	0.061	0.040	0.020	0.020	0.020	0.020	0.020	0.020	0.000								
0.25	0.000	0.016	0.048	0.075	0.087	0.084	0.072	0.055	0.036	0.018	0.000	0.000	0.000	0.000	0.055	0.036	0.018	0.018	0.018	0.018	0.018	0.018	0.000								
0.30	0.000	0.010	0.031	0.053	0.066	0.066	0.058	0.046	0.031	0.015	0.000	0.000	0.000	0.000	0.046	0.031	0.015	0.015	0.015	0.015	0.015	0.015	0.000								
0.35	0.000	0.005	0.019	0.035	0.046	0.049	0.044	0.036	0.024	0.012	0.000	0.000	0.000	0.000	0.036	0.024	0.012	0.012	0.012	0.012	0.012	0.012	0.000								
0.40	0.000	0.003	0.010	0.021	0.029	0.033	0.031	0.026	0.018	0.009	0.000	0.000	0.000	0.000	0.026	0.018	0.009	0.009	0.009	0.009	0.009	0.009	0.000								
0.45	0.000	0.001	0.004	0.010	0.016	0.020	0.020	0.017	0.013	0.007	0.000	0.000	0.000	0.000	0.017	0.013	0.007	0.007	0.007	0.007	0.007	0.007	0.000								
0.50	0.000	-0.000	0.001	0.003	0.007	0.010	0.011	0.010	0.008	0.004	0.000	0.000	0.000	0.000	0.010	0.008	0.004	0.004	0.004	0.004	0.004	0.004	0.000								
0.60	0.000	-0.001	-0.002	-0.002	-0.001	0.000	0.002	0.002	0.002	0.001	0.000	0.000	0.000	0.000	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.000								
0.70	0.000	-0.001	-0.002	-0.002	-0.002	-0.002	-0.002	-0.001	-0.000	0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000								
0.80	0.000	-0.000	-0.001	-0.002	-0.002	-0.002	-0.002	-0.001	-0.001	-0.001	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000								
0.90	0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000								
1.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	0.000	0.000	0.000	0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000								
1.25	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000								
1.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000								
1.75	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000								
2.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000								
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000								

Figure 38. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.04	REYNOLDS NUMBER = 50.0						CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50	0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.007	1.007
-0.40	0.986	0.987	0.989	0.992	0.997	1.001	1.005	1.008	1.011	1.012	1.013
-0.30	0.972	0.974	0.979	0.987	0.995	1.003	1.010	1.015	1.019	1.022	1.023
-0.25	0.960	0.963	0.971	0.982	0.994	1.005	1.014	1.021	1.026	1.029	1.030
-0.20	0.942	0.947	0.961	0.978	0.994	1.008	1.020	1.029	1.035	1.038	1.039
-0.15	0.913	0.923	0.947	0.973	0.996	1.015	1.029	1.039	1.046	1.050	1.051
-0.10	0.864	0.866	0.931	0.972	1.003	1.026	1.042	1.053	1.060	1.064	1.065
-0.05	0.771	0.826	0.919	0.980	1.018	1.043	1.059	1.070	1.077	1.081	1.082
0.00	0.000	0.725	0.938	1.014	1.049	1.069	1.083	1.092	1.097	1.100	1.102
0.05	0.000	0.625	0.919	1.037	1.081	1.100	1.110	1.116	1.120	1.122	1.123
0.10	0.000	0.546	0.877	1.038	1.104	1.129	1.138	1.142	1.144	1.146	1.146
0.15	0.000	0.487	0.829	1.024	1.115	1.152	1.164	1.168	1.169	1.170	1.170
0.20	0.000	0.445	0.785	1.002	1.117	1.168	1.187	1.192	1.193	1.193	1.193
0.25	0.000	0.414	0.747	0.977	1.112	1.178	1.205	1.214	1.216	1.217	1.216
0.30	0.000	0.391	0.716	0.953	1.103	1.183	1.220	1.234	1.238	1.239	1.239
0.35	0.000	0.375	0.691	0.932	1.092	1.185	1.231	1.250	1.257	1.259	1.259
0.40	0.000	0.362	0.671	0.912	1.081	1.184	1.239	1.264	1.274	1.278	1.278
0.45	0.000	0.351	0.654	0.896	1.070	1.182	1.245	1.276	1.290	1.295	1.296
0.50	0.000	0.343	0.641	0.882	1.060	1.179	1.249	1.286	1.303	1.310	1.312
0.60	0.000	0.332	0.621	0.860	1.043	1.172	1.254	1.301	1.325	1.335	1.338
0.70	0.000	0.324	0.608	0.845	1.031	1.166	1.256	1.310	1.341	1.355	1.359
0.80	0.000	0.320	0.601	0.835	1.022	1.160	1.255	1.316	1.351	1.368	1.374
0.90	0.000	0.319	0.598	0.831	1.016	1.156	1.254	1.318	1.357	1.377	1.383
1.00	0.000	0.319	0.598	0.830	1.014	1.153	1.252	1.318	1.358	1.379	1.386
1.25	0.000	0.322	0.602	0.833	1.016	1.153	1.250	1.315	1.355	1.376	1.383
1.50	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.381
1.75	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.00	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380
2.75	0.000	0.323	0.603	0.835	1.018	1.154	1.250	1.314	1.353	1.374	1.380

Figure 39. Computer output

VELOCITY IN Y-DIRECTION			TIME = 0.04			REYNOLDS NUMBER = 50.0						CASE II
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
-0.95		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75		0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.000
-0.50		0.000	0.002	0.005	0.006	0.007	0.007	0.007	0.006	0.004	0.002	0.000
-0.40		0.000	0.005	0.009	0.012	0.013	0.014	0.013	0.010	0.007	0.004	0.000
-0.30		0.000	0.010	0.018	0.023	0.026	0.025	0.022	0.018	0.013	0.006	0.000
-0.25		0.000	0.015	0.027	0.033	0.035	0.034	0.030	0.024	0.016	0.008	0.000
-0.20		0.000	0.023	0.040	0.048	0.049	0.045	0.039	0.030	0.021	0.011	0.000
-0.15		0.000	0.039	0.061	0.069	0.067	0.060	0.050	0.038	0.026	0.013	0.000
-0.10		0.000	0.071	0.098	0.100	0.091	0.077	0.062	0.047	0.031	0.016	0.000
-0.05		0.000	0.149	0.161	0.143	0.119	0.097	0.075	0.056	0.037	0.018	0.000
0.00		0.000	0.178	0.201	0.177	0.144	0.114	0.087	0.063	0.041	0.020	0.000
0.05		0.000	0.119	0.179	0.180	0.155	0.125	0.096	0.069	0.045	0.022	0.000
0.10		0.000	0.071	0.138	0.161	0.151	0.127	0.099	0.072	0.047	0.023	0.000
0.15		0.000	0.042	0.101	0.134	0.137	0.122	0.098	0.073	0.048	0.024	0.000
0.20		0.000	0.026	0.073	0.108	0.120	0.112	0.093	0.071	0.047	0.023	0.000
0.25		0.000	0.017	0.053	0.086	0.102	0.100	0.086	0.067	0.045	0.023	0.000
0.30		0.000	0.012	0.039	0.068	0.085	0.088	0.078	0.062	0.042	0.021	0.000
0.35		0.000	0.009	0.030	0.054	0.071	0.076	0.070	0.057	0.039	0.020	0.000
0.40		0.000	0.006	0.023	0.043	0.059	0.065	0.062	0.051	0.036	0.018	0.000
0.45		0.000	0.005	0.018	0.035	0.049	0.055	0.054	0.045	0.032	0.017	0.000
0.50		0.000	0.004	0.014	0.028	0.040	0.047	0.047	0.040	0.029	0.015	0.000
0.60		0.000	0.003	0.009	0.019	0.028	0.033	0.034	0.030	0.022	0.012	0.000
0.70		0.000	0.002	0.006	0.012	0.018	0.023	0.024	0.022	0.017	0.009	0.000
0.80		0.000	0.001	0.003	0.006	0.010	0.013	0.015	0.014	0.011	0.006	0.000
0.90		0.000	0.000	0.000	0.001	0.003	0.005	0.006	0.006	0.005	0.003	0.000
1.00		0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.000	0.001	0.001	0.001	0.000
1.25		0.000	-0.000	-0.001	-0.001	-0.002	-0.002	-0.002	-0.002	-0.002	-0.001	0.000
1.50		0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
1.75		0.000	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
2.00		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2.75		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 40. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.06	REYNOLDS NUMBER = 50.0						CASE II	
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X												
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001	1.001
-0.50	0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.007	1.007	1.007
-0.40	0.986	0.987	0.989	0.992	0.997	1.001	1.005	1.008	1.011	1.012	1.013	1.013
-0.30	0.972	0.974	0.979	0.986	0.995	1.003	1.010	1.015	1.019	1.022	1.023	1.023
-0.25	0.960	0.963	0.971	0.982	0.994	1.005	1.014	1.021	1.026	1.029	1.030	1.030
-0.20	0.942	0.947	0.961	0.978	0.994	1.008	1.020	1.029	1.035	1.038	1.039	1.039
-0.15	0.913	0.923	0.947	0.973	0.996	1.015	1.029	1.039	1.046	1.050	1.051	1.051
-0.10	0.864	0.886	0.931	0.972	1.003	1.026	1.042	1.053	1.060	1.064	1.065	1.065
-0.05	0.771	0.826	0.918	0.980	1.018	1.043	1.059	1.070	1.077	1.081	1.082	1.082
0.00	0.000	0.725	0.938	1.014	1.049	1.069	1.083	1.092	1.097	1.101	1.102	1.102
0.05	0.000	0.625	0.919	1.036	1.081	1.100	1.110	1.116	1.120	1.122	1.123	1.123
0.10	0.000	0.545	0.877	1.038	1.104	1.129	1.138	1.142	1.145	1.146	1.146	1.146
0.15	0.000	0.487	0.829	1.024	1.115	1.152	1.164	1.168	1.169	1.170	1.170	1.170
0.20	0.000	0.444	0.785	1.002	1.117	1.168	1.187	1.193	1.194	1.194	1.194	1.194
0.25	0.000	0.414	0.747	0.977	1.111	1.178	1.205	1.215	1.217	1.217	1.217	1.217
0.30	0.000	0.391	0.715	0.953	1.102	1.183	1.220	1.234	1.238	1.239	1.239	1.239
0.35	0.000	0.374	0.690	0.931	1.091	1.184	1.231	1.251	1.258	1.260	1.260	1.260
0.40	0.000	0.361	0.669	0.911	1.080	1.184	1.239	1.265	1.276	1.279	1.280	1.280
0.45	0.000	0.351	0.653	0.894	1.069	1.181	1.245	1.277	1.291	1.296	1.298	1.298
0.50	0.000	0.342	0.639	0.880	1.059	1.178	1.250	1.287	1.305	1.312	1.314	1.314
0.60	0.000	0.330	0.618	0.856	1.041	1.171	1.255	1.303	1.328	1.339	1.342	1.342
0.70	0.000	0.322	0.603	0.839	1.026	1.164	1.257	1.314	1.346	1.361	1.366	1.366
0.80	0.000	0.315	0.592	0.826	1.015	1.157	1.257	1.322	1.360	1.379	1.385	1.385
0.90	0.000	0.310	0.583	0.816	1.005	1.152	1.257	1.328	1.371	1.394	1.401	1.401
1.00	0.000	0.307	0.577	0.808	0.998	1.147	1.257	1.333	1.380	1.406	1.414	1.414
1.25	0.000	0.301	0.566	0.795	0.986	1.139	1.256	1.340	1.395	1.426	1.436	1.436
1.50	0.000	0.299	0.564	0.792	0.982	1.135	1.254	1.341	1.400	1.433	1.444	1.444
1.75	0.000	0.300	0.565	0.793	0.983	1.135	1.253	1.339	1.398	1.432	1.443	1.443
2.00	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441
2.75	0.000	0.301	0.566	0.794	0.984	1.136	1.253	1.339	1.397	1.430	1.441	1.441

Figure 41. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.06	REYNOLDS NUMBER = 50.0							CASE II
X	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
-0.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75	0.000	0.001	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.000
-0.50	0.000	0.002	0.005	0.006	0.006	0.007	0.007	0.007	0.006	0.004	0.002	0.000
-0.40	0.000	0.005	0.009	0.012	0.012	0.013	0.014	0.013	0.010	0.007	0.004	0.000
-0.30	0.000	0.010	0.018	0.023	0.023	0.026	0.025	0.022	0.018	0.013	0.007	0.000
-0.25	0.000	0.015	0.027	0.033	0.033	0.035	0.034	0.030	0.024	0.016	0.008	0.000
-0.20	0.000	0.023	0.040	0.048	0.048	0.049	0.045	0.039	0.030	0.021	0.011	0.000
-0.15	0.000	0.039	0.061	0.069	0.069	0.067	0.060	0.050	0.038	0.026	0.013	0.000
-0.10	0.000	0.071	0.098	0.100	0.100	0.091	0.077	0.062	0.047	0.031	0.016	0.000
-0.05	0.000	0.149	0.161	0.143	0.143	0.119	0.097	0.076	0.056	0.037	0.018	0.000
0.00	0.000	0.178	0.201	0.177	0.177	0.144	0.114	0.087	0.064	0.041	0.020	0.000
0.05	0.000	0.119	0.179	0.180	0.180	0.155	0.125	0.096	0.069	0.045	0.022	0.000
0.10	0.000	0.071	0.138	0.161	0.161	0.151	0.127	0.099	0.073	0.047	0.023	0.000
0.15	0.000	0.042	0.101	0.134	0.134	0.137	0.122	0.098	0.073	0.048	0.024	0.000
0.20	0.000	0.026	0.073	0.108	0.108	0.120	0.112	0.094	0.071	0.047	0.024	0.000
0.25	0.000	0.017	0.053	0.086	0.086	0.102	0.101	0.087	0.067	0.045	0.023	0.000
0.30	0.000	0.012	0.040	0.068	0.068	0.086	0.088	0.079	0.063	0.043	0.022	0.000
0.35	0.000	0.009	0.030	0.054	0.054	0.072	0.077	0.071	0.057	0.040	0.020	0.000
0.40	0.000	0.006	0.023	0.044	0.044	0.060	0.066	0.063	0.052	0.036	0.019	0.000
0.45	0.000	0.005	0.018	0.036	0.036	0.050	0.057	0.055	0.047	0.033	0.017	0.000
0.50	0.000	0.004	0.015	0.029	0.029	0.042	0.049	0.049	0.042	0.030	0.016	0.000
0.60	0.000	0.003	0.010	0.020	0.020	0.030	0.037	0.038	0.033	0.024	0.013	0.000
0.70	0.000	0.002	0.007	0.015	0.015	0.022	0.028	0.029	0.027	0.020	0.011	0.000
0.80	0.000	0.001	0.006	0.011	0.011	0.017	0.022	0.023	0.021	0.016	0.009	0.000
0.90	0.000	0.001	0.004	0.009	0.009	0.013	0.017	0.018	0.017	0.013	0.007	0.000
1.00	0.000	0.001	0.003	0.007	0.007	0.010	0.013	0.015	0.014	0.011	0.006	0.000
1.25	0.000	0.000	0.001	0.003	0.003	0.005	0.006	0.007	0.007	0.005	0.003	0.000
1.50	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000	0.000	0.001	0.001	0.000	0.000
1.75	0.000	-0.000	-0.000	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	-0.001	0.000
2.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 42. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.10	REYNOLDS NUMBER = 50.0						CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50	0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.007	1.007
-0.40	0.986	0.987	0.989	0.992	0.997	1.001	1.005	1.008	1.011	1.012	1.013
-0.30	0.972	0.974	0.979	0.980	0.995	1.003	1.010	1.015	1.019	1.022	1.023
-0.25	0.960	0.963	0.971	0.982	0.994	1.005	1.014	1.021	1.026	1.029	1.030
-0.20	0.942	0.947	0.961	0.978	0.994	1.008	1.020	1.029	1.035	1.038	1.039
-0.15	0.913	0.923	0.947	0.973	0.996	1.015	1.029	1.039	1.046	1.050	1.051
-0.10	0.864	0.886	0.931	0.972	1.003	1.026	1.042	1.053	1.060	1.064	1.065
-0.05	0.771	0.826	0.918	0.980	1.018	1.043	1.059	1.070	1.077	1.081	1.082
0.00	0.000	0.725	0.938	1.014	1.049	1.069	1.083	1.092	1.097	1.101	1.102
0.05	0.000	0.625	0.919	1.036	1.081	1.100	1.110	1.116	1.120	1.122	1.123
0.10	0.000	0.545	0.877	1.038	1.104	1.129	1.138	1.142	1.145	1.146	1.146
0.15	0.000	0.487	0.829	1.024	1.115	1.152	1.164	1.168	1.169	1.170	1.170
0.20	0.000	0.444	0.785	1.002	1.117	1.168	1.187	1.193	1.194	1.194	1.194
0.25	0.000	0.414	0.747	0.977	1.111	1.178	1.205	1.215	1.217	1.217	1.217
0.30	0.000	0.391	0.715	0.953	1.102	1.183	1.220	1.234	1.238	1.239	1.239
0.35	0.000	0.374	0.690	0.931	1.091	1.184	1.231	1.251	1.258	1.260	1.260
0.40	0.000	0.361	0.669	0.911	1.080	1.184	1.239	1.265	1.276	1.279	1.280
0.45	0.000	0.351	0.653	0.894	1.069	1.181	1.245	1.277	1.291	1.296	1.298
0.50	0.000	0.342	0.639	0.880	1.059	1.178	1.250	1.287	1.305	1.312	1.314
0.60	0.000	0.330	0.618	0.856	1.041	1.171	1.255	1.303	1.328	1.339	1.343
0.70	0.000	0.322	0.603	0.839	1.026	1.164	1.257	1.314	1.346	1.362	1.366
0.80	0.000	0.315	0.592	0.826	1.014	1.157	1.257	1.322	1.361	1.380	1.386
0.90	0.000	0.310	0.583	0.815	1.005	1.151	1.257	1.329	1.372	1.395	1.402
1.00	0.000	0.306	0.576	0.807	0.997	1.147	1.257	1.333	1.381	1.407	1.415
1.25	0.000	0.299	0.563	0.792	0.983	1.138	1.256	1.342	1.398	1.430	1.440
1.50	0.000	0.294	0.555	0.782	0.974	1.132	1.255	1.347	1.409	1.445	1.457
1.75	0.000	0.291	0.550	0.776	0.968	1.128	1.255	1.350	1.416	1.455	1.468
2.00	0.000	0.289	0.546	0.771	0.964	1.125	1.254	1.353	1.421	1.462	1.475
2.75	0.000	0.287	0.543	0.767	0.961	1.123	1.254	1.355	1.426	1.469	1.483

Figure 43. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.10	REYNOLDS NUMBER = 50.0							CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X												
-0.95	0.000	0.000	0.000	0.000	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.000	0.000
-0.50	0.000	0.002	0.005	0.006	0.007	0.007	0.007	0.006	0.004	0.002	0.000	0.000
-0.40	0.000	0.005	0.009	0.012	0.013	0.014	0.013	0.010	0.007	0.004	0.000	0.000
-0.30	0.000	0.010	0.018	0.023	0.026	0.025	0.022	0.018	0.013	0.007	0.000	0.000
-0.25	0.000	0.015	0.027	0.033	0.035	0.034	0.030	0.024	0.016	0.008	0.000	0.000
-0.20	0.000	0.023	0.040	0.048	0.049	0.045	0.039	0.030	0.021	0.011	0.000	0.000
-0.15	0.000	0.039	0.061	0.069	0.067	0.060	0.050	0.038	0.026	0.013	0.000	0.000
-0.10	0.000	0.071	0.098	0.100	0.091	0.077	0.062	0.047	0.031	0.016	0.000	0.000
-0.05	0.000	0.149	0.161	0.143	0.119	0.097	0.076	0.056	0.037	0.018	0.000	0.000
0.00	0.000	0.178	0.201	0.177	0.144	0.114	0.087	0.064	0.041	0.020	0.000	0.000
0.05	0.000	0.119	0.179	0.180	0.155	0.125	0.096	0.069	0.045	0.022	0.000	0.000
0.10	0.000	0.071	0.138	0.161	0.151	0.127	0.099	0.073	0.047	0.023	0.000	0.000
0.15	0.000	0.042	0.101	0.134	0.138	0.122	0.098	0.073	0.048	0.024	0.000	0.000
0.20	0.000	0.026	0.073	0.108	0.120	0.112	0.094	0.071	0.047	0.024	0.000	0.000
0.25	0.000	0.017	0.053	0.086	0.102	0.101	0.087	0.067	0.045	0.023	0.000	0.000
0.30	0.000	0.012	0.040	0.068	0.086	0.088	0.079	0.063	0.043	0.022	0.000	0.000
0.35	0.000	0.009	0.030	0.055	0.072	0.077	0.071	0.057	0.040	0.020	0.000	0.000
0.40	0.000	0.006	0.023	0.044	0.060	0.066	0.063	0.052	0.036	0.019	0.000	0.000
0.45	0.000	0.005	0.019	0.036	0.050	0.057	0.055	0.047	0.033	0.017	0.000	0.000
0.50	0.000	0.004	0.015	0.029	0.042	0.049	0.049	0.042	0.030	0.016	0.000	0.000
0.60	0.000	0.003	0.010	0.020	0.030	0.037	0.038	0.033	0.025	0.013	0.000	0.000
0.70	0.000	0.002	0.007	0.015	0.023	0.028	0.030	0.027	0.020	0.011	0.000	0.000
0.80	0.000	0.002	0.006	0.011	0.017	0.022	0.023	0.022	0.016	0.009	0.000	0.000
0.90	0.000	0.001	0.004	0.009	0.014	0.017	0.019	0.018	0.014	0.007	0.000	0.000
1.00	0.000	0.001	0.004	0.007	0.011	0.014	0.016	0.015	0.011	0.006	0.000	0.000
1.25	0.000	0.001	0.002	0.005	0.007	0.009	0.010	0.009	0.007	0.004	0.000	0.000
1.50	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.006	0.005	0.003	0.000	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.004	0.004	0.004	0.003	0.002	0.000	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.003	0.003	0.003	0.002	0.001	0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 44. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.14	REYNOLDS NUMBER = 50.0						CASE II	
	Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X												
-0.95		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75		0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50		0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.007	1.007
-0.40		0.986	0.987	0.989	0.992	0.997	1.001	1.005	1.008	1.011	1.012	1.013
-0.30		0.972	0.974	0.979	0.986	0.995	1.003	1.010	1.015	1.019	1.022	1.023
-0.25		0.960	0.963	0.971	0.982	0.994	1.005	1.014	1.021	1.026	1.029	1.030
-0.20		0.942	0.947	0.961	0.978	0.994	1.008	1.020	1.029	1.035	1.038	1.039
-0.15		0.913	0.923	0.947	0.973	0.996	1.015	1.029	1.039	1.046	1.050	1.051
-0.10		0.864	0.886	0.931	0.972	1.003	1.026	1.042	1.053	1.060	1.064	1.065
-0.05		0.771	0.826	0.918	0.980	1.018	1.043	1.059	1.070	1.077	1.081	1.082
0.00		0.000	0.725	0.938	1.014	1.049	1.069	1.083	1.092	1.097	1.101	1.102
0.05		0.000	0.625	0.919	1.036	1.081	1.100	1.110	1.116	1.120	1.122	1.123
0.10		0.000	0.545	0.877	1.038	1.104	1.129	1.138	1.142	1.145	1.146	1.146
0.15		0.000	0.487	0.829	1.024	1.115	1.152	1.164	1.168	1.169	1.170	1.170
0.20		0.000	0.444	0.785	1.002	1.117	1.168	1.187	1.193	1.194	1.194	1.194
0.25		0.000	0.414	0.747	0.977	1.111	1.178	1.205	1.215	1.217	1.217	1.217
0.30		0.000	0.391	0.715	0.953	1.102	1.183	1.220	1.234	1.238	1.239	1.239
0.35		0.000	0.374	0.690	0.931	1.091	1.184	1.231	1.251	1.258	1.260	1.260
0.40		0.000	0.361	0.669	0.911	1.080	1.184	1.239	1.265	1.276	1.279	1.280
0.45		0.000	0.351	0.653	0.894	1.069	1.181	1.245	1.277	1.291	1.296	1.298
0.50		0.000	0.342	0.639	0.880	1.059	1.178	1.250	1.287	1.305	1.312	1.314
0.60		0.000	0.330	0.618	0.856	1.041	1.171	1.255	1.303	1.328	1.339	1.343
0.70		0.000	0.322	0.603	0.839	1.026	1.164	1.257	1.314	1.346	1.362	1.366
0.80		0.000	0.315	0.592	0.826	1.014	1.157	1.257	1.322	1.361	1.380	1.386
0.90		0.000	0.310	0.583	0.815	1.005	1.151	1.257	1.329	1.372	1.395	1.402
1.00		0.000	0.306	0.576	0.807	0.997	1.147	1.257	1.333	1.381	1.407	1.415
1.25		0.000	0.299	0.563	0.792	0.983	1.138	1.256	1.342	1.398	1.430	1.440
1.50		0.000	0.294	0.555	0.782	0.974	1.132	1.255	1.347	1.409	1.445	1.457
1.75		0.000	0.291	0.550	0.776	0.968	1.128	1.255	1.350	1.416	1.455	1.468
2.00		0.000	0.289	0.546	0.771	0.964	1.125	1.254	1.353	1.422	1.462	1.476
2.75		0.000	0.286	0.541	0.765	0.959	1.122	1.254	1.356	1.428	1.471	1.486

Figure 45. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.14	REYNOLDS NUMBER = 50.0						CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.000
-0.50	0.000	0.002	0.005	0.006	0.007	0.007	0.007	0.006	0.004	0.002	0.000
-0.40	0.000	0.005	0.009	0.012	0.013	0.014	0.013	0.010	0.007	0.004	0.000
-0.30	0.000	0.010	0.018	0.023	0.026	0.025	0.022	0.018	0.013	0.007	0.000
-0.25	0.000	0.015	0.027	0.033	0.035	0.034	0.030	0.024	0.016	0.008	0.000
-0.20	0.000	0.023	0.040	0.048	0.049	0.045	0.039	0.030	0.021	0.011	0.000
-0.15	0.000	0.039	0.061	0.069	0.067	0.060	0.050	0.038	0.026	0.013	0.000
-0.10	0.000	0.071	0.098	0.100	0.091	0.077	0.062	0.047	0.031	0.016	0.000
-0.05	0.000	0.149	0.161	0.143	0.119	0.097	0.076	0.056	0.037	0.018	0.000
0.00	0.000	0.178	0.201	0.177	0.144	0.114	0.087	0.064	0.041	0.020	0.000
0.05	0.000	0.119	0.179	0.180	0.155	0.125	0.096	0.069	0.045	0.022	0.000
0.10	0.000	0.071	0.138	0.161	0.151	0.127	0.099	0.073	0.047	0.023	0.000
0.15	0.000	0.042	0.101	0.134	0.138	0.122	0.098	0.073	0.048	0.024	0.000
0.20	0.000	0.026	0.073	0.108	0.120	0.112	0.094	0.071	0.047	0.024	0.000
0.25	0.000	0.017	0.053	0.086	0.102	0.101	0.087	0.067	0.045	0.023	0.000
0.30	0.000	0.012	0.040	0.068	0.086	0.088	0.079	0.063	0.043	0.022	0.000
0.35	0.000	0.009	0.030	0.055	0.072	0.077	0.071	0.057	0.040	0.020	0.000
0.40	0.000	0.006	0.023	0.044	0.060	0.066	0.063	0.052	0.036	0.019	0.000
0.45	0.000	0.005	0.019	0.036	0.050	0.057	0.055	0.047	0.033	0.017	0.000
0.50	0.000	0.004	0.015	0.029	0.042	0.049	0.049	0.042	0.030	0.016	0.000
0.60	0.000	0.003	0.010	0.020	0.030	0.037	0.038	0.033	0.025	0.013	0.000
0.70	0.000	0.002	0.007	0.015	0.023	0.028	0.030	0.027	0.020	0.011	0.000
0.80	0.000	0.002	0.006	0.011	0.017	0.022	0.023	0.022	0.016	0.009	0.000
0.90	0.000	0.001	0.004	0.009	0.014	0.017	0.019	0.018	0.014	0.007	0.000
1.00	0.000	0.001	0.004	0.007	0.011	0.014	0.016	0.015	0.011	0.006	0.000
1.25	0.000	0.001	0.002	0.005	0.007	0.009	0.010	0.009	0.007	0.004	0.000
1.50	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.006	0.005	0.003	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.004	0.005	0.004	0.003	0.002	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.003	0.003	0.003	0.002	0.001	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Figure 46. Computer output

VELOCITY IN X-DIRECTION				TIME = 0.18	REYNOLDS NUMBER = 50.0						CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
X											
-0.95	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
-0.75	0.999	0.999	0.999	0.999	1.000	1.000	1.000	1.001	1.001	1.001	1.001
-0.50	0.993	0.993	0.994	0.996	0.998	1.000	1.002	1.004	1.006	1.007	1.007
-0.40	0.986	0.987	0.989	0.992	0.997	1.001	1.005	1.008	1.011	1.012	1.013
-0.30	0.972	0.974	0.979	0.986	0.995	1.003	1.010	1.015	1.019	1.022	1.023
-0.25	0.960	0.963	0.971	0.982	0.994	1.005	1.014	1.021	1.026	1.029	1.030
-0.20	0.942	0.947	0.961	0.978	0.994	1.008	1.020	1.029	1.035	1.038	1.039
-0.15	0.913	0.923	0.947	0.973	0.996	1.015	1.029	1.039	1.046	1.050	1.051
-0.10	0.864	0.886	0.931	0.972	1.003	1.026	1.042	1.053	1.060	1.064	1.065
-0.05	0.771	0.826	0.918	0.980	1.018	1.043	1.059	1.070	1.077	1.081	1.082
0.00	0.000	0.725	0.938	1.014	1.049	1.069	1.083	1.092	1.097	1.101	1.102
0.05	0.000	0.625	0.919	1.036	1.081	1.100	1.110	1.116	1.120	1.122	1.123
0.10	0.000	0.545	0.877	1.038	1.104	1.129	1.138	1.142	1.145	1.146	1.146
0.15	0.000	0.487	0.829	1.024	1.115	1.152	1.164	1.168	1.169	1.170	1.170
0.20	0.000	0.444	0.785	1.002	1.117	1.168	1.187	1.193	1.194	1.194	1.194
0.25	0.000	0.414	0.747	0.977	1.111	1.178	1.205	1.215	1.217	1.217	1.217
0.30	0.000	0.391	0.715	0.953	1.102	1.183	1.220	1.234	1.238	1.239	1.239
0.35	0.000	0.374	0.690	0.931	1.091	1.184	1.231	1.251	1.258	1.260	1.260
0.40	0.000	0.361	0.669	0.911	1.080	1.184	1.239	1.265	1.276	1.279	1.280
0.45	0.000	0.351	0.653	0.894	1.069	1.181	1.245	1.277	1.291	1.296	1.298
0.50	0.000	0.342	0.639	0.880	1.059	1.178	1.250	1.287	1.305	1.312	1.314
0.60	0.000	0.330	0.618	0.856	1.041	1.171	1.255	1.303	1.328	1.339	1.343
0.70	0.000	0.322	0.603	0.839	1.026	1.164	1.257	1.314	1.346	1.362	1.366
0.80	0.000	0.315	0.592	0.826	1.014	1.157	1.257	1.322	1.361	1.380	1.386
0.90	0.000	0.310	0.583	0.815	1.005	1.151	1.257	1.329	1.372	1.395	1.402
1.00	0.000	0.306	0.576	0.807	0.997	1.147	1.257	1.333	1.381	1.407	1.415
1.25	0.000	0.299	0.563	0.792	0.983	1.138	1.256	1.342	1.398	1.430	1.440
1.50	0.000	0.294	0.555	0.782	0.974	1.132	1.255	1.347	1.409	1.445	1.457
1.75	0.000	0.291	0.550	0.776	0.968	1.128	1.255	1.350	1.416	1.455	1.468
2.00	0.000	0.289	0.546	0.771	0.964	1.125	1.254	1.353	1.422	1.462	1.476
2.75	0.000	0.286	0.541	0.765	0.959	1.122	1.254	1.356	1.428	1.471	1.486

Figure 47. Computer output

VELOCITY IN Y-DIRECTION				TIME = 0.18	REYNOLDS NUMBER = 50.0							CASE II
Y	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	
X												
-0.95	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-0.75	0.000	0.001	0.001	0.001	0.002	0.002	0.002	0.001	0.001	0.001	0.000	0.000
-0.50	0.000	0.002	0.005	0.006	0.007	0.007	0.007	0.006	0.004	0.002	0.000	0.000
-0.40	0.000	0.005	0.009	0.012	0.013	0.014	0.013	0.010	0.007	0.004	0.000	0.000
-0.30	0.000	0.010	0.018	0.023	0.026	0.025	0.022	0.018	0.013	0.007	0.000	0.000
-0.25	0.000	0.015	0.027	0.033	0.035	0.034	0.030	0.024	0.016	0.008	0.000	0.000
-0.20	0.000	0.023	0.040	0.048	0.049	0.045	0.039	0.030	0.021	0.011	0.000	0.000
-0.15	0.000	0.039	0.061	0.069	0.067	0.060	0.050	0.038	0.026	0.013	0.000	0.000
-0.10	0.000	0.071	0.098	0.100	0.091	0.077	0.062	0.047	0.031	0.016	0.000	0.000
-0.05	0.000	0.149	0.161	0.143	0.119	0.097	0.076	0.056	0.037	0.018	0.000	0.000
0.00	0.000	0.178	0.201	0.177	0.144	0.114	0.087	0.064	0.041	0.020	0.000	0.000
0.05	0.000	0.119	0.179	0.180	0.155	0.125	0.096	0.069	0.045	0.022	0.000	0.000
0.10	0.000	0.071	0.138	0.161	0.151	0.127	0.099	0.073	0.047	0.023	0.000	0.000
0.15	0.000	0.042	0.101	0.134	0.138	0.122	0.098	0.073	0.048	0.024	0.000	0.000
0.20	0.000	0.026	0.073	0.108	0.120	0.112	0.094	0.071	0.047	0.024	0.000	0.000
0.25	0.000	0.017	0.053	0.086	0.102	0.101	0.087	0.067	0.045	0.023	0.000	0.000
0.30	0.000	0.012	0.040	0.068	0.086	0.088	0.079	0.063	0.043	0.022	0.000	0.000
0.35	0.000	0.009	0.030	0.055	0.072	0.077	0.071	0.057	0.040	0.020	0.000	0.000
0.40	0.000	0.006	0.023	0.044	0.060	0.066	0.063	0.052	0.036	0.019	0.000	0.000
0.45	0.000	0.005	0.019	0.036	0.050	0.057	0.055	0.047	0.033	0.017	0.000	0.000
0.50	0.000	0.004	0.015	0.029	0.042	0.049	0.049	0.042	0.030	0.016	0.000	0.000
0.60	0.000	0.003	0.010	0.020	0.030	0.037	0.038	0.033	0.025	0.013	0.000	0.000
0.70	0.000	0.002	0.007	0.015	0.023	0.028	0.030	0.027	0.020	0.011	0.000	0.000
0.80	0.000	0.002	0.006	0.011	0.017	0.022	0.023	0.022	0.016	0.009	0.000	0.000
0.90	0.000	0.001	0.004	0.009	0.014	0.017	0.019	0.018	0.014	0.007	0.000	0.000
1.00	0.000	0.001	0.004	0.007	0.011	0.014	0.016	0.015	0.011	0.006	0.000	0.000
1.25	0.000	0.001	0.002	0.005	0.007	0.009	0.010	0.009	0.007	0.004	0.000	0.000
1.50	0.000	0.000	0.002	0.003	0.005	0.006	0.007	0.006	0.005	0.003	0.000	0.000
1.75	0.000	0.000	0.001	0.002	0.003	0.004	0.005	0.004	0.003	0.002	0.000	0.000
2.00	0.000	0.000	0.001	0.001	0.002	0.003	0.003	0.003	0.002	0.001	0.000	0.000
2.75	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Other Programs

Other problems of the same general nature as the one discussed were also coded. The following pages list some of these and briefly describe the results which were obtained from them.

Steady state solution

To obtain a feel for the problem of inlet flow in a channel, a steady state solution was coded. For this condition, Equations 23 and 24 reduce to

$$\nabla^2 \omega = R \frac{\partial(\omega, \phi)}{\partial(x, y)},$$

and

$$\nabla^2 \phi = -\omega.$$

The numerical method used to solve both of these equations was the method of successive displacement by points, described in Varga (50) and Forsythe and Wasow (13). The finite difference equations used were

$$\begin{aligned} \omega_{i,j}^{[k+1]} = & \frac{1}{4} \left[\omega_{i+1,j}^{[k]} + \omega_{i-1,j}^{[k+1]} + \omega_{i,j+1}^{[k]} + \omega_{i,j-1}^{[k+1]} \right] \\ & + \frac{R}{16} \left[(\omega_{i,j+1}^{[k]} - \omega_{i,j-1}^{[k+1]}) (\phi_{i+1,j}^{[k]} - \phi_{i-1,j}^{[k]}) \right. \\ & \left. - (\omega_{i+1,j}^{[k]} - \omega_{i-1,j}^{[k+1]}) (\phi_{i,j+1}^{[k]} - \phi_{i,j-1}^{[k]}) \right], \end{aligned}$$

$$\begin{cases} 1 \leq i \leq M \\ 1 \leq j \leq N \end{cases}, \quad 60)$$

and

$$\phi_{i,j}^{[k+1]} = \frac{1}{4} \left[\phi_{i+1,j}^{[k]} + \phi_{i-1,j}^{[k+1]} + \phi_{i,j+1}^{[k]} + \phi_{i,j-1}^{[k+1]} + h^2 \omega_{i,j}^{[k+1]} \right],$$

$$\left\{ \begin{array}{l} 1 \leq i \leq M \\ 1 \leq j \leq N \end{array} \right. . \quad 61)$$

Rather than the symmetric boundary conditions used for the time dependent solution, the vorticity and stream function over the entire channel was calculated. Thus the boundary conditions were

$$\left. \begin{array}{l} \phi_{0,j} = jh \\ \omega_{0,j} = 0 \end{array} \right\}, \quad 0 < j < N+1, \quad 62a)$$

$$\left. \begin{array}{l} \phi_{M+1,j}^{[k+1]} = \phi_{M,j}^{[k]} \\ \omega_{M+1,j}^{[k+1]} = \omega_{M,j}^{[k]} \end{array} \right\}, \quad 0 < j < N+1, \quad 62b)$$

$$\left. \begin{array}{l} \phi_{i,0} = 0 \\ \omega_{i,0}^{[k+1]} = \omega_{i,0}^{[k]} + \frac{1}{2} \left[-\frac{2\phi_{i,1}^{[k]}}{h^2} - \omega_{i,0}^{[k]} \right] \end{array} \right\}, \quad 0 < i < M+1, \quad 62c)$$

$$\left. \begin{aligned} \phi_{i,N+1} &= (N+1)h \\ \omega_{i,N+1}^{[k+1]} &= \omega_{i,N+1}^{[k]} + \frac{1}{2} \left[\frac{2\phi_{i,N}^{[k]}}{h^2} - \frac{2(N+1)}{h} - \omega_{i,N+1}^{[k]} \right] \end{aligned} \right\}, \quad 0 < i < M+1, \quad 62d$$

These boundary conditions should be compared with Equations 50 and 53 which were used for the symmetric case.

These equations were solved in a straightforward manner by solving Equation 60 for $\omega_{i,j}^{[k+1]}$, $1 \leq i \leq M$, $1 \leq j \leq N$, then Equation 61 for $\phi_{i,j}^{[k+1]}$, $1 \leq i \leq M$, $1 \leq j \leq N$. Finally the boundary values for the vorticity were corrected using the boundary Equations 62c and 62d and the process repeated until convergence was achieved.

The program was run only a few times to test the procedure. It required a large number of iterations and so was not run to complete convergence. The solution did appear to be approaching the correct solution as evidenced by comparison with Schlichting's results (40).

Much faster convergence could be expected from an over-relaxation method, but this was not attempted.

Preliminary time dependent solution

As a preliminary solution for Equations 23 and 24, Equation 23 was approximated by the finite difference equation

$$\begin{aligned} & \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n-1}}{2\Delta\tau} - \frac{R}{4h^2} \left[\Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \right] \\ &= \frac{1}{h^2} \left[\omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n - 2\omega_{i,j}^{n+1} - 2\omega_{i,j}^{n-1} \right] \end{aligned}$$

When solved for $\omega_{i,j}^{n+1}$, this results in the explicit equation

$$\omega_{i,j}^{n+1} = \left[\frac{1}{1 + \frac{4}{\alpha}} \right] \left[\omega_{i,j}^{n-1} + \frac{R}{2\alpha} \{ \Delta_Y(\omega^n) \Delta_X(\phi^n) - \Delta_X(\omega^n) \Delta_Y(\phi^n) \} \right. \\ \left. + \frac{2}{\alpha} \{ \omega_{i+1,j}^n + \omega_{i-1,j}^n + \omega_{i,j+1}^n + \omega_{i,j-1}^n - 2\omega_{i,j}^{n-1} \} \right] .$$

63)

Although this equation is explicit and therefore easily solved, it does require the vorticity field at two previous time planes, and, so, is somewhat wasteful of storage.

Equations 63 and 61 were solved with the nonsymmetric boundary conditions 62 and the two sets of initial conditions

$$\left. \begin{aligned} \phi_{i,j}^0 &= jh \\ \omega_{i,j}^0 &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 \leq i \leq M+1 \\ 0 \leq j \leq N+1 \end{aligned} \right. , \\ \left. \begin{aligned} \phi_{i,j}^1 &= jh \\ \omega_{i,j}^1 &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 \leq i \leq M+1 \\ 0 \leq j \leq N+1 \end{aligned} \right. , \\ \left. \begin{aligned} \omega_{i,j}^1 &= 0 \end{aligned} \right\} , \left\{ \begin{aligned} 0 \leq i \leq M+1 \\ 1 \leq j \leq N \end{aligned} \right. , \\ \omega_{i,0}^1 &= -2h , \quad 1 \leq i \leq M , \\ \omega_{i,N+1}^1 &= 2h , \quad 1 \leq i \leq M .$$

This method was found to be divergent except for very small time increments and so was not used.

Extrapolation

For a given value of the y-coordinate, j, the stream function at the downstream end of the region can be expanded by Newton's forward difference formula as shown in Jeffreys and Swirles (21, pp. 262 ff.) to give

$$\phi_{M+1,j}^n = P_k(\phi_{M,j}^n, \phi_{M-1,j}^n, \dots, \phi_{M-k-1,j}^n) + R_k$$

where P_k is a kth order polynomial passing through the k+1 points $\phi_{M,j}^n, \phi_{M-1,j}^n, \dots, \phi_{M-k-1,j}^n$, and R_k is the remainder term given by

$$R_k = \frac{1}{(k+1)!} \left. \frac{\partial^{k+1} \phi}{\partial x^{k+1}} \right)_{\xi,j}^n h[2h][3h] \cdots [(k+1)h] = h^{k+1} \left. \frac{\partial^{k+1} \phi}{\partial x^{k+1}} \right)_{\xi,j}^n$$

where $M-k-1 \leq \xi \leq M+1$. Thus the stream function can be extrapolated to any order of accuracy desired if the derivatives are bounded.

When this extrapolation technique was used instead of the downstream boundary conditions given by Equation 50b or 52b, the convergence of the stream function was much slower. This is to be expected since this technique lessens the diagonal dominance of the matrix of the SBOR method.

It was found that convergence was more rapid with

comparable accuracy if the value of h was decreased or M increased rather than use the extrapolation technique. However, for problems for which $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \omega}{\partial x}$ do not approach zero within a reasonable distance, this technique would be valuable.

Alternative vorticity boundary condition

The boundary values of the vorticity along the wall of the channel were found to be

$$\omega_{i,0}^n = \frac{2(\phi_{i,0}^n - \phi_{i,1}^n)}{h^2} + O(h) \quad , \quad \begin{array}{l} 1 \leq i \leq M \\ n > 0 \end{array} \quad ,$$

in the discussion preceding Equations 49 and 50. When the stream function is expanded to one higher order than was used to derive this result it is found that

$$\phi_{i,1}^n = \phi_{i,0}^n + h \left. \frac{\partial \phi}{\partial y} \right|_{i,0}^n + \frac{h^2}{2} \left. \frac{\partial^2 \phi}{\partial y^2} \right|_{i,0}^n + \frac{h^3}{6} \left. \frac{\partial^3 \phi}{\partial y^3} \right|_{i,0}^n + O(h^4) \quad .$$

Since

$$\begin{aligned} - \omega_{i,j}^n &= \left. \frac{\partial^2 \phi}{\partial x^2} \right|_{i,j}^n + \left. \frac{\partial^2 \phi}{\partial y^2} \right|_{i,j}^n \quad , \\ - \left. \frac{\partial \omega}{\partial y} \right|_{i,j}^n &= \left. \frac{\partial^3 \phi}{\partial x^2 \partial y} \right|_{i,j}^n + \left. \frac{\partial^3 \phi}{\partial y^3} \right|_{i,j}^n \quad , \end{aligned}$$

and

$$\left. \frac{\partial \phi}{\partial y} \right)_{i,0}^n = 0 ,$$

$$\left. \frac{\partial^2 \phi}{\partial x^2} \right)_{i,0}^n = 0 ,$$

$$\left. \frac{\partial^3 \phi}{\partial x^2 \partial y} \right)_{i,0}^n = 0 ,$$

it follows that

$$\phi_{i,1}^n = \phi_{i,0}^n - \frac{h^2}{2} \omega_{i,0}^n - \frac{h^3}{6} \left. \frac{\partial \omega}{\partial y} \right)_{i,0}^n + O(h^4) .$$

Also

$$\left. \frac{\partial \omega}{\partial y} \right)_{i,0}^n = \frac{1}{h} [\omega_{i,1}^n - \omega_{i,0}^n] + O(h) ,$$

so that

$$\omega_{i,0}^n = \frac{3(\phi_{i,0}^n - \phi_{i,1}^n)}{h^2} - \frac{\omega_{i,1}^n}{2} + O(h^2) .$$

This boundary condition was used instead of Equation 50c for case I. The maximum variation was about one digit in the second decimal place of the velocities, or less than 1 percent, except at the leading edge of the walls where the vorticity

gradients were large. However, since the increase in computation is negligible, for future solutions it is recommended that this higher order boundary condition be utilized.

Suddenly accelerated flat plate

A small effort was made to calculate the flow field around a suddenly accelerated flat plate. The boundary and initial conditions used were Equations 49 and 50 except that Equation 50d was replaced by

$$\left. \begin{aligned} \phi_{i,N+1}^n &= \phi_{i,N}^n + h \\ \omega_{i,N+1}^n &= 0 \end{aligned} \right\}, \quad \left\{ \begin{aligned} 0 &< i < M+1 \\ n &> 0 \end{aligned} \right.$$

This boundary condition is equivalent to saying that u at a distance from the flat plate is equal to one.

Only a small amount of effort was expended on this problem, but it was found that convergence was very much slower than for the channel problem. More work should be done on this problem.

