STATE OF IOWA
1932

Courses of Study for High Schools

MATHEMATICS

Issued by the Department of Public Instruction
AGNES SAMUELSON, *Superintendent*

This book is the property of the district

Published by
THE STATE OF IOWA
Des Moines

STATE OF IOWA
1931

Courses of Study for High Schools

MATHEMATICS

Issued by the Department of Public Instruction
AGNES SAMUELSON, *Superintendent*

THIS BOOK IS THE PROPERTY OF THE DISTRICT

Published by
THE STATE OF IOWA
Des Moines

CONTENTS

FOREWORD

This course of study is one of a series of curriculum publications to be presented the high schools of the state from time to time by the Department of Public Instruction. It has been prepared by a subject committee of the Iowa High School Course of Study Commission working under the immediate direction of an Executive Committee. If it is of concrete guidance to the teachers of the state in improving the outcomes of instruction, the major objective of all who have contributed to its construction will have been realized.

From the start the need of preparing working materials based upon cardinal objectives and adaptable to classroom situations was emphasized. The use of the course of study in the development of proper pupil attitudes, ideals, habits, and skills was the criterion for selecting and evaluating subject matter material. At the same time it was important to consider the relation of the single course of study unit to the variety of textbooks used in the high schools of the state. The problem before the committees was that of preparing suitable courses of study representing the best in educational theory, practice, and research, and organized in such a way as to guide the teachers in using the textbook to greater advantage in reaching specified outcomes of instruction.

The selection of texts in this state is a function of the local school boards. The Department of Public Instruction and the committees do not recommend any particular text as essential to the working success of this course of study. The titles listed on the following pages are not to be interpreted as having official endorsement as against other and newer publications of value. They were found upon investigation to be in most common use in the high schools of the state at the time the units were being prepared; a follow-up survey might show changes.

Although many valuable studies have been made in the effort to determine what to teach and how to teach it, and to discover how children learn, these problems have not been solved with finality. For that reason and because no fixed curriculum can be responsive to changing needs, this course of study is to be considered as a report of progress. Its revision in accordance with the enriched content and improved procedures constantly being developed is a continuous program of the Department of Public Instruction. Your appraisal and evaluation of the material as the result of your experience with it are sincerely requested.

ACKNOWLEDGMENTS

The Department of Public Instruction takes this opportunity of thanking the many college specialists, school administrators, and classroom teachers who have helped with this program. Without the active coöperation of the educational forces of the state it could not have even been attempted. It has had the coöperation both in general and specific ways. The support given by the Iowa State Teachers Association and the High School Principals' Section has enabled the Executive Committee to meet and also to hold meetings with the Commission as a whole and with the chairman of subject committees.

Special acknowledgment is given the Executive Committee for its significant leadership in organizing the program and to Dr. T. J. Kirby for his valuable services in directing its development. Sincere gratitude is also expressed to the various committees for their faithful and skillful work in completing the subject matter reports assigned them and to Dr. C. L. Robbins for his careful and painstaking work in editing the manuscripts. The state is deeply indebted to the High School Course of Study Commission for its expert and gratuitous service in this enterprise. Credit is due the publishers for making their materials accessible to the committees and to all who served in advisory or appraisal capacities. Many of their names may not have been reported to us, but we acknowledge our appreciation to every one who has shown an interest in this significant program.

In the following committee list, the positions held by members are given as of the school year 1928-1929.

IOWA HIGH SCHOOL COURSE OF STUDY COMMISSION

Executive Committee

Thomas J. Kirby, Professor of Education, State University of Iowa, Iowa City, Executive Chairman

A. J. Burton, Principal, East High School, Des Moines

H. M. Gage, President, Coe College, Cedar Rapids

M. S. Hallman, Principal, Washington Senior High School, Cedar Rapids

O. R. Latham, President, Iowa State Teachers College, Cedar Falls

E. E. Menefee*, Superintendent, Public Schools, Hawarden

Theodore Saam,** Superintendent, Public Schools, Council Bluffs

F. H. Chandler*, Superintendent, Public Schools, Sheldon

MATHEMATICS

Ira S. Condit, Head, Department of Mathematics, Iowa State Teachers College, Cedar Falls, Chairman

W. E. Beck, Principal, High School, Iowa City

Marion Daniells, Associate Professor of Mathematics, Iowa State College, Ames

---

* Superintendent Chandler was appointed in 1929 to fill the vacancy created by the resignation of Superintendent Menefee.

** Resigned

Mary E. Joiner, Principal, High School, Maquoketa
Ruth Lane, Teacher of Mathematics, University High School, Iowa City
Ida McKee, Head of Mathematics, Newton High School, Newton
Geraldine Rendleman, Teacher of Mathematics, Woodrow Wilson Junior High, Des Moines
Jennie Taylor, Teacher of Mathematics, High School, Shenandoah

Agnes Samuelson
Superintendent of Public Instruction

GENERAL INTRODUCTION

At the first general meeting of the various subject committees a suggestive pattern for the courses of study, embodying the fundamental needs for teaching, was projected. Four crucial factors that should be emphasized in any course of study to make it an instrument that would cause teachers to consult it for guidance in the performance of their daily work were set forth as follows: objective, teacher procedures, pupil activities, and evidences of mastery.

Objectives—The meaning of objectives as here used is those concepts which are set up for pupils to achieve. As used in current practice, there is a hierarchy of objectives as shown by the fact that we have objectives of general education, objectives for various units of our educational system such as those proposed by the Committee on Cardinal Principles, objectives for subjects, objectives for a unit of instruction, and objectives for a single lesson. In each level of this hierarchy of objectives a constant element is expressed or implied in the form of knowledge, a habit, an attitude, or a skill which the pupils are expected to acquire.

In the entire field of secondary education no greater problem confronts us than that of determining what these fundamental achievements are to be. What shall be the source of those objectives, is a problem of too great proportions for discussion here,-but it is a problem that each committee must face in the construction of a course of study. A varying consideration of objectives by the various committees is evident in the courses of study they have prepared. The value of the courses varies in terms of the objectives that have been set up, according to the value of the objective in social life, according to the type of mental techniques which they stimulate and exercise, and according to the objectivity of their statement.

Pupil Activities—In our educational science we are attaching increasing significance to self-activity on the part of the learner. Recognition is made of the fundamental principal that only through their own activity pupils learn and that the teacher's rôle is to stimulate and direct this activity. No more important problem faces the curriculum-maker than that of discovering those fundamental activities by which pupils learn. In a well-organized course of study, that series of activities, in doing which pupils will attain the objectives set up, must be provided. These activities must not be chosen in a random fashion, but care must be taken that appropriate activities for the attainment of each objectives are provided.

Teacher Procedures—With the objectives determined and the activities by which pupils learn agreed upon, the function of the teacher in the pupil's learning process must be considered. In a course of study there should appear those teacher procedures of known value which make learning desirable, economical, and permanent. Here our educational science has much to offer. Where research has demonstrated with a high degree of certitude that a given technique is more effective in the learning process than others, this technique should be included in a course of study. Common teaching errors with suggested procedures to replace them may be included. Pupil difficulties which have been discovered through research should be mentioned and methods of proven value for meeting these difficulties should be included. Suggested ways of utilizing pupils' experiences should be made. And as important as any other feature is the problem of motivating learning. Whatever our educational research has revealed that stimulates the desires of pupils to learn should be made available in a course of study. Valuable types of testing should be incorporated as well as effective type assignment. The significance of verbal illustrations as evidence of comprehending the principle at issue should be featured as a procedure. Where there is a controlling procedure of recognized value such as is recognized in general science—bringing the pupil into direct contact with the phenomena studied—forceful effort for the operation of this procedure should be made.

Evidences of Mastery—What are to be the evidences of mastery of the objectives set up? There are all degrees of mastery from the memoriter repetition of meaningless terms up to a rationalized comprehension that shows grasp of both the controlling principles involved and the basic facts necessary to a clear presentation of the principles. These evidences of mastery may be in the form of dates *to be known,* formulae *to be able to use,* types of problems *to be able to solve,* quality of composition *to produce,* organization of materials *to be made,* floor talks *to be able to give,* papers *to be able to write.*

In no part of educational procedure is there need for more effort than in a clear determination of those evidences, by which a well-informed teaching staff can determine whether a pupil has a mastery of the fundamental objectives that comprise a given course. As we clarify our judgments as to what comprises the essential knowledge, habits, attitudes, and modes of thinking involved in a certain course, we can set forth with more confidence the evidences of mastery. Teachers are asking for the evidences of mastery that are expected of pupils, and courses of study should reveal them.

While these four elements constitute the basic pattern, the principle of continuity from objective to pupil activity, to teacher procedure, to evidence of mastery was stressed. The maker of a course of study must bear in mind that what is needed is an objective having accepted value; a pupil activity, in performing which, pupils gain a comprehension of the objective that is now being considered; that a teacher procedure is needed which evidence has shown is best adapted to stimulating pupils to acquire this objective for which they are striving; and that evidences of mastery must be incorporated into the course by which to test the degree of comprehension of the objective now being considered.

The courses of study vary in the degree to which these four fundamental features have been objectified and in the degree to which the principle of continuity from objective to evidence of mastery has been cared for. On the whole they will provide effective guides which teachers will use.

Realizing that these courses of study were prepared by school men and women doing full time work in their respective positions, one fully appreciates the professional zeal with which they worked and the splendid contribution to high school education which they made.

THOMAS J. KIRBY,
Chairman of the Executive Committee

COURSE OF STUDY FOR MATHEMATICS

I. Introduction

1. Why study mathematics?
   a. It is a basic element in human thinking
   b. Modern civilization is dependent upon machinery and processes made possible by the formulations of mathematics
   c. Men are not content to live like machines in a machine age
   d. By mathematical thinking we are able to establish definite relationships with the world about us

II. Educational value of the study of mathematics

1. Practical
   a. Arithmetic is one of the results of man's struggle for existence. His efforts to gain control of his surroundings, and to deal with the quantitative relations of his world, involve measurement. To record these measurements, number systems were invented. These systems developed into a science of number and an art of computation which in time was called arithmetic. In our daily life we use arithmetic constantly, either consciously or unconsciously, as all our activities are relative and involve measurement
   b. Algebraic formulas express important quantitative relationships and develop ability to think in general terms
   c. Graphic representation accompanies training in mathematics from the earliest years. Its importance is evident as shown by its frequent use in the columns of magazines, newspapers, and books
   d. Geometric or space relationship lies at the foundation of the fine and the practical arts; the painter, the sculptor, the architect, and the builder all express themselves in space relationships and give us the aesthetic values and the practical conveniences of our modern civilization
2. Disciplinary
   a. Functional thinking. Thinking of quantities in relation to each other
   b. Ability to analyze complex situations
   c. Ability to generalize
   d. A disposition to be "true to one's matured judgments"
3. Cultural (Report of National Committee)
   a. Appreciation of beauty in the geometrical forms of nature, art, and industry
   b. Ideals of perfection as to logical structure, precision of statement and of thought, logical reasoning (as exemplified in the geometric demonstration), discrimination between the true and the false, etc.
   c. Appreciation of the power of mathematics—of what Byron expressively called "the power of thought, the magic of the mind"—and the rôle that mathematics and abstract thinking, in general,

have played in the development of civilization; in particular, in science, in industry, and in philosophy

III. Objectives

1. Knowledge
   a. Of essential subject matter of algebra and geometry
   b. Of the relation of the above subjects to each other, of their dependence upon arithmetic and of their forward look toward the higher mathematics
   c. Of the relation of the above subjects to the world in which we live
   d. Of the procedure in each subject
2. Appreciations
   a. Of the relationship of mathematics to other fields of knowledge
   b. Of mathematics as a part of general culture
   c. Of mathematics as a world force—the extent to which our civilization depends upon it
   d. Of mathematics *per se*
3. Habits
   a. Of logical analysis in dealing with problem situations
   b. Of concise expression
   c. Of selecting the important elements in a problem situation
   d. Of rapid and accurate calculation
   e. Of checking results
   f. Of originality in thinking
   g. Of intellectual honesty
   h. Of neatness and accuracy in written work
4. Skills
   a. In handling essential computations
   b. In expressing problem situations in mathematical language
   c. In organizing statistical data
   d. In graphic representation
   e. In using instruments of precision
   f. In oral expression of mathematical situations

IV. Organization of courses

1. Statistics secured from the State Department of Education in October, 1928, relative to Iowa high schools are as follows:

Four-year high schools (no junior high schools)	594
Six-three-three plan	52
Six-two-four plan	160
Seven-two-three plan	33
Less than four years high school work	134

2. By means of a questionnaire, information relative to courses offered in mathematics was secured from 102 Iowa high schools, including all but one of the county seat schools. Eighty-nine or over 87 per cent of these schools teach algebra in the ninth year and geometry in the tenth
3. Since 754 of the 973 high schools have the eight-four type of organization, and since in the great majority of the larger high schools,

algebra is taught in the ninth year and geometry in the tenth, the committee presents outlines as follows:

a. Algebra
   1) Elementary (ninth year)
   2) Advanced (one semester, eleventh year)

b. Geometry
   1) Plane (tenth year)
   2) Solid (one semester, eleventh or twelfth year)

4. The aim of the committee has been to prepare a practical working outline in accordance with the best and most recent investigations in the teaching of high school mathematics as represented in the Report of the National Committee, and in the subsequent reorganization of subject matter and procedure.

   The course of study is a guide book for the teacher and is suitable for any Iowa high school. There is a wealth of supplementary material in the bibliographies which the committee hopes will be used freely by teachers and, under their guidance, by the pupils.

   Attention has been called to the importance of mathematics in our social order as well as in our educational system. The responsibility of getting this across to the pupil rests upon the teacher who must decide whether he will follow the easy way of routine procedure or the more strenuous path of carefully planned units of work, well taught, drilled, and tested. Special attention is called to the suggestions for awakening enthusiastic interest and coöperation by the use of clubs, plays, bulletin boards, and various recreational features, designed to assist in creating a mathematical atmosphere.

   The committee wishes to acknowledge its indebtedness to the teachers who have assisted in the preparation of this course of study by giving us the benefit of their counsel based upon successful experience. We hope that it will, in some measure, aid in the improvement of mathematics teaching in Iowa and be the initial movement toward a better coördination of effort.

Ira S. Condit, Chairman
W. E. Beck
Marian E. Daniells
Mary E. Joiner
Ruth O. Lane
Ida McKee
Geraldine Rendleman
Jennie Taylor

NINTH YEAR ALGEBRA

A. Introduction

1. Importance of subject

   Algebra in a limited sense is the shorthand of mathematics. There was a trace of algebraic symbolism in the Ahmes Papyrus (c. 1550 B. C.). The earliest treatise on algebra was that of Diophantus (c. 250 A. D.). His algebra was written in words with abbreviations for frequently recurring words or ideas. The Hindus were first to recognize the existence of absolutely negative quantities. Since the middle of the seventeenth century the principal features of present day algebraic symbolism have been in use. An Arabian work of the ninth century was the first to appear under the name "algebra." The Arabic title is two words meaning "restoration" and "opposition" evidently referring to operations upon an equation. In later times algebra has been called "the science of the equation." To-day the formula, the equation, and the graph, are considered the important features of algebra. Some special functions of algebra are:

   a. To broaden and strengthen the processes of arithmetic
   b. To develop devices useful in computation
   c. To develop the equation and to apply it in a wide range of solutions in which algebraic methods are more efficient than those of arithmetic
   d. To furnish valuable material useful in the later study of mathematics and science

2. Characteristics which give the subject an entity of its own

   a. It expands the number system of arithmetic to include negative and imaginary numbers
   b. Its use of general symbols in formulas in addition to the symbols of arithmetic with their fixed values

B. Educational value of subject

1. Why study it?

   a. It is a necessary tool in the learning of the physical sciences
   b. It is required for entrance to colleges and professional schools
   c. It increases ability to read and understand scientific literature and much of the material in our newspapers and magazines, such as formulas and graphs

2. Forms of thinking or controls that are acquired from its study

   a. Ability to state in a working formula the results of experimentation in any field of knowledge. This means
   b. Control of one's world through functional relationships

C. Objectives

1. Knowledge (order not fixed)

   a. Formulas

b. Equations of first degree with one unknown
c. Graphs (statistical and of mathematical laws)
d. Directed numbers
e. Fundamental operations with algebraic polynomials
f. Algebraic fractions
g. Fractional equations
h. Ratio, proportion, and variation
i. Numerical trigonometry
j. Simultaneous linear equations
k. Powers and roots
l. Quadratic equations (simpler forms)
m. Vocabulary of algebra
n. Application of algebraic notions in the solution of problems

Note: The committee has used as a working basis the topics recommended by the National Committee on Mathematical Requirements.

2. Appreciations
   a. General or algebraic thinking as a factor in the intellectual developments of the race
   b. The expansion of arithmetical thinking by the use of general number
   c. Graphic or visual representation as an aid in acquiring knowledge
   d. Ratio, proportion, and variation, as involved in our number thinking from its earliest manifestations in childhood
   e. The equation as an expression of the balance which must be maintained in all our activities
   f. Algebra in its relation to the other branches of mathematics and to such subjects as surveying, physics, engineering, and astronomy
   g. The eternal nature of the truths of algebra
   h. The value of a careful hypothesis
   i. The need for a careful evaluation of data
   j. The value of algebra *per se*
3. Habits
   a. Accuracy and neatness in oral and written expression
   b. Clear thinking about all the elements involved in a problem situation
   c. Feeling responsible for a correct result. Developing self-confidence by checking results
   d. Estimating results by looking over all the elements involved in a problem
   e. Trying to find general laws governing particular manifestations—the scientific attitude
   f. Self-reliance and devotion to truth
   g. Growth in algebraic skills
4. Skills
   a. The skills to be developed will be found fully presented in the main body of the course of study

NINTH YEAR ALGEBRA

FIRST SEMESTER

Note: In the following outlines typical situations are presented. No effort is made to present all details. Excellent modern textbooks render this unnecessary. Upper sections will be able to do more difficult work than that suggested under "Evidences of Mastery."

I. SYMBOLISM

Unit Objective

To acquire facility in the use of algebraic symbolism

Specific Objectives

1. To use algebra shorthand in representing words, phrases, and sentences about quantities

Teacher Procedures

1. Point out the convenience of symbolism or shorthand in many lines of work, such as: stenography, telegraphy, radio, stocks and bonds. Emphasize that algebra is an extension of arithmetic. **Draw continually upon arithmetic for illustrations**
2. Present the shorthand way of expressing relations between numbers. Note particularly the way of indicating multiplication and division
3. For practice in the use of algebra shorthand prepare statements to be completed, selection exercises and true-false exercises
4. A prognostic test in algebra may be used here. See bibliography

Pupil Activities

1. Pupils will write illustrations of uses of symbolism
2. Pupils will solve type problem. Start with a number n. Represent a number six times as large as n; a number five more than n; a number three less than n; a number one-fourth as much as n
3. Pupils will make a table to show the phrases and sentences commonly used for such expressions as $x + y$; $x - y$; xy; x/y or $\frac{x}{y}$

   Example

Algebra shorthand	Word statement
$x + y$	x plus y; the sum of x and y; x increased by y; y more than x; y added to x

Evidences of Mastery

1. Ability to recognize the value of symbolism in every day affairs
2. Ability to complete a statement such as: If John has m marbles, and Tom has three times as many as John, then Tom has .... marbles
3. Ability to select the correct algebra shorthand expression in exercises such

as: Our basketball team played n games and won 10. Then the number of games lost was 10n; n + 10; n — 10; 10 — n

4. In a given exercise, if the word statements are in one column, and the algebra shorthand expressions in another column but not matched, the pupil should be able to match the expressions

II. THE FORMULA

Unit Objective

To acquire facility in translating verbal statements into algebraic formulas and in translating algebraic formulas into verbal statements

Specific Objectives

1. To construct formulas for situations within the pupil's understanding
2. To evaluate formulas

Teacher Procedures

1. Show that the formula is a further use of the algebra shorthand previously studied
2. As the first test of mastery of algebra is in physics, the teacher should ever keep in mind the notation and type of equation to be found in high school physics textbooks
   (See **Inventory Tests for Mathematics of High School Physics** in Bibliography)
3. Show how to make direct substitution in the formula by erasing a letter and putting its numerical value in that place
4. Show how to simplify the expression by performing the operations indicated by the algebra shorthand
5. Explain the order of operations to be used whenever the problem involves more than one operation
6. Teach the formula as an expression of relationship between quantities and discuss constants and variables

Pupil Activities

1. Pupils will develop the formula for interest. Write the rule used in arithmetic for finding interest. Use this rule to find interest on $1800 at 5% for 2 years. Write the answer in the form i = $........ Similarly find the interest on $1800 at 5% for t years; on $1800 at r% for t years; on $p at r% for t years. Compare the word rule with this algebra shorthand rule for finding interest
2. State other word rules used in arithmetic, and translate them directly into algebra shorthand
3. Pupils will solve such a type problem as: p = 2l + 2w. Find p if l is 6 and w is 3; if l is 8 and w is 3. Make a table of these values.

l	w	p
6	3	18
8	3	22
13	3	32

They will notice the increase in value of p as l increases; and l decreases. Similarly notice the change in value of p if w increases; if w decreases. Draw figures to show what is happening to the value of p

Evidences of Mastery

1. Ability to write formulas for such statements as: Write an expression for the perimeter (p) of a rectangle if the length is l and the width is w; for the area (A); for distance (D) in terms of rate (R) and time (T); for rate in terms of distance and time
2. Ability to select the correct formula for such statements as: Find the cost (c) of one article if n articles cost T.

   $c = Tn$; $c = n + T$; $c = T - n$; $c = \frac{T}{n}$; $c = \frac{n}{T}$
3. Ability to evaluate a formula such as: $A = p + \frac{prt}{100}$ if p is 300, r is 4, and t is 2. Evaluate the formula $c = 4w + 3$ referring to the cost of sending a telegram. Evaluate the formula $w = 5\frac{1}{2}(h - 60) + 110$ which tells approximately your normal weight if your height is in inches and h is above 60
4. Ability to select the correct value for such formulas as: $s = 3n + 7$. If n is 2, then s = 12, 20, 13, 10
5. An understanding of the dependence of one quantity upon the other quantities in a formula

III. SIMPLE EQUATIONS

Unit Objective

To acquire ability to: (a) solve equations by the use of the four principal axioms; (b) set up equations from verbal problems

Specific Objectives

1. To acquire the meaning of the simple equation and the ability to solve it by means of the four principal axioms used in solving equations
2. To use the equation in working word problems

Teacher Procedures

1. Teach the equation as a question written in algebra shorthand. In writing this shorthand do not limit practice to x and y
2. Compare the equation to the balance scales. Use a pair of balance scales to explain the four axioms, and show the symbol or shorthand for each statement
3. Explain what is meant by the solution of the equation, and the form for writing it. Show how the correct solution will maintain the "balance" of the equation, and the incorrect solution will destroy the "balance"
4. Complete word statements about the exact use of the axiom should be required in all oral discussions
5. Avoid the use of transposition rule since this hinders full appreciation of the addition and subtraction axioms.
6. Show that this work is a further use of the algebra shorthand previously studied
7. Present a definite plan for analysis of the problem, such as:
   List unknown quantities
   Determine the key-unknown (Other unknowns can be expressed in terms of this key-unknown)

Make an equation by using a formula or by translating the word statement into algebra shorthand. Solve the equation and obtain answers for all unknown quantities listed

8. Point out that the answers must satisfy the original statement of the problem. This check should be in arithmetic form
9. Emphasize the need for system and form in writing a solution of a problem. Aid pupils in deciding upon a model form for solutions

Pupil Activities

1. Pupils will translate such an equation as 4a = 20 into a word statement, as: Four times a is 20. How much is a? Or, if 4 apples cost 20c how much does one apple cost? Write the answer to the question in shorthand form, a = 5
2. Pupils will examine the equation and its solution (4a = 20, a = 5) to determine which axiom was used
3. Pupils will indicate the exact use of an axiom in solving an equation as:

4a = 20	b + 5 = 12
a = 5 .... D_4	b = 7 .... S_5
x − 3 = 5	½y = 6
x = 8 .... A_3	y = 12 .... M_2

4. Pupils will show the check of their solution by substitution in the original equation
5. Pupils will solve such problems as: Tom and Harry together have 75c. Tom has 15c more than twice as much as Harry. How much has each boy? Unknown quantities are:

   Number of cents Tom has; number of cents Harry has. Key—unknown is
   Harry's number of cents. Let m = Harry's number of cents. Then
   2m + 15 = number of cents Tom has
   3m + 15 = 75
   3m = 60 .... S_{15}
   m = 20 .... D_3
   m = 20 — number of cents Harry has
   2m + 15 = 55 — number of cents Tom has. Therefore, Harry has 20c and Tom has 55c

   Check
   20c and 55c together equal 75c
   55c is 15c more than twice 20c

Evidences of Mastery

1. Ability to select the correct solution for such equations as: If 2x — 6 = 1, then x = 5, 3½, 7, 4, or 14
2. Ability to solve and check such equations as 8a + 3 = 15; $\frac{x}{2} + 1 = 4$
3. Ability to use any letter of the alphabet rather than to limit usage to x and y
4. Ability to use equations in working simple problems such as:
   a. Farmer Brown can plow one of his fields with a tractor in 3 days. It would take his neighbor 15 days with a team of horses. How long will it take Mr. Brown if his neighbor helps him with the team?

b. A boy wanted to cut a piece of wire 11 ft. long into two lengths so that one piece would be but two-thirds as long as the other. Into what two lengths must he cut it?

c. How much water should be added to 10 gal. of a 20% solution to make an 8% solution?

d. A man can allow his three children all together $2 a week for spending money. James needs 50 cents a week more than Ruth, and Charles requires only half as much as Ruth. What allowance will each child receive?

IV. GRAPHS

Unit Objective

To acquire ability to: (a) read the common forms of graphs; (b) construct statistical graphs; (c) construct and interpret formula graphs

Specific Objectives

1. To understand statistical graphs, including pictograms, circle graphs, bar graphs, and line graphs
2. To construct common types of statistical graphs
3. To construct and interpret the formula graph

Teacher Procedures

1. Provide samples of area pictograms, volume pictograms, circle graphs, bar graphs, broken line and curved line graphs
2. Point out the unreliability of area and volume pictograms in making accurate comparisons at sight
3. Lead discussion of methods used in making different types of graphs
4. Stress accuracy
5. Point the suitability of: (a) pictogram to show growth over period of years, also for attractive advertising; (b) circle graph to show relation of each item to the sum of all items; (c) bar graph to compare values if one value has not changed or grown into the next value; (d) broken line and curved line graphs to compare successive related values, the curved line especially to show a gradual change in successive values
6. Recall the method for locating sets of values for the bar graphs, and line graphs
7. Introduce notion of coördinate axes by using the local street which is the dividing line for east and west as one coördinate axis and the street that divides the town north and south as the other axis. Also discuss location of townships from arbitrarily established base and meridian lines. Call attention to latitude and longitude. Point out how the graph shows the dependence of one quantity upon the other
8. Show that new information can be obtained from the graph, while the table presents a limited amount of information

   Note: Use tests to check work. See bibliography on tests and measurements

Pupil Activities

1. Pupils will collect samples of graphs from current magazines and papers
2. Pupils will make tables of the information presented by the graphs. Notice that it is easier to compare quantities in the graph than in the table
3. They will state the facts obtainable from each given graph

4. They will use protractor and ruler to determine accuracy of circle and bar graphs discussed in class
5. They will examine given tables and choose the type of graph best suited for the pictured representation of such data
6. They will provide themselves with ruler, compasses, protractor, and squared paper
7. They will choose a convenient scale for each graph constructed and mark each graph so that it may be easily read
8. They will solve such problems as: From the formula $I = \frac{Prt}{100}$, make a graph showing relationship between P and I, when r is 5 and t is 2. The axes should be marked P and I

When P is	Then I will be
112	11.2
130	13
150	15
175	17.5

Choose suitable scales, and locate points to represent all sets of values excepting one. Draw line through points located

9. They will see if the point representing the set of values not yet located, will lie on this line
10. They will choose any point on the line, read the coördinates for this point, and see if they will fit in the table of values for P and I
11. From the graph, they will tell value of P when I is 12; tell the value of I when P is 160

Evidences of Mastery

1. Ability to gather facts from graphs commonly found in papers, magazines, and advertisements
2. Ability to construct accurate line graphs, bar graphs, and circle graphs for suitable data; and to construct attractive pictograms
3. Ability to construct graphs for the common formulas
4. Ability to use the graph to determine the relationship between quantities in the formula
5. Ability to use the graph to obtain new information about the quantities in the formula

V. POSITIVE AND NEGATIVE NUMBERS

(Directed or Signed Numbers)

Unit Objective

To acquire skill in the use of directed or signed numbers

Specific Objectives

1. To acquire an understanding of directed or signed numbers
2. To acquire skill in adding signed numbers
3. To acquire skill in subtracting signed numbers
4. To acquire skill in multiplying signed numbers
5. To acquire skill in dividing signed numbers

Teacher Procedures

1. Call attention to the zero-point as the starting point for the + and — readings
2. Extend exercises to include points won and lost in games; money received and spent; altitudes above and below sea level; and thermometer readings
3. Explain that the + and — signs are the algebra shorthand for oppositeness in quantities, hence the name "directed numbers." Also that directed numbers may be called signed numbers
4. The presentation of some familiar interpretations is left entirely to the teacher. Suggested interpretations include thermometer readings; financial situations involving money lost or owed, and money received; games where points were won and lost
5. Explain meaning of absolute value of a number
6. Point out that rules are time savers
7. Extend the chosen interpretation, to explain subtraction
8. Stress the time saving element in using the rule. The change in sign should be done mentally
9. As in addition and subtraction the presentation of a familiar interpretation to explain multiplication is left to the teacher. Since multiplication is a short method of addition, any of these suggested interpretations may be chosen. (If the financial interpretation is given it must be clear that the present time is the zero point, time in the future is + and time in the past is —)
10. Show how to determine the sign of the product when there is an odd number of negative factors; when there is an even number of negative factors
11. Introduce the product of equal factors and the use of the exponent
12. Teach division as the inverse of multiplication
13. The teacher must accept the responsibility of building up in pupils the ability to add and subtract with perfect scores. By constant use of testing and reteaching this high degree of attainment may be secured

Pupil Activities

1. Pupils will locate various readings on a thermometer, as —4°, 20°, —5°, etc.
2. They will work exercises about changing temperatures, as: Start with reading of 60°. Show where mercury would be after rising 8°; after falling 10°, etc.
3. They will make a number scale on a horizontal line and locate various positive and negative numbers such as 3, —4, 8, —6, etc. Compare with latitude and longitude
4. Pupils will draw number scales, horizontal and vertical
5. They will show that to add +3 and —4, they must locate positive 3, then move 4 spaces in negative direction
   Develop rules
6. They will explain how to add —2, —5, +7, +10, —18. Use the number scale, making each move as indicated. Then make all positive moves in succession, followed by all negative moves in succession. Develop rule.
7. They will solve such type problems as: To subtract —6 from +5. What number must be added to —6 to make +5?

To subtract +3 from —9. What number must be added to +3 to make —9

8. They will use the number scale, locate the subtrahend and then count to the minuend. Result must describe direction and number of spaces counted. Also use thermometer scale
9. They will recall that multiplication in arithmetic is a short process of addition
10. They will use the number scale to multiply +3 by —4. This means (—4) taken 3 times in its own direction. To multiply —3 by —4 means (—4) taken 3 times in a direction opposite to its own direction. Similarly use the number scales to find the product of (+3) and (+4); (−3) and (+4); etc., and so develop rules for signs in multiplication
11. They will find the product of several factors, by multiplying the first two together, that result by the third, etc.
12. They will use products of equal factors and the exponent to show this operation
13. They will recall from arithmetic that the dividend divided by the divisor equals the quotient, and that the divisor times the quotient plus the remainder equals the dividend
14. They will show that (+8) ÷ (—2) means what number multiplied by (—2) will = +8
15. After practice with the multiplication method they will acquire the law of signs for division

Evidences of Mastery

1. Ability to use + and — signs as algebra shorthand to express oppositeness in quantities
2. Ability to locate positive and negative numbers on the number scale
3. Ability to add horizontally and vertically such groups as: (—5a + 11a — 16a + 32a + 18a — 35a)

   +3a
   —8a
   —7a
   +10a
   —1a
   ———

4. An understanding of the new use of the + sign and the — sign to qualify a number
5. Ability to add with unerring accuracy positive and negative monomials
6. Ability to use addition of signed numbers as a tool to solve equations such as are found in the text used
7. Ability to perform subtractions with signed numbers, either horizontally or vertically with unerring accuracy
8. Ability to use addition and subtraction as a tool in solving equations such as: (a) 6b — 3 = —9; (b) 10 = 3c — 12
9. Ability to find the product of two or more signed numbers with accuracy and speed.
10. Ability to perform division with signed numbers with accuracy and speed

11. Ability to use division of signed numbers as a tool to solve such equations as
    a. 3m = —9
    b. —5k = 40
    c. 3y — 9 = 15
    d. 2m — 7 = 3m — 5

VI. FUNDAMENTAL OPERATIONS WITH POLYNOMIALS

Unit Objective

To acquire facility in the fundamental operations with polynomials

Specific Objectives

1. To acquire ability to add and subtract polynomials
2. To acquire ability to multiply polynomials; monomials; polynomials by monomials; binomials by binomials
3. To acquire ability to divide polynomials; monomials; polynomials by monomials; polynomials by binomials

Teacher Procedures

1. Introduce the terms, monomial, binomial, polynomial, factor, coefficient, like terms. Discuss examples like: Add 3a —7b +c and —9a +3b —4c Subtract 2m+ 5n — 3k from m + 3n + 4k
2. Recall squares and cubes of numbers in arithmetic to explain the use of exponents as algebra shorthand. Point out that there are three things to be taken care of in multiplication of monomials—the signs, the coefficients, and the exponents. Show how to form a rule for raising a monomial to any power
3. The pictured representation will help to show how to multiply the polynomial 2b + 5 by 3b. Recall the formula for the area of a rectangle

	2b	5	
3b	$6b^2$	15b	3b

   The area of the whole figure is 3b. (2b + 5). It is equal to the sum of the 2 areas, $6b^2$ and 15b .·. 3b (2b + 5) = $6b^2$ + 15b
4. Show pictured representation:

	2x	3y
3x	$6x^2$	9xy
y	2xy	$3y^2$

   The area of the entire figure (2x + 3y) (3x + y) = the sum of the four areas
   $6x^2 + 9xy + 2xy + 3y^2$ or $6x^2 + 11xy + 3y^2$
   Then show the usual arrangement of the work for such multiplication and show the application of the rule
5. Division of polynomials may be taught as the inverse of multiplication. The check should be considered as a necessary part of the work

6. Point out the similarity of long division in arithmetic and division by binomials in algebra. Stress the importance of arrangement of terms in both dividend and divisor. Use arrangement by ascending as well as descending powers
7. Omit such types as $\frac{a^5 - b^5}{a - b}$, etc.

Pupil Activities

1. Pupils will write illustrations of new terms learned
   Illustration
   $328 = 3(10)^2 + 2(10) + 8$
   Working assigned exercises
2. Pupils will write the multiplication problems the long way
   $x^2 \cdot x^3 = (x \cdot x)\ (x \cdot x \cdot x) = x^5$
   $-x^2 \cdot y^2 = -(+x \cdot +x) \cdot (y \cdot y) = -x^2y^2$
   They will use the law of exponents for multiplication. Type problem. To multiply $-\ 5x^2$ by $+\ 3xy^4$
3. They will show by arithmetic $36 \times 4 = 144$. This may be written
   $30 + 6$
   4
   $\overline{120 + 24} = 144$
   similarly $2b + 5$ multiplied by $3b$ means $6b^2 + 15b$, but in this case the terms are unlike and so cannot be combined. Develop the rule
4. Pupils will examine the results obtained in the area problem, to see how $2x + 3y$ is multiplied by $3x + y$
5. After examination of results obtained by the multiplication method, pupils will form a rule for exponents in division
6. They will work some long division problems in arithmetic, and check the results. Use exercises and checks in division of polynomials by binomials to review all the fundamental operations

Evidences of Mastery

1. Ability to add and subtract with unerring accuracy monomials and polynomials of the type found in textbooks. Future progress in algebra depends upon this ability
2. Ability to do mentally 25 exercises such as $2a^2b \times -3ab$; $(-2a^2)^3$; and $\left(-\frac{2}{3} x^2w^3t\right)^4$ without error
3. Ability to do mentally 15 exercises such as
   $-2xy\ (4x^2 - 3xy + y^2)$; $3(r - 3t) + 7(r - 2t)$ without error
4. Ability to work 10 exercises such as $(4x + 3y)\ (2x - y)$ without error
5. Ability to work mentally 25 exercises such as $\frac{-15a^3b^5}{-ab^2}$ without error
6. Ability to work mentally 15 exercises such as $\frac{16x^3 - 6x^2 - 4x}{2x}$ without error
7. Ability to work 6 exercises such as
   $(8x^3 - 125) \div (2x - 5)$ and $(6x^2 - xy - 2y^2) \div (3x - 2y)$ without error

SECOND SEMESTER

VII. SPECIAL PRODUCTS AND FACTORING

Unit Objective

To acquire short ways of finding certain special products and to find the factors of special products

Specific Objectives

1. To acquire the ability
   To expand mentally expressions like $a(x + y)$
   To factor special products like $ax + ay$
   To expand mentally binomials like $(x \pm y)^2$
   To factor $x^2 + 2xy + y^2$
   To expand mentally expressions like $(x + y)(x - y)$
   To factor $x^2 - y^2$
   To expand mentally expressions like $(2x + 3y)(3x + 2y)$
   To factor $6x^2 + 13xy + 6y^2$ and other similar quadratic trinomials

Teacher Procedures

1. Review meaning of factor, common factor, prime factor; also of monomial, binomial, trinomial, and polynomial
2. Show pupil the procedure both in expanding and in factoring. Drill on similar exercises. Use diagram on board
3. Have pupils drill on lists of exercises. Use diagrams on board
4. Ask pupil to multiply binomial forms and study result. Drill on similar forms
5. Ask pupil to expand binomial expressions by multiplication and then study result in order to get a general rule
6. Instruct pupil in trial and error method of factoring. This is a good exercise for developing correct judgment. Use drills, reviews, and tests on mixed lists of exercises. Keep to the simpler forms in this year's work. Cut the textbook lists carefully if necessary. There is no value in complicated exercises in factoring

Pupil Activities

1. Pupils will resolve some arithmetic numbers into prime factors
2. Pupils will expand mentally forms like $a(x + y)$; $b(m - n)$. Then reverse the process and factor the result of the expansions
3. They will learn by multiplication the form of the squares. Reverse the process to find factors
4. They will learn by multiplication the formula $(x + y)(x - y) = x^2 - y^2$. They will factor like exercises by inspection
5. They will expand by multiplication the assigned exercises, discover general principle for expanding similar exercises mentally, factor by method of trial and error, and factor mentally

Evidences of Mastery

1. Ability
   To expand mentally 20 exercises such as $c(dx - y)$, $a^2x(a^3 - 2ax^2 + 1)$ without error
   To factor $cdx - cy$; $rs - rt$; $3m^4n^2 - 6m^2n^4 + 3m^2n^2$
   To expand mentally 10 exercises such as $(2x - 3y)^2$; $(s + 2t)^2$ without error
   To factor the resulting forms
   To expand mentally 10 exercises such as $(x + y)(x - y)$; $(4r^2t^2 - 3y)(4r^2t^2 + 3y)$ without error
   To factor the resulting forms
   To expand mentally 10 exercises such as $(2x + 3y)(3x + 2y)$ and similar products of binomials without error
2. Ability to factor 10 exercises such as $6x^2 - 7xy - 3y^2$ without error

VIII. ALGEBRAIC FRACTIONS

Unit Objective

To acquire ability to deal with algebraic fractions

Specific Objectives

1. To acquire an understanding of the meaning of an algebraic fraction
2. To acquire an understanding of the signs connected with a fraction
3. To acquire an understanding of the laws of fractions
4. To acquire the ability to
   Add algebraic fractions
   Subtract algebraic fractions
   Multiply algebraic fractions
   Divide algebraic fractions

Teacher Procedures

1. Explain that, as in arithmetic, there are various ways of considering a fraction. For example: $\frac{2}{3}$ may be considered as $\frac{1}{3}$ of 2; as 2 of the 3 equal parts of unity; as the ratio of 2 to 3; or as an indicated division. In algebra, however, it is simpler to think of a fraction as an indicated division. Thus

   $\frac{a}{b}$ means a divided by b
2. The teacher will consider arithmetical fractions whose value is 1 such as $\frac{4}{4}$ and show all the changes in sign possible. She will then do the same with an algebraic fraction such as $\frac{a - b}{b^2 - a^2}$
3. After pupils have developed the laws of arithmetical fractions show how the same laws apply to algebraic fractions
4. Show how the work is simplified by first separating the numerator and denominator into prime factors and then dividing by factors common to both
5. Stress the fact that fractions should be reduced until no common factor remains in the numerator and denominator
6. Show that a complex fraction is another way of indicating a division of one fraction by another; show that either the numerator or denominator or both may be fractions, show, that when there is no parenthesis, multiplication or

division is to be done first and then addition and subtraction. Illustrate with arithmetic examples

7. Show how the least common denominator may be found by first separating each denominator into its prime factors and then taking the product of all the factors of the first denominator and all the factors of the second *not* found in the first, and all the factors of the third not found in the second, etc.
8. Call attention to the usefulness of the principle regarding changes in the signs (laws of fractions) connected with a fraction
9. The teacher's particular task is to guide pupils into a ready use of the two principles
   a. Multiplying both numerator and denominator of a fraction by the same number does not change the value of the fraction
   b. Dividing both numerator and denominator of a fraction by the same number does not change the value of the fraction
   c. Pupils must be able to use these principles in such problems as
   $\frac{a}{a^2 - b^2} - \frac{b}{b - a}$ in order to get $\frac{a}{a^2 - b^2} + \frac{b}{a - b}$
10. The teacher should correct all errors in handling fractions that may have carried over from arithmetic

    Note: Continue use of tests of skill. See bibliography

Pupil Activities

1. Pupils will make a list of algebraic fractions
2. From these examples the pupil will discover the important principle that any two of the three signs connected with a fraction may be changed without altering the value of the fraction
3. By the use of arithmetic fractions, pupils will establish the fact that multiplying or dividing both terms of a fraction by the same number does not alter the value of the fraction
4. Pupils will frequently restate the law of fractions used when they divide both numerator and denominator by the same factor or multiply numerator and denominator by the same number
5. Pupils will multiply arithmetic fractions and thus derive the rule for multiplying algebraic fractions
6. Pupils will divide arithmetic fractions and from that derive the rule for dividing algebraic fractions
7. Pupils will add simple arithmetic fractions and then fractions in which letters are gradually introduced
8. Pupils will subtract fractions in arithmetic and thus derive the rule for subtracting algebraic fractions

Evidences of Mastery

1. An understanding of the meaning of an algebraic fraction manifested by ability to change the form of such expressions as: $\frac{12}{16}$; $\frac{mk^2}{mk}$; $\frac{a - b}{a^2 - b^2}$
2. Ability to make and explain the changes that may be made in the signs connected with a fraction, such as $\frac{a}{b - a} = \frac{-a}{a - b} = -\frac{a}{a - b}$
3. Ability to show by the laws of fractions, that though the appearance of a fraction is changed its value is not changed

4. Ability to answer continuous oral and written questioning will reveal whether or not pupils are familiar with the laws governing fractions
5. Ability to multiply and divide with accuracy, fractions such as are found in the texts
6. Ability to add and subtract with accuracy, fractions such as are found in texts

IX. FRACTIONAL EQUATIONS

Unit Objective

To acquire ability to solve fractional equations and to use them in problems

Specific Objectives

1. To acquire the ability to solve fractional equations
2. To acquire ability to solve verbal problems involving fractional equations
3. To acquire ability in ratio and proportion by changing of fractions and fractional equations to ratios and to proportions
4. To understand the meaning of direct and inverse variation and to solve problems in variations

Teacher Procedures

1. Call *frequent* attention to the fact that when one member of an equation has been changed only in appearance, as by grouping of terms, obviously the other member of the equation must not be changed except in appearance. Also note that in such a change no law of the equation has been used
2. *Frequently* explain why it is that when a fraction is preceded by a minus sign, the sign of each term in the numerator must be changed when multiplication is performed
3. After fractional equations, in which each member consists solely of a fraction, are well mastered the words ratio and proportion should be discussed together with the ways of expressing ratio and proportion. There is little need for knowledge of these words at this stage except that the pupils will see them in various connections and probably have already used them in arithmetic
4. A variable is a quantity that is constantly changing, while a constant is a quantity that does not change. In $C = \pi D$, C and D are variables and π is constant. Point out that when a change in one variable produces a corresponding change in the other, there is a direct variation. When two variables are so related that their quotient is a constant we know they vary directly
5. Show pupils that the formula for the area of a rectangle, whose area is constant but dimensions vary, is an example of inverse variation. In general, whenever two variables are so related that their product is a constant, they are said to vary inversely
6. Give class a number of illustrations of inverse and direct variation such as the following and ask them to decide which is which
   a. Teeter board
   b. Trains running between two towns at different speeds
   c. Distance a body falls compared with time of falling
   d. Pressure of gas

   Note: Use tests of progress. See bibliography

Pupil Activities

1. Pupils will solve numerous fractional equations such as are found in texts

2. Pupils will work with formulas and with a few simple literal equations
3. Pupils will solve problems such are found in the text
4. Pupils will write all proportions as fractional equations
5. Typical example (of which many should be solved)

   $C = \pi D$

   Substitute for D increasingly large values and find out how C changes. Find out what happens to C when D is doubled. Substitute for D decreasing values and find out what happens to C. Do the same thing with C and find out what effect it has on D

   Also $c = 2\pi r$ or $\frac{c}{r} = 2\pi$

   $a = \pi r^2$ or $\frac{a}{r^2} = \pi$

   $A = \frac{1}{2}ba$ or $\frac{A}{ba} = \frac{1}{2}$

   $A = S^2$ or $\frac{A}{S^2} = 1$

   By substituting values in these formulas (preferably in the second form) pupils will discover this constant relationship
6. In the formula for the area of a rectangle when the area is 100, then $lw = 100$. Pupils assign different values to w and discover that as w increases in value that l decreases and vice versa
7. In the formula for distance, rate, and time ($d = rt$) discuss the effect of changes in r and t with d remaining unchanged
8. Also discuss $L = 2\pi ra$ (L = lateral area of right cylinder) $V = \pi r^2 a$

Evidences of Mastery

1. Ability to solve and check fractional equations such as are found in all texts, with accuracy, including a few such as:
   a. $i = prt$

      Solve for r; for p; for t

   b. $V = \pi Lr^2$

      Solve for L; for r

   c. $M = \frac{bh^3}{3}$

      Solve for b
2. Ability to solve work problems, mixture problems, and number problems, involving fractional equations. See lists in text
3. Ability to solve and check exercises like the following:

   $\frac{x}{a} = \frac{c}{d}$ and $\frac{a}{a + 2} = \frac{b}{b - 3}$
4. Ability to solve verbal problems such as are found in modern texts

X. SYSTEMS OF LINEAR EQUATIONS

Unit Objective

To acquire ability to use systems of linear equations

Specific Objectives

1. To acquire skill in graphing a linear equation of two variables such as:

To solve graphically D = 30T
(D = distance; T = time and 30 (mi.) = rate per hour)
To solve graphically equations like y = 5x; y = 3x + 2

2. To acquire an understanding of systems of equations through the graph
3. To acquire ability to solve systems of linear equations in two unknowns by algebraic method

Teacher Procedures

1. Ask pupil to graph D = 30T as in formula graphs previously discussed
2. Introduce the plan of coördinate axes (x-axes and y-axes) and drill upon the location of points by two known facts. Illustrate by latitude and longitude and location of townships. Call attention to the form of the graph (a straight line) to explain the name linear equation
3. Ask pupil to graph on same pair of axes such simple equations as:
   x + y = 5
   x — y = 3
   Extend to more complicated cases such as: 2x — 3y = 6; 3x + 2y = 6
4. Discuss solution and bring out the fact that linear equations whose graphs have one point in common are called consistent. Assign for graphing other pairs of consistent equations. Call attention to the fact that it is often difficult to determine the exact value of x and y at the point of intersection and therefore the graphic solution is not entirely satisfactory
5. Discuss systems of equations to bring out the value of the constant term; the value of the signs; how to tell from the appearance of two equations whether the lines will intersect, be parallel, coincide, or go through the origin. Also whether they will intersect one or both axes and where they will intersect the axes
6. Teach two methods of solution, elimination by addition or subtraction and also by substitution
7. Teach how to check by substituting the values for the variables in both of the *original* equations
8. Show the importance of checking in the solution of verbal problems. Check in the problem, not in the equation.

Pupil Activities

1. Pupil will draw a graph as directed. Tabulations such as
   $\frac{T = 1, 2, 3, \ldots}{D = 30, 60, 90, \ldots}$. Also draw graph of A = 5W (A = area, W = width)
2. Pupil will make tables of values and using cöordinate paper and will draw graphs of assigned equations such as:
   y = 5x; y = 3x + 2; x + y = 5; 3x — y = 9; 3x + 4y = 12
3. Pupil will make table of values and graph equations as directed, discovering that there is but one point of intersection. Should read coördinates of point of intersection and check by substitution in both equations
4. Pupil will graph:
   2x + y = 5   x + y = 1
   2x — y = —1   3x — 5y = 27
5. Pupil will graph systems of equations such as are found in modern texts
6. Pupil will solve equations in two unknowns including verbal problems. Upper sections may handle some literal coefficients and three variables if desired

Part of the time saved by cutting down the number of complicated problems listed in the older texts may be spent by pupils more profitably on appreciation of simultaneous equations which can best be learned by more extended use of the graph

Evidences of Mastery

1. Ability to solve graphically such equations as $D = 30T$ and $A = 5W$
2. Ability to solve graphically equations like $y = 3x + 2$. To explain the name linear equation
3. Ability to tell from the appearance of systems of equations in two variables, the nature of the graphs, and to solve by the graphs such systems as
   a. $3x - 2y = 8$
      $2x + 3y = 14$
   b. $x + y = 8$
      $x - y = 0$
   c. $2p - 3q = 6$
      $p + q = 8$
4. Ability to solve problems like the following
   a. $2x - 5y = 20$
      $5x + 4y = 17$
   b. $\frac{2}{3}x + y = 15$
      $x + \frac{1}{2}y = 10$
   c. A man has $1800 at interest. For one part he receives 4% for the other 5%. His income from this money is $82.00 per year. How is the money divided?

XI. NUMERICAL TRIGONOMETRY

This topic may be taught any time after fractional equations have been introduced. If similar triangles have been treated intuitively in the seventh or eighth grade the topic can profitably be introduced here and practice can be given on equations. Some teachers may prefer to teach this work following similar triangles in plane geometry. It fits in very well there. However, for those pupils who drop out of school after the ninth grade or do not study geometry, a useful and motivating topic has been lost. In this matter a school may find it necessary to follow the plan of the textbooks used. There are advantages either way. The important thing is that introduction to trigonometry should be taught thoroughly in either the ninth or tenth grades. If taught in the ninth year the plan outlined in the plane geometry section can be followed with a few modifications and omissions.

XII. POWERS AND ROOTS

Unit Objective

To acquire skill in the use of powers and roots

Specific Objectives

1. To acquire the ability to find the square roots of arithmetic numbers
   a. From a graph
   b. By computation
   c. By tables
2. To acquire the ability to find the square root of an algebraic expression

Teacher Procedures

Note: Some practice here as elsewhere should be provided on material somewhat more difficult than the degree of difficulty of that for which complete mastery is sought. The final test should then be of the degree of difficulty for which complete mastery is sought

1. Drill on meaning and use of powers and roots by use of easy examples. The greater part of the drill will naturally be on second degree roots and powers
2. Ask pupils to draw on coördinate paper the graph of $A = S^2$ (A = area, S = side of square). Then show them how to read exact and approximate roots from the graph
3. Teach the process of finding square roots by division. Estimate root, divide by the estimate, find the arithmetic average of the estimate and the quotient. This process is adequate for all roots necessary at this time. Use Pythagorean Theorem

   Note: Teach the regular arithmetic process if it is preferred. There is little difference in the degree of difficulty
4. Teach use of square root tables in text
5. Drill in squaring and finding mentally the square roots of such exercises as:

   $(x + y)^2 = ?$

   $x^2 + 2xy + y^2 = ?$

   $(2x + 1)^2 = ?$

   $4x^2 + 4x + 1 = ?$ etc.

   Note: The general method of extracting square roots of polynominals is not worth the time needed to teach it at this point. However there should be drill, review, and testing in powers and roots. Special attention should be given to simplifying common radical forms. Graphs of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ will serve to bring out the fact that these roots can be represented exactly by geometry
6. Teach and drill on material that will be met with in the solution of the quadratic equation

Pupil Activities

1. Pupils will learn the meaning and use of powers and roots by working simple examples such as are found in the texts
2. Pupils will construct graphs and read the square root of numbers from the graph

Evidences of Mastery

1. Ability to define and use: power, root, square root, principal root, rational, irrational, exponent, index of a root, radicand. Ability to evaluate such simple expression as

   8^3, $\sqrt[3]{8}$,

   $\sqrt{400}$, 4^2

   $\sqrt{4}$, $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{3}}$, but not the complicated cases
2. Ability to read exact and approximate roots from a graph
3. Ability to find square roots by computation; by use of tables
4. Ability to find square roots of algebraic expressions which are perfect squares

XIII. QUADRATIC EQUATIONS

Unit Objective

To acquire a working knowledge of quadratic equations

Specific Objectives

1. To acquire the ability to recognize and to solve readily simple quadratic equations
   a. By use of graph
   b. By completing the square
   c. By factoring

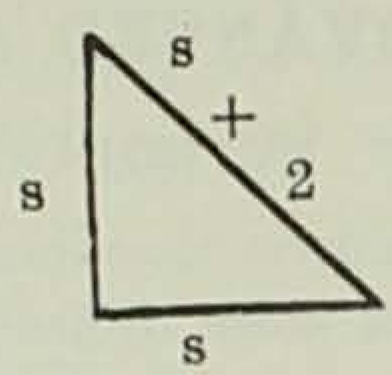
Teacher Procedures

1. Give problems of this type: Find the side of a square whose diagonal is 2 feet longer than a side
2. Call attention to changes of value of $s^2 - 4s - 4$ as s changes in value. Ask pupil to estimate values of s at points where curve crosses the x-axis. Assist pupil in estimates and in checking by substitution
3. Assign an equation with exact roots like $x^2 - 3x - 10 = 0$. Discuss the strong and weak points in the graphic solution. It emphasizes dependence of quantities and will be useful in future work as in engineering
4. Review previous work on graphing linear equations with two variables. Lead up to use of
   $y = f(x)$ as in $y = x^2 - 5x + 6$
5. Assign a completion exercise such as
   $(x + 1)^2 = x^2 + 2x + 1$
   $(x + 2)^2 = x^2 + 4x + ?$
   $(x - 4)^2 = x^2 - 8x + ?$
   Complete square in $x^2 + cx + ?$
6. Assign an equation such as $s^2 - 4s - 4 = 0$ and direct the pupil in the solution by completing the square
7. Assign an equation such as $2x^2 - x - 3 = 0$. Direct pupil in solution
8. Show that in some quadratic equations like $x^2 - 3x - 10 = 0$, the first member of the equation can be solved by factoring. Show that when the product of two factors is zero, either factor may equal zero. Check these values
9. See list of tests in bibliography for tests, drills, and reviews on simple forms of quadratics. Select from lists of exercises in textbook those which best illustrate the solution presented. Omit the more complicated and difficult exercises except with the upper group of pupils

   Note: 1. The quadratic equation is frequently half-taught in the ninth year or omitted entirely. With the selection of material indicated in this course of study, it is possible to find time to bring pupils to a good degree of proficiency in the solution of simple quadratics. The solutions by completing the square and by factoring should be used as convenient. The solution by the formula should be left for the third semester except with the upper group of pupils

   2. For tests of the year's achievement see bibliography of tests and measurements

Pupil Activities

1. Pupils will draw such a figure as 

and develop

$s^2 + s^2 = s^2 + 4s + 4$
$s^2 = 4s + 4$
$s^2 - 4s - 4 = 0$
Tabulation

s	—2	—1	0	1	2	3	4	5	6
$(s^2 - 4s - 4)$	8	1	—4	—7	—8	—7	—4	1	8

Draw graph on coördinate paper. Estimate values of s and substitute in equation. Estimated values are between —1 and 0 and between 4 and 5. Find exact roots and check

2. Pupils will complete squares in assigned exercises such as
$x^2 - 12x + ?$
$y^2 + 10y + ?$
3. Pupils will use addition axiom in completing the square in such equations as $s^2 - 4s = 4$. It is suggested that the one method of completing the square be taught:
$2x^2 - x - 3 = 0$
Divide by 2: $x^2 - \frac{1}{2}x - \frac{3}{2} = 0$
Completing square: $x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{3}{2} + \frac{1}{16} = \frac{25}{16}$
$x - \frac{1}{4} = \pm \frac{5}{4}$
$x = \frac{1}{4} \pm \frac{5}{4} = \frac{3}{2}$ or -1
Check values in the original equation
4. Pupils will solve many quadratic equations until they develop a high degree of accuracy and speed

Evidences of Mastery

1. Ability to find by a graph the roots of quadratics such as
$x^2 + x - 6 = 0$
$x^2 = 4$
$x^2 + 3x - 4 = 0$
$y = x^2 - 4$
$y = x^2 + 5x + 6$
2. Ability to solve by completing the square such equations as:
$3x^2 + 7x + 1 = 0$
$7t^2 + 2 = 15t$
$6y^2 + 15y + 6 = 0$
3. Ability to solve by factoring such equations as:
$y^2 - 10y + 21 = 0$
$6m^2 - 11m - 10 = 0$
$(t^2 - 4) + 5(t - 2) = 0$

ADVANCED ALGEBRA

INTRODUCTION

Third semester algebra should be something more than a dull gray review of the algebra of the ninth year with more difficult exercises and problems. This course should be planned for pupils who have elected to take it because they desire a better technique in algebraic expression. The review should be motivated and new material should be introduced.

These pupils are looking forward to later courses in high school or college where ability to formulate and evaluate scientific data is of prime importance. This applies not only to natural and physical science, but also to social science, psychology, and business administration. We should also keep in mind the pupils who take mathematics because they like the subject for its own sake.

"The average citizen needs

1. The ability to recognize problems in his environment;
2. The ability to recognize the quantitative features of these problems;
3. The habit of suspending judgment in quantitative matters, which habit leads to the practice of gathering all the factors bearing on a situation and of weighing their relative values;
4. The ability to state a problem properly;
5. The ability to reduce the solution of a problem to a single thought embracing a general plan;
6. A clear conception of the comparison process;
7. Skill in the translation process: in deciding when to multiply or divide, in selecting formulas; in writing equations;
8. Knowledge of quantities and units of measurement;
9. Manipulative skill and accuracy." Ligda, *Teaching of Elementary Algebra*, p. 227

I. ALGEBRAIC LANGUAGE

Unit Objective

To acquire the ability to answer the question, "Why take a third semester of algebra?"

Specific Objectives

1. To acquire a knowledge of why algebra is studied and where it functions
2. To acquire the ability to write verbal statements in algebraic form

Teacher Procedures

1. Lead in the discussion of reasons for the study of algebra set forth in the introduction to this course of study. Illustrate the points made
2. Present statements like: Write m per cent of n dollars
3. Present equation forms like: A rectangle is 2 ft. longer than it is wide. Its perimeter is 32 ft. What are its dimensions?
4. Recall and revive formulas
   a. $i = prt$
   b. $v = \pi r^2 n$
   (Assign a list of such formulas and give much practice in solving for the several variables)

Pupil Activities

1. Pupils will consider reasons for taking the course; will tell their plans for the future and how this work helps in carrying out their plans
2. Pupils will formulate algebraic statements for verbal problems given by the teacher such as:

   Express in symbols

   a. The number which is 3 times a
   b. The number which is 4 times larger than b
   c. The number which is 5 smaller than b
   d. The number which is half as large as n
   e. The number which exceeds n by 11
   f. The number which exceeds twice a by 5
   g. If James is y years old to-day, what will represent his age five years ago? What will represent his age 3 years from now?
   h. A boy has d dimes and 5 more nickles than dimes. How many nickles has he? What is the value of the nickles in cents?
   i. A man is traveling r miles an hour; another man travels 3 miles an hour faster. How many miles will the second man go in 6 hours? in h hours? in m minutes?
   j. If n represents a number, what will represent the next larger consecutive number? the next smaller consecutive number?
   k. If n represents an even integer, what will represent the next larger consecutive even integer? (Change even to odd)
   l. Express the fact that x° is the complement of y°. Write the supplement to x°

Evidences of Mastery

1. Ability to set forth in oral or written form the advantages of a further knowledge of algebra
2. Ability to state three or more reasons for the study of algebra
3. Ability to state three or more situations in which algebra functions
4. Ability to translate verbal statements and formulas into algebraic language, such as

   a. The sum of the cubes of two numbers
   b. The larger part of v if x is the smaller part
   c. The number which is five more than a
   d. The result of diminishing q by 6
   e. The excess of 25 over t
   f. One-half the sum of two numbers
   g. R per cent of P dollars
   h. The complement of a°; the fact that a, b, and c are the three angles of a triangle
   i. The difference between $\frac{1}{3}$ and $\frac{1}{5}$ the same number
   j. The number of cents in n nickles and d dimes
   k. The distance travelled in t hours at a rate of 35 miles per hour; at a rate of r miles per hour
   l. If u is the units digit and t the tens digit of a number of two digits, represent the number
   m. If u is the units digit, t the tens digit, and h the hundredths digit of a

number of three digits, represent the number. Represent a number whose digits are the same but in reverse order

n. The numerator of a fraction is three times the denominator, represent the fraction

II. USE OF FUNDAMENTAL OPERATIONS

Unit Objective

To acquire increased ability in the fundamental operations of algebra

Specific Objectives

1. To revive the skill needed for this course in positive and negative numbers
2. To restore the skill needed for this course in addition, subtraction, multiplication, and division of directed numbers. Same for polynomials
3. To be able to use the parenthesis

Teacher Procedures

1. Use illustrations to show use of numbers below zero as well as above, then follow by explanation of how to indicate each
2. Call attention to the change in value of a term if negative sign is omitted
3. Introduce the history of the origin of our number system
4. Use some arithmetic examples to obtain skill in the use of the laws of signs. Sufficient examples should be given to be sure that each pupil knows the laws and is habituated in the use of them
5. Call attention to the correct order of operations in simplifying algebraic expressions. Illustration from arithmetic: Does $30 + 20 \div 5 - 2 \times 4 = 32$ or 26? Numerical evaluation of algebraic expressions will help pupils. Give us many examples as are needed. Emphasize necessity of being accurate
6. Be sure that the meaning of terms is understood, also arrangement of terms, then give sufficient examples in each of the four processes to develop the necessary skill
7. Present such examples as $5a - (3a - 2)$, $6x\ (3x - 1)$, $(3b - 1)^2$, $\frac{(a - b)(a + b)}{a - b}$, and show that a parenthesis is used when we wish to treat several terms as a single quantity
8. Explain details of each process, and show how to proceed if one parenthesis is included within another
9. Emphasize its importance and show how a quantity is inserted in and removed from a parenthesis

Pupil Activities

1. Pupils will recall reading of a thermometer, latitude and longitude, bank balance, etc. Consider value of number with and without sign to show the meaning of the term absolute value
2. Pupils will make practical examples suggested above, to find from them a law of signs, first for addition, then by its use and by comparison of it with the other processes find laws for each
3. Pupils will demonstrate the ability to give fundamental definitions which are necessary, such as: similar terms, difference between term and factor, and between coefficient and exponent
4. Pupils will practice removal of parentheses beginning with the innermost one,

the outermost one, or any one. Also form the habit of removing all parentheses at once

Evidences of Mastery

1. Ability to use positive and negative numbers, such as: —15°, +40°; Latitude —42° Longitude +90°
2. Ability to add, subtract, multiply, and divide, algebraic expressions with accuracy and speed, such as:

$$\left(\frac{1}{6}u + \frac{1}{3}v - \frac{1}{8}\right) + \left(\frac{1}{4}u + \frac{3}{10}v + \frac{3}{4}\right) + \left(\frac{1}{3}u + v + \frac{1}{2}\right)$$

$$[.2x + .5y - z + k] - [.8x - y - .3z + m]$$

$$(3x^2 - 4x^2y + 7xy^2 - 2y^3) \div (x^2 - xy + 2y^2)$$

3. Ability to insert and to remove a parenthesis, without changing the value of the expression, such as:

$$x - [3 + 6x - (4 - 3x) + 7] - 16$$

III. LINEAR EQUATIONS IN ONE UNKNOWN

Unit Objective

To acquire greater skill in solving linear equations in one unknown

Specific Objectives

1. To acquire the ability to understand the meaning of an equation and kinds of equations
2. To acquire the ability to solve integral, literal, and fractional equations
3. To acquire the ability to analyze and solve verbal problems

Teacher Procedures

1. By examples present the idea of an equation; the distinction between equations of condition and identity; what a literal equation is; and what is meant by the roots of an equation
2. Give the class for solution examples of various kinds of equations beginning with the simplest integral equation, with emphasis on accuracy, particularly with the signs. Be sure that each understands that only one root is possible, and that he can test its correctness by substitution
3. Drill on literal equations and formulas which are not too complicated
4. As an introduction to the work with verbal problems, make it clear that the ultimate aim of algebra is to be able to solve problems, no matter how many unknowns they involve. Skill can be gained in translation from these problems even though they are not always taken from life experiences. Demonstrate process of translation from the problem into the equation. The equation is the mathematical statement of the interpretation of the problem
5. Emphasize, as a first step, the need for a careful reading and understanding of the problem
6. Check answers in the problem as originally stated
7. Among problems valuable because they (a) are as practical as possible, (b) develop skill in translation, and (c) develop idea of relationship, are number, lever, area, coin, interest, investment, digit, and distance-rate-time problems

Pupil Activities

1. Pupils will revive the equations used in the previous course necessary in this course, such as
   a. $3x = 6$
   b. $5y = 6$
   c. $-9a = 2$
   d. $8(x - 5) = 5(x - 2)$
   e. $4r = 1 - 2(r - 1)$
   f. $\frac{1}{2}(12x - 10) = \frac{1}{3}(9x - 12)$
2. Pupils will revive the axioms and their applications to equations. Observe that rules in formula form are equations and that it is necessary to solve them, not only for a numerical value, but for any letter, in terms of the other parts of the formula, such as
   a. $ax = 2$
   b. $ax = a^2$
   c. $ax = ab$
   d. $ax = b$
   e. $bx + c = d$
   f. Find the value of $F = \frac{W}{r + 1}$ when $r = 5$ and $W = 30$
   g. Find the value of $E = C(R + n)$ when $C = 8$, $n = 112$, and $R = 12$
   h. Find the value of $d = \frac{n + 2}{p}$ when $n = 16$ and $p = 1.5$
   i. Find the value of $w = P \frac{2R}{R - r}$ when $P = 140$, $R = 12$, and $r = 4$
   j. Find the value of $T = 2\pi r(r + h)$ when $h = 8$, $r = 2.5$, and $\pi = 3.14$
3. Pupils will observe that the problems which must be solved are not in equation form but can be stated thus, thereby making a solution possible. This involves translation into algebraic language or symbols. This translation requires thought and develops skill that is necessary to solve problems such as
   a. Find three consecutive numbers whose sum is 78
   b. A man sold his automobile for $1260, which was 40% less than the car cost. What was the cost of the car?
   c. One pipe can fill a tank in 10 hours; another in 3 hours; and a third can empty the tank in 8 hours. How long will it take to fill the tank if all three pipes are running?
   d. Two supplementary angles have the ratio of 9 to 1. How many degrees are there in each?
   e. The first angle of a triangle is 3 times the second, and the third is 5° less than the second. How many degrees are there in each angle?
   f. Mabel finds in her bank 10 more nickels than dimes. Altogether she has $5.00. How many coins of each kind has she?
   g. John and his father were working for a contractor. John received 30c less per hour than his father. They both together received $12.00 for a day's work of 8 hours. How much did each receive per hour?

Evidences of Mastery

1. Ability to name equations of different kinds, such as
   a. $3(a - b) = 3a - 3b$

b. $3x - 5 = 11$

c. $Wr = PR$

2. Skill in the solution of linear equations with one unknown, such as

a. $2q - (.3q + 2) = 6(.4q - .7) - (q - 4.6)$

b. $\frac{5a + 1}{4} - \frac{4a + 6}{9} = 2$

c. $T = \pi r(r + 1)$ for $r = 1.5$, $1 = 14.5$, and $\pi = 3.14$

d. $ax = b + c$

e. $acx = a^2c^2 - ac^3$

3. Ability to solve verbal problems involving equations with one unknown, such as A grocer mixes some coffee selling at 40c per pound with some that sells at 70c per pound to make 120 pounds worth 60c per pound. How many pounds of each did he use?

IV. FACTORING

Unit Objective

To acquire greater skill in factoring

Specific Objectives

1. To revive the types learned in first year algebra which are extended in this course, such as
   a. The monomial factor $ax + bx + x$
   b. The binomial factor $ax - bx - ay + by$ or the trinomial factor $ax + bx - cx - ay - by + cy$
   c. The trinomial square $a^2 + 2ab + b^2$
   d. Polynomials reducible to the trinomial square $(a - b)^2 - 2c(a - b) + c^2$
   e. The difference of two squares $a^2 - b^2 = (a - b)(a + b)$
   f. Polynomials reducible to the difference of two squares $(a - b)^2 - c^2$; $(a - b)^2 - (x - y)^2$; $a^2 - 6a + 9 - x^2 + 2xy - y^2$
   g. (1) The general trinomial $ax^2 + (a + b)x + b$
      (2) $acx^2 + (ad + bc)x + bd$
2. To be able to factor the new cases of $a^3 + b^3$ and $a^3 - b^3$
3. To be able to use the remainder theorem in the following
   a. To find the remainder when a polynomial is divided by $x - a$
   b. To use synthetic division as an aid to the factor theorem
   c. To be able to solve equations by factoring and the factor theorem

Teacher Procedures

1. Have pupils revive definitions; the principle of factoring, that when one factor is found the other is the result of division and that factoring is the process of finding two or more numbers when their product is given, in such cases as
   a. $18 = 9 \times 2 = 3 \times 3 \times 2$
   b. $ax^2 - 3ax + 5bx + x = x(?)$
2. Present an expression such as $ac - bc - am + bm$ in which a binomial factor must be formed by grouping the terms
3. Have the pupils develop that $(a + b)(a + b) = a^2 + 2ab + b^2$ and that $(a - b)^2 = a^2 - 2ab + b^2$ from which the factors of $a^2 + 2ab + b^2$ and of $a^2 - 2ab + b^2$ are found. The factors should always be checked

4. Have the pupils develop $(a - b - c)^2 = a^2 - 2ab + b^2 - 2ac + 2bc + c^2$. To factor such an expression show how it breaks up into the trinomial square $(a - b)^2 - 2c(a - b) + c^2$ and factors into $[(a - b) - c]^2$. Another method is to let $(a - b) = x$ and the expression becomes $(x)^2 - 2c(x) + c^2$ which is $(x - c)^2$ and replacing x by its value gives $(a - b - c)^2$
5. Have pupils develop $(a + b)(a - b) = a^2 - b^2$ from which the factors of $a^2 - b^2$ are found
6. In the case of $a^2 - b^2$ have the pupils think of a and b as two numbers, a^2 and b^2 as the squares of these numbers, then this case applies to such expressions as:

   $(x - y)^2 - z^2$ where $x - y$ is one number and z is the other and the factors are $(x - y + z)(x - y - z)$. A further use of this is seen in $(a - b)^2 - (x - y)^2 = (a - b + x - y)(a - b - x + y)$. The substitution method may also be used
7. Present the more difficult case of reduction: $a^4 + a^2 + 1$
8. a. Have the pupils develop $(x - 3)(x + 5) = x^2 + 2x - 15$ and $(2a - 3b)(3a + 5b) = 6a^2 + ab - 15b^2$

   b. Present the expression $(a + b^2) + 5(a + b) - 24$. Using the substitution method $(x^2) + 5(x) - 24$ in which $x = (a + b)$. This becomes $(x + 8)(x - 3)$ or $(a + b + 8)(a + b - 3)$. With some practice the substitution will be unnecessary but at first it helps to identify the expression as a trinomial
9. Present by division that $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. To put this in words—the sum of the cubes of two numbers may be divided by the sum of the numbers and the result is the square of the first number minus the product of the two numbers plus the square of the second number

   And a similar rule for the difference of the cubes of two numbers. In $8a^3 - 27$, following the rules gives $\frac{8a^3 - 27}{2a - 3} = (2a)^2 + (2a)(3) + (3)^2 = 4a^2 + 6a + 9$ or in factored form $8a^3 - 27 = (2a - 3)(4a^2 + 6a + 9)$. The pupil should be warned against writing $8a^3 - 27 = (2a - 3)^3$ or $8a^3 - 27 = (2a - 3)(4a^2 + 12a + 9)$
10. Show the pupil that when a polynomial is divided by $x - a$ the remainder is the same as when a is substituted for x in the polynomial. Use examples where the remainder is zero and lead up to the fact that when the remainder after substituting a for x is zero, $x - a$ is a factor. This is known as the factor theorem
11. Recall the principle of factoring that when one expression is exactly divisible by another, the divisor is a factor of the original expression
12. Present other examples to bring out the fact that when an expression is divisible by $x - a$ then $x - a$ is a factor and this theorem then becomes the factor theorem
13. Show how the binomial divisors are determined
14. Show the pupils that where one factor is found it reduces the degree of the equation by one. When the quotient becomes a quadratic it can be factored by the usual methods
15. Show the pupil how the factor theorem using long division is long and tedious.

Present the process known as synthetic division to assist in the use of the factor theorem

16. Develop synthetic division and drill the pupils in its use until they are able to solve assigned exercises
17. Present the method of solving equations by factoring, introducing the factor theorem

Pupil Activities

1. Pupils will define a factor; will determine the difference between a factor and a term; what sign of operation separates factors. Factor 27, 98, 250. What are prime factors? Pupils will practice in removing a common monomial factor
2. Pupils will factor $ax - bx - ay + by$, and $ax + bx - cx - ay - by + cy$
3. Pupils will expand orally with absolute accuracy twenty exercises such as
   a. $(a - 5)^2$ c. $(2a - 3b)^2$
   b. $(2a + 1)^2$ d. $(5a^2b + 6xy^3)^2$
4. Pupils will find the factors of ten trinomials such as
   a. $x^2 - 12x + 36$ c. $9b^6 - 30b^3c^2 + 25c^4$
   b. $4a^4 - 4a^2 + 1$
5. Pupils will square orally with absolute accuracy ten trinomials such as the following
   a. $x - y - z$ c. $2a - 3b + c$
   b. $a + b - c$
   Form a rule for the square of a trinomial, or of any polynomial
6. Pupils will factor such expressions as
   a. $(x - y)^2 - 4(x - y) + 4$
   b. $16(c - d)^2 - 24m(c - d) + 9m^2$
   c. $25q^2 - 60q(m + n) + 36(m + n)^2$
7. Pupils will expand orally with absolute accuracy twenty such expressions as
   a. $(a^2 - 1)(a^2 + 1) =$
   b. $(3x^2y - c^2d)(3x^2y + c^2d) =$
8. Pupils will factor twenty such expressions as
   a. $25 - 4b^2$
   b. $121a^2b^6 - 81m^2n^{10}$
9. Pupils will expand orally ten such expressions as
   a. $[(a - b) + c] [(a - b) - c]$
   b. $[2(a - b) - 3] [2(a - b) + 3]$
   c. $[6 - (m + n)] [6 + (m + n)]$
   d. $(a - b - c) (a + b + c)$
10. Pupils will factor ten such expressions as
    a. $36x^2y^4 - 25(a - 2b)^2$
    b. $9m^4(p + 2q)^2 - 25x^4y^2x^{10}$
    c. $121m^4(a + b)^2 - 225n^6(x - y)^2$
    d. $9c^2 + 20mn + 4d^2 - 12cd - 25m^2 - 4n^2$
    e. $a^2 - x^2 - y^2 + b^2 + 2ab + 2xy$
11. Pupils will determine orally what must be added to each of the following to make them trinomial squares
    a. $a^4 + a^2 + 1$ d. $a^4 + 4b^4$
    b. $x^4 + a^2x^2 + a^4$ e. $1 + 64x^4$
    c. $4x^4 - 13x^2 + 1$

12. Pupils will factor the above expressions and also such expressions as
    a. $a^4 - 6a^2b^2 + b^4$
    b. $100x^4 - 61x^2 + 9$
    c. $1 + 64x^4$
    d. $9x^4 + 3x^2y^2 + 4y^4$
13. Pupils will expand orally and with absolute accuracy 20 exercises such as $(5x - 2d)(6x + 5d)$
14. Pupils will factor 20 exercises such as
    a. $3x^2 - 14x + 8$
    b. $16x^2 - 6xy - 27y^2$
    c. $25a^4 - 41a^2b^2 + 16b^4$
15. Pupils will factor such expressions as
    a. $3(x - y)^2 + 7(x - y)z - 6z^2$
    b. $4x(x^2 + 3x)^2 - 8x(x^2 + 3x) - 32x$
16. Pupils will determine how many terms in $(a + y)^2 - 4(x + y) - 12$. Let $(x + y) = a$, factor and complete the exercise
17. Pupils will determine
    a. $\frac{a^3 + b^3}{a - b} =$
    b. $\frac{a^3 - b^3}{a - b} =$
    c. $a^3 + b^3 = (\quad)(\quad)$
    d. Also try $\frac{a^3 - b^3}{a + b} =$
    e. $\frac{a^3 - b^3}{a - b} =$
    f. Then $a^3 - b^3 = (\quad)(\quad)$
    g. $\frac{a^3 - 8}{a - 2} =$
    h. $\frac{x^3 + 27}{x + 3} =$
    i. Factor twenty such expressions as:
       $27 - a^3$
       $8a^3 + 125$
       $216x^3y^6 - 1000a^9b^3$
18. Pupils will work exercises such as the following
    a. Divide $a^2 - 7a + 10$ by $a - 6$
       Substitute $a = 6$ in $a^2 - 7a + 10$
    b. Divide $a^2 - 7a + 10$ by $a - 5$
       Substitute $a = 5$ in $a^2 - 7a + 10$
    c. Divide $a^2 - 7a + 10$ by $a + 2$
       Substitute $a = 2$ in $a^2 - 7a + 10$
    d. Divide $5x^3 - 11x^2 - 6x - 10$ by $x - 3$
       Substitute $x = 3$ in $5x^3 - 11x^2 - 6x - 10$
    e. Divide $5x^3 - 11x^2 - 6x - 10$ by $x + 1$
       Substitute $x = -1$ in $5x^3 - 11x^2 - 6x - 10$
    f. Divide $ax^2 + bx + c$ by $x - a$
       Substitute $x = a$ in $ax^2 + bx + c$
    g. Divide $a^3 + 4a^2 + a - 6$ by $a + 2$
       Factor $a^3 + 4a^2 + a - 6$
    h. Divide $a^3 - 4a^2 + a + 6$ by $a - 3$; $a + 1$; $a - 2$
       Factor $a^3 - 4a^2 + a + 6$
    i. Find the folowing products
       $(x - 3)(x + 2)$
       $(a + 5)(a - 11)$
       $(x - 1)(x - 2)(a + 1)$
       $(a - 2)(a + 2)(a + 1)$
       Note what determines the final numerical term of the product. To factor

$a^3 + 4a^2 + a - 6$, the pupil will observe that the possible factors are $x \pm 1, x \pm 2, x \pm 3, x \pm 6$

19. Pupils will work, using synthetic division, such exercises as
    a. Divide $a^3 + 4a^2 + a - 6$ by $a - 1$
    b. Divide $x^3 - 16x^2 + 27x + 36$ by $x - 3$
    c. Divide $y^3 + 6y^2 - 5y - 30$ by $y + 6$
    d. Divide $a^3 - 13a - a^2 - 14$ by $a + 2$
    e. Divide $x^4 - 10x^3 + 24x^2 + 2x - 8$ by $x - 4$
20. Pupil will solve such equations as
    a. $y^3 - 6y^2 + 11y - 6 = 0$
    b. $a^3 + 2a^2 - 5a - 6 = 0$
    c. $y^3 - 7y + 6 = 0$
    d. $3x^3 + 8x^2 + 3x - 2 = 0$
    e. $2c^3 - c^2 - 7c + 6 = 0$
    f. $x^3 - 6x^2 + 25 = 0$
    g. $2x^4 + x^3 - 14x^2 + 5x + 6 = 0$

    Note that an equation of the first degree has one root; of second degree, 2 roots; of third degree, 3 roots; of nth degree, n roots.

Evidences of Mastery

1. Ability to state the prime factors of such expressions as
   $125;\ 130;\ 51;\ 3a^2cx - 6ac^2 + 9axc^2 - 12\,a^2c^2\,x^2$
2. Ability to extend the principle of the monomial factor to that of finding the binominal factor that is common in an expression, such as
   $mnpq + 2 + pq + 2mn$
3. Ability to square a binomial orally
4. Ability to square a polynomial by inspection, such as
   $(2a - 3b^2 + 4c^2 - 52)^2$
5. Ability to factor such expressions as
   $16(x - y)^2 - 40(x - y)(c + d) + 25\,(c + d)^2$
6. Ability to factor the difference of two squares, such as
   $625m^6 - 169x^2y^4$
7. Ability to factor such expressions as
   a. $16(a - b)^2 - 25c^2$
   b. $81m^4 - 25(x - y)^2$
   c. $49(x - y)^2 - 36(a + b)^2$
   d. $4a^2 - 12ab + 9b^2 - 9x^2 + 30xy - 25y^2$
8. Ability to recognize certain trinomials that can be factored as the difference of two squares, such as
   $16a^4 - 28a^2b^2 + b^4$
9. Ability to factor the general trinomial, such as
   a. $x^4 - 9x^2 + 8$
   b. $32a^2 + 4ab - 45b^2$
   c. $24x^3 + 10x^2y^2 - 18xy^4$
10. Ability to use the substitution method to simplify complex expressions, such as
    $(3x - 5y)^2 - 2(3x - 5y) - 35$
11. Ability to factor $a^3 + b^3$ and $a^3 - b^3$ and formulate a working rule for factoring such expressions as
    a. $343a^6f^9 - 1$

b. $1728x^3y^{12} + 8a^6b^3$

12. Ability to use the remainder theorem
13. Ability to use synthetic division as a short way of finding factors
14. Ability to solve equations by factoring and the factor theorem

V. FRACTIONS

Unit Objective

To acquire skill in the use of algebraic fractions

Specific Objectives

1. To revive the earlier work in algebraic fractions needed in this course
2. To revive the work in fractional equations including formulas and verbal problems studied in first year algebra and needed in this course
3. To acquire the ability to solve literal fractional equations
4. To acquire the ability to solve verbal problems involving fractions

Teacher Procedures

1. Fractions have always been considered difficult. The idea of a fraction expressing division simplifies much that is seemingly difficult about fractions. The teacher should revive the details of addition and subtraction of fractions necessary for this course. Care should be taken in multiplication and division that the principles of division are used in the simplification of fractions. Care should be taken in making clear the three signs of a fraction and the changes in sign that can be made without changing the value of the fraction. The teacher should revive the work of first year algebra and advance to more difficult types. A higher standard should be set and insisted upon in this work. Emphasize the need for accuracy and for care in distinction between factor and term
2. The teacher should show the pupil how to clear an equation of fractions
3. Present the equation $ax = b$ to be solved for x. The following will result: $x = \frac{a}{b}$; $x = b - a$; $x = a - b$; $x = \frac{b}{a}$. Go slowly here as this is the real test whether the work is being done mechanically or with a knowledge of the axioms used. Lead up to the complicated cases through the very easiest ones and good results will be obtained. This work is often neglected and consequently the teacher of physics criticizes the mathematics teacher for not doing his work well
4. It is interesting to note that pupils can make very good problems and in solving one another's problems they become interested. This interest, once aroused, will usually last for a time if the pupil is led into the more difficult situations slowly. Quite often the trouble is in not reading the problem through until it is understood. Again the problem may not be stated in clear concise English. The words "more than," "excess," "is greater than," are not always understood. Most problems should be read several times before any attempt is made to solve them

Pupil Activities

1. Pupils will work exercises to be found in the modern texts on
   a. Reduction of fractions to lowest terms
   b. Changing fractions to a common denominator
   c. Addition and subtraction of fractions

d. Multiplication and division of fractions

e. Solution of fractional equations

f. Exercises found in physics, such as

$\frac{x + 20}{\frac{3}{4}}$; $\frac{41}{11} \div q$; $\frac{1}{6} \div \frac{1}{q}$; $\frac{211}{mn^4} \div q$

$\frac{x}{4} \div 2$; $\frac{v}{t} \div \frac{61}{42}$; $\frac{(-1)}{q} \div \frac{10}{11}$; $q \div 10q$

$40 \div \frac{1}{q}$; $\frac{w}{\frac{2}{3}}$; $\frac{10^6}{\frac{8}{9}}$; $\frac{w}{\frac{w}{2}}$; $\frac{q}{10} = 1$; $\frac{3}{2}q = 15$

2. Pupils will solve literal equations such as

a. $3x = 6$

b. $ax = 2$

c. $ax = ab$

d. $bx + c = d$

e. $ax + bx = c$

f. $ax + bx = a^2 - b^2$

g. $cx - dx = c^2 - d^2$

h. $mx - nx = m^3 - n^3$

i. $a^2y - b^2y = a^4 - b^4$

j. $ax - 3x = a^2 - 9$

k. $\frac{a}{x} = 1$; $\frac{-3}{x} = 9$

l. $\frac{a}{x} = p$; $\frac{1}{x} = \frac{1}{2}$

m. $\frac{a}{x} = \frac{b}{a}$; $\frac{b}{x - a} = m$

n. $\frac{ax}{b} + \frac{bx}{a} = \frac{a}{b} - \frac{b}{a}$

o. $\frac{a + x}{a - 2x} = \frac{a - x}{a + 2x}$

p. $\frac{a + b}{x} = \frac{c + d}{5}$

3. Pupils will solve exercises similar to the following found in physics tests

a. $S = vt$. Solve for v; for t

b. $\frac{8.82v}{x} = 3.32$. Solve for x

c. $\frac{1}{q} = \frac{1}{f} + \frac{1}{p}$. Solve for f; p; q

4. Pupils will solve problems such as are to be found in modern textbooks

Evidences of Mastery

1. Ability to add and subtract fractions, such as

a. $\frac{2}{x + 4} - \frac{x - 3}{x^2 - 4x + 16} - \frac{x^2}{x^3 + 64}$

b. $\frac{2b + a}{x + a} - \frac{2b - a}{a - x} - \frac{4bx - 2a^2}{x^2 - a^2}$

c. $\frac{3x + 2}{x^2 - 5x + 6} + \frac{x}{8x - x^2 - 15} - \frac{4 - x}{7x - x^2 - 10}$

2. Ability to multiply and divide fractions, such as

a. $\frac{6x^2 - 5x - 4}{2x^2 + 7x - 4} \cdot \frac{6x^2 + x - 2}{2x^2 + 3x + 1} \cdot \frac{2x^2 + 5x - 12}{9x^2 - 6x - 8}$

b. $\frac{a^3y - ax^2y}{a^3x^2 + a^2x^2y} \div \frac{a^2y - 2axy + x^2y}{a^2 + ay}$

c. $\frac{x^2 - (a - 1)^2}{a^2 - (x + 1)^2} \cdot \frac{(a + x)^2 - 1}{(a - x)^2 - 1} \div \frac{a + x - 1}{a - x - 1}$

3. Ability to solve fractional equations, such as

a. $\frac{x - 2}{x^3 - 8} = \frac{3}{x - 2} + \frac{1 - 3x}{x^2 + 2x + 4}$

b. $\frac{x + 1}{2x - 3} - \frac{x^2 + 10}{4x^2 - 9} = \frac{2}{2x + 3} - \frac{x - 1}{6 - 4x}$

4. Ability to solve equations involving literal quantities, such as

a. $\frac{ax}{3a + b} + \frac{bx}{3a - b} = \frac{3a^2x + b^2}{9a^2 - b^2}$

b. $\frac{px + qx}{pq} - \frac{p + q}{p^2 - pq} - \frac{x}{p} - \frac{px - qx}{pq} = 0$

c. $\frac{4x - a}{2x - a} - 1 = \frac{x + a}{x - a}$

5. Ability to handle simple verbal problems and to acquire skill in analyzing and solving the more difficult ones, such as
   a. A letter going from Boston to Chicago (900 miles) was carried for a certain time on a train at 50 miles per hour, and for the same length of time in an airplane at 130 miles per hour. How long did the trip take?
   b. A girl has worked a certain number of problems and has $\frac{2}{3}$ of them right. If she should work 9 more problems and get 8 of them right, her average would be .75. How many problems has she worked?

VI. FUNCTIONALITY—GRAPHS

Unit Objective

To acquire ability in the use of functional relationships and their graphic representation

Specific Objectives

1. To be able to understand functional relationships and dependence
2. To be able to use the graph as a method of showing how one variable depends upon another
3. To be able to use statistical graphs of an advanced type
4. To be able to make a graph of a set of linear equations

Teacher Procedures

1. Recall simple equations already used and show how quantities have been dependent upon one another throughout the work. Discuss such problems as
   a. Take a contract to supply a football team with suits. What will the total cost depend upon?
   b. What does the time required for an auto to travel 100 miles depend upon?
   c. What does the area of a rectangle depend upon?
   d. What does the cost of wheat per bushel depend upon?
   e. What does the time for a man to do a piece of work depend upon?
   f. From these simple relations, it is evident that everything in the world is dependent upon some other thing. Mathematically stated, the one thing is a function of the other
   g. Present a few formulas and show how the quantities are functionally related, such as

$A = \frac{1}{2}ab$ where A, a, and b are called variables. $\frac{1}{2}$ is called a constant. In this case, since A is directly expressed in terms of a, b, they are called independent variables and A the dependent variable

2. Present the two variables a and b, where a is always equal to 2b. Show how to place the values found in pupil procedure on graph paper (a as a function of b)
3. Present two variables where the one is always 5 more than the other
4. Introduce simple linear equations and have graphs made of these equations, always bringing to the front the functional relationships and dependence or independence of the variables
5. Present an equation such as $y = 5x - 1$, bringing out the relation between where the graph crosses the x — axis and the root of the equation
   It is important to point out that there are many values of x and y for the equation $y = 5x - 1$, but only one value of x for which $y = 0$ or $5x - 1 = 0$. Hence, only one point of crossing the x-axis
6. Show that the graph is very useful in illustrating the relations of various statistical data where there is a definite functional relationship
7. Give a temperature graph as a first illustration as it is a situation within the experience of all pupils
8. Have the pupils bring in examples of graphs met with in their reading
9. Mention or have reports in class on self-recording apparatus for temperature; pressure of steam in boilers; pressure of water in city mains; the chart of a hospital patient. In particular, the chart of the temperature of a typhoid fever patient is very interesting
10. The teacher is cautioned to spend a limited amount of time on the material of the 9th grade algebra. Much more advanced material should be introduced
11. Show that it is sometimes necessary to find a common solution for a set of two or more equations
12. Set down a series of values for $x + y = 7$. Do the same for $x - y = 3$. How many pairs in the former set are to be found in the latter set? This common pair of values is a solution for the set of equations
13. Explain the following statements
    a. The expressions $x + y = 7$ and $x - y = 3$ are called functions of x and y
    b. In the expressions $y = 7 - x$ and $y = x - 3$, y is a function of x
    c. In the expressions $x = 7 - y$ and $x = y + 3$, x is a function of y
    d. Present the equation $x = 7 - y$ in the functional notation $f(y) = 7 - y$; also the following $f(x) = 7 - x$. If $f(x) = x^2 - 3x + 5$, then $f(y) = y^2 - 3y + 5$, and $f(a) = a^2 - 3a + 5$

Pupil Activities

1. The pupils will work the following
   a. In $x + 5 = 6$
      $x = 1$
      $x + 5 = 7$
      $x = 2$
      $x + 5 = 8$
      $x = 3$
   What does the value of x depend directly upon?

b. Express the cost of x suits at \$6.00 per suit
c. Express the time for an auto to travel 100 miles at the rate of x miles per hour
d. If the area = A, base = b, and altitude = a of a rectangle, express the area
e. If the wheat crop is short 100,000,000 bushels, what effect does it have on the price per bushel? What effect if there is a bumper crop?
f. What does the height of the water in a stream depend upon? The swiftness of the current? The kind of auto I will buy? Is distance a function of rate and time?
g. The price of wheat per bushel is a function of the size of the crop. It is governed by the law of supply and demand
h. Use the formula for the area of a rectangle $A = ab$. This shows the functional relation between A, a, and b
i. Area of a triangle $A = \frac{1}{2}ab$ shows what?
j. Area of a circle $A = \pi r^2$ shows what?
k. Give other examples
l. Pick out the constant quantities, the dependent and independent variables in the following

$I = prt$	$A = \pi r^2$	$P = 2(a + b)$	$V = \frac{4}{3}\pi r^3$
$D = rxt$	$V = lwh$	$C = 2\pi r$	$F = m \cdot a$

2. Pupils will work such problems as:
a. Given $a = 2b$. Make a table for values of a when we give b the values 0, 1, 2, 3, 4, 5. Placing these on coördinate axis will show how this relation works out
b. Given $y = x + 5$ (y is a function of x). Make a table of values of y when x has the values —5, —4, —3, —2, —1, 0, 1, 2, 3, 4, 5. Construct the graph
c. Make graphs of $y = 5x - 1$

$$y = \frac{2x + 5}{2}$$
$$3y = 3x - 8$$
$$2x + 3y = 6$$
$$3x - 4y = 12$$

Where does the graph of $y = 5x - 10$ cross the x-axis?
d. Graph the following and then solve and check on this principle

Graph $y = 3x - 6$	Solve $3x - 6 = 0$
$2y = 3x + 12$	$3x + 12 = 0$
$3y - 2x = -11$	$2x - 11 = 0$
$5x - 3y = 15$	$5x - 15 = 0$

3. Pupils will show on graph paper the changes in temperature recorded in a western town one day in November

Time A.M.	Temp.	Time P.M.	Temp.
1	50°	1	70°
2	48°	2	75°
3	46°	3	80°
4	45°	4	80°
5	45°	5	75°

6	49°	6	73°
7	50°	7	70°
8	52°	8	65°
9	55°	9	62°
10	60°	10	60°
11	65°	11	55°
12	68°	12	54°

Give an estimate of the temperature at 6:30 A.M. and 8:00 P.M. Make a graph of:

Hour 1 — 2 — 3 — 4 — 5 — 6 — 7 — 8 — 9 — 10

Temp. 3°, 1°, 0°,— 3°,— 4°,— 2°,— 1°, 0°, 2°, 6°

Find graphs in newspapers and magazines and bring to class for interpretation

Obtain local school statistics and make graphs of

Temperature of school room

Comparison of attendance record with total enrollment from day to day

Per cent of pupils graduating in comparison with number entering school

4. Pupils will solve such problems as
   a. To find two numbers whose sum is 7 and whose difference is 3. The first condition gives $x + y = 7$. Make a graph of this. The second condition gives $x - y = 3$. Make a graph of this on the same axis used in $x + y = 7$. Study this for a moment. Are the lines parallel; perpendicular; do they cross? How does this compare with the common pair of values found in the two tabulations?
   b. Graph $2x + y = 13$

      $x - y = 2$

      Do these lines cross? Where?

      Can this set of equations be solved in another way to check the result?

      Will substitution check it?

Evidences of Mastery

1. Ability to think functionally
2. Ability to use the graph as an instrument to show geometrically just how one variable depends upon another
3. Ability to show the relation between the roots of a linear equation and the values of x at the points where the graph crosses the x-axis
4. Ability to use the graph as in statistics
5. Ability to read and interpret graphs as well as construct them
6. Ability to make a graph of a set of linear equations in two unknowns

VII. SYSTEMS OF LINEAR EQUATIONS

Unit Objective

To acquire an understanding of and ability to use systems of linear equations

Specific Objectives

1. To see the usefulness of using more than one equation with more than one unknown
2. To revive the solution of systems of linear equations in two unknowns as taught in first year algebra
   (a) Solution by addition or subtraction
   (b) Solution by substitution
   (c) Inconsistent and dependent cases

3. To acquire skill in solving miscellaneous sets of equations by any method
4. To acquire skill in solving equations and problems involving three unknowns
5. To acquire skill in the use of determinants in the solution of systems of equations
   a. To understand the term determinant
   b. To be able to use determinants in the solution of equations in two unknowns
   c. To be able to use determinants to detect inconsistent and dependent systems of equations
   d. To be able to use determinants of the third order

Teacher Procedures

1. The teacher will suggest a verbal problem which is solved more readily by the use of two equations with two unknowns such as

   Mix two kinds of candy costing 18c and 30c a pound respectivly so that the mixture can be sold at 24c a pound and make a gross profit of 20% on the cost of the candy. How many pounds of each should be used to make up 30 pounds of the mixture?

2. The teacher will present the solution of a system of equations by addition and subtraction. The teacher will have some of these equations graphed as well as solved and the connection pointed out
3. The teacher will present the solution of a system of equations by substitution. This method is not given as a more simple process but as a preparation for more advanced cases where such substitutions are necessary. It is very useful when the coefficient of one of the unknowns is unity as in

   $x - 2y = 2$
   $2x + 3y = 32$

4. The teacher will present equations to solve, such as

   $2x - y = 6$
   $4x - 2y = 2$
   $3x - 2y = 6$
   $6x - 4y = 10$

   and turn to the graph for an explanation of their peculiarities. This makes use of the graph to diagnose critical situations as is often done in advanced mathematics

   Sets of fractional equations that simplify into such sets make good examples

5. The teacher will present systems of equations simple, fractional, and literal; also problems resulting in systems of equations. The type where the unknowns are in the denominator is frequently met with in physics
6. The teacher will present problems involving three unknowns and point out the method of solving a system of three equations in three unknowns
7. The teacher will present the subject of determinants as a discovery that the expression $ac - bd$ could be arranged in a square such as

$$\begin{array}{c} - \\ \begin{vmatrix} a & d \\ b & c \end{vmatrix} = ac - bd \\ + \end{array}$$

which should be expanded by multiplying the elements on the two diagonals, subtracting the product of the elements on the secondary diagonal from those on the primary

8. The teacher will show how determinants are used in the solution of a system of equations, such as

$2x - 5y = -16$
$3x + 7y = 5$

where $$x = \frac{\begin{vmatrix} -16 & -5 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & +7 \end{vmatrix}} = \frac{(-16)(7) - (5)(-5)}{(2)(7) - (3)(-5)} = \frac{-112 + 25}{14 + 15} = -3$$

and $$y = \frac{\begin{vmatrix} 2 & -16 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & 7 \end{vmatrix}} = \frac{(2)(5) - (3)(-16)}{(2)(7) - (3)(-5)} = \frac{10 + 48}{14 + 15} = 2$$

9. The teacher will present such systems as

$x - 2y = -2$
$3x - 6y = -4$

where $$x = \frac{\begin{vmatrix} -2 & -2 \\ -4 & -6 \end{vmatrix}}{\begin{vmatrix} 1 & -2 \\ 3 & -6 \end{vmatrix}} = \frac{12 - 8}{-6 + 6} = \frac{4}{0}$$

since division by zero is impossible this system must be inconsistent

also $3x - 6y = -6$
$x - 2y = -2$

where $$x = \frac{\begin{vmatrix} -6 & -6 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix}} = \frac{12 - 12}{-6 + 6} = \frac{0}{0}$$

The graph shows this system to be independent. However, in both cases, the denominator is zero. In the general system

$ax + by = c$
$dx + ey = f$

the solution is impossible if

$$\begin{vmatrix} a & b \\ d & e \end{vmatrix} = 0$$

and the equations are either inconsistent or dependent

10. The teacher will present determinants of the third order and their application to the solution of equations, such as

$$x + y + z = 9$$
$$2x + 3y + z = 17$$
$$x + 2y + 2z = 16$$

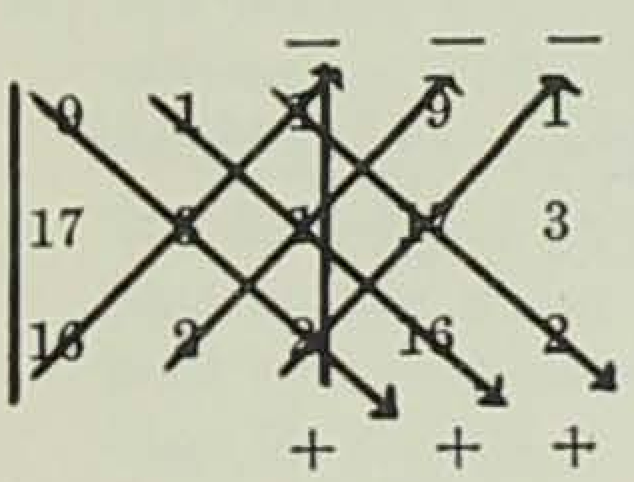
It is interesting to note that in the general system

$$ax + by + cz = d$$
$$ex + fy + gz = h$$
$$ix + jy + kz = l$$

the set is impossible if

$$\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix} = 0$$

It is suggested that the following method of expanding a determinant of the third order be taught



$$= 9\cdot3\cdot2 + 1\cdot1\cdot16 + 1\cdot17\cdot2$$
$$- 16\cdot3\cdot1 - 2\cdot1\cdot9 - 2\cdot17\cdot1 = 4$$

Pupil Activities

1. Pupils will make statements from assigned problems using two unknowns, such as:

   A man invests a certain amount of mony at 4% interest and another amount at 5%, thereby obtaining an annual income of $3100.00 If the first amount had been invested at 5% and the second at 4%, the annual income would have been $3200.00. Represent the annual income in each case and write the equation of condition

2. Pupils will work systems of equations by the addition and subtraction method, such as are to be found in the modern textbooks
3. Pupils will work systems of equations by the method of substitution, such as are to be found in the modern textbooks
4. Pupils will try to work such systems as

   a. $3x - 2y = 6$, $6x - 4y = 10$ b. $2x - y = 6$, $4x - 2y = 12$

   turning to the graph for an explanation of the apparent absurd results
5. Pupils will work systems of equations and problems involving two unknowns such as are to be found in the modern textbooks
6. Pupils will solve such equations and problems involving three unknowns as are to be found in the modern textbooks

7. Pupils will evaluate determinants, such as

a. $\begin{vmatrix} 3 & 6 \\ 5 & 7 \end{vmatrix}$ c. $\begin{vmatrix} a & 5 \\ b & -3 \end{vmatrix}$

b. $\begin{vmatrix} -6 & 4 \\ 5 & -6 \end{vmatrix}$ d. $\begin{vmatrix} ac & ef \\ bd & -gh \end{vmatrix}$

8. Pupils will solve systems of equations by determinants, such as:

a. $x + 2y = 12$
$3x - y = 1$

b. $8p + 5q = 5$
$3p - 2q = 29$

c. $\frac{1}{x} + \frac{2}{y} = 12$ Note: In (c) solve first for $\frac{1}{x}$ and $\frac{1}{y}$
$\frac{3}{x} - \frac{1}{y} = 1$

9. Pupils will work and explain such sets as

a. $3x - y = 7$
$6x - 2y = 11$
using determinants

b. $3x - y = 2$
$6x - 2y = 4$

10. Pupils will solve by determinants such systems of equations and problems in three unknowns as are to be found in the modern textbooks

Evidences of Mastery

1. Ability to state problems by use of two unknowns and two equations
2. Ability to solve systems of linear equations by addition or subtraction, such as

a. $\frac{1}{2}x + \frac{5}{4}y = 2$
$\frac{1}{6}x - \frac{5}{3}y = \frac{3}{2}$

b. $\frac{3}{x - 1} + \frac{4}{y - 1} = 0$
$\frac{5}{2x - 3} - \frac{7}{2y + 13} = 0$

c. $mx + ay = m^2 + n$
$x + amy = m + mn$

3. Ability to solve systems of equations by substitution, such as

a. $8a + 2b = 5$
$2a - b = -1$

b. $5x + 2y = 4$
$7x - 3y = 23$

4. Ability to show why some systems do not have a solution or may have many solutions such as:

a. $5x - 3y = 4$
$10x - 6y = 10$

b. $5x - 3y = 4$
$10x - 6y = 8$

5. Ability to solve with accuracy and speed systems of equations or problems involving two unknowns such as

a. $2ax + 2by = 4a^2 + b^2$
$x - 2y = 2a - b$

b. $\frac{x - y}{3} - \frac{2x + y}{2} = 0$
$\frac{x + 2y}{2} - \frac{x}{4} = \frac{11}{4}$

c. An airplane traveled 200 miles with a wind of a certain velocity in 4 hours and returned against a wind of the same velocity in 6⅔ hours. What was the rate of the wind and of the airplane?

6. Ability to solve systems of equations in three unknowns and problems leading to systems of three equations in three unknowns, such as

a. $x + .2y - .3z = 1$
$3x - y + .2z = 8.5$
$x + .8y + z = 11$

b. $\frac{3}{x} + \frac{2}{y} + \frac{1}{z} = \frac{5}{2}$
$\frac{1}{x} - \frac{1}{y} + \frac{2}{z} = \frac{-5}{2}$
$\frac{2}{x} + \frac{3}{y} - \frac{2}{z} = 6$

c. The sum of the digits of a certain number of three digits is 11. If the number be divided by the sum of its hundreds' and its units digits, the quotient is 20 and the remainder is 6. The units digit exceeds the sum of the hundreds' and tens' digits by one. Find the number

7. Ability to translate into thought, such expressions as

a. $\begin{vmatrix} 3 & 7 \\ 5 & -2 \end{vmatrix}$

b. $\begin{vmatrix} 2a & -c \\ 3c & -5a \end{vmatrix}$

8. Ability to use determinants of the second order in solving such equations as
$3x - 5y = 7$
$2x + 3y = 11$

9. Ability to use determinants to test a system of equations for inconsistency or dependence

10. Ability to use determinants of the third order in solving such equations as

a. $m + 6n + 3p = 8$
$3m + 4n = -3$
$5m + 7n = 1$

b. $\frac{1}{x} - \frac{1}{y} = a$
$\frac{1}{x} - \frac{1}{z} = b$
$\frac{1}{y} + \frac{1}{z} = c$

VIII. RADICALS AND EXPONENTS

Unit Objective

To acquire ability to understand and manipulate radical expressions and exponents

Specific Objectives

1. To revive the simplification of radical forms taught in the first course in algebra and extend the processes as needed in this course
2. To acquire skill in the four fundamental operations with radicals
3. To revive the process of extracting the square root of numbers as taught in arithmetic and apply it to algebraic expressions
4. To acquire a knowledge of imaginary numbers and how to handle them in the cases of simplification and the four fundamental processes
5. To acquire skill in solving equations involving simple radicals
6. To revive the laws of exponents taught in the first course in algebra and extend that knowledge as needed in this course

Teacher Procedures

1. The teacher will present and emphasize three conditions that must be satisfied before a radical is in its simplest form as exemplified in such as
   a. $\sqrt{50} = 5\sqrt{2}$; $\sqrt{27a^3} = 3a\sqrt{3a}$
   b. $\sqrt{\frac{3}{8}} = \frac{1}{4}\sqrt{6}$; $\sqrt{\frac{a}{5b}} = \frac{1}{5b}\sqrt{5ab}$
   c. $\sqrt[4]{25} = \sqrt{5}$; $\sqrt[4]{121a^2b^2} = \sqrt{11ab}$
2. The teacher will present the meaning of similar radicals and show how addition and subtraction of radicals is accomplished as exemplified in such as
   a. $\sqrt{27} + \sqrt{12} = 3\sqrt{3} + 2\sqrt{3} = 5\sqrt{3}$
   b. $\sqrt{18} - \sqrt{50} = 3\sqrt{2} - 5\sqrt{2} = -2\sqrt{2}$
3. The teacher will show how to multiply and divide radicals as exemplified in such as
   a. $\sqrt{3ab} \times \sqrt{6abc} = \sqrt{18a^2b^2c} = 3ab\sqrt{2c}$
   b. $\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{\frac{18}{2}} = \sqrt{9} = 3$
   c. $\sqrt{32} \div 3\sqrt{2} = \frac{1}{3}\sqrt{\frac{32}{2}} = \frac{1}{3}\sqrt{16} = \frac{4}{3}$
4. The teacher will show how to rationalize the denominator in the case of quadratic surds, such as
   a. $\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{9}} = \frac{2}{3}\sqrt{3}$
   b. $\frac{6}{(3 + \sqrt{5})} = \frac{6(3 - \sqrt{5})}{9 - 5} = \frac{6(3 - \sqrt{5})}{-4} = \frac{-9 + 3\sqrt{5}}{2}$
   c. $\frac{3 + \sqrt{2}}{3 - \sqrt{2}} = \frac{(3 + \sqrt{2})^2}{9 - 2} = \frac{11 + 6\sqrt{2}}{7}$
5. The teacher will present the arithmetic method of extracting the square root of numbers. Also a good method of approximation
6. The teacher will apply the arithmetic method to algebraic polynomials, such as
   $x^4 + 10x^2 + 19x^2 - 30x + 9$
7. The teacher will show how the arithmetic method is but a simple case of the algebraic and how the latter helps to understand the former
8. The teacher will present the meaning of an imaginary number; the symbol for $\sqrt{-1}$, and the methods used to add; subtract; multiply; divide; and simplify expressions involving imaginary expressions, such as
   a. $i = \sqrt{-1}$; $\sqrt{-50a^2b} = 5ai\sqrt{2b}$
   b. $\sqrt{-8} + \sqrt{-18}$; $\sqrt{-72a^2}$
   c. $\sqrt{-3} \cdot \sqrt{-4}$; $(3 + \sqrt{-1})(3 - \sqrt{-1})$
   d. $\frac{\sqrt{18}}{\sqrt{-3}}$; $\frac{3 + \sqrt{-5}}{3 - \sqrt{-5}}$
9. The teacher will present methods of solving such equations as
   a. $\sqrt{x + 5} = 2$
   b. $\sqrt{4x + 1} = \sqrt{x + 16}$

c. $\sqrt{2x + 1} - \sqrt{x} = 1$
d. $\sqrt{x + 6} - 1 = \sqrt{3x + 7}$
e. $\sqrt{x^2 - x + 2} + x = 4$
Check all solutions

10. The teacher will present the following relations
   a. $a^{\circ} = 1$
   b. $a^{-n} = \frac{1}{a^n}$
   c. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
11. The teacher will, by numerous examples, lead pupils to acquire skill in handling exponential expressions, such as are to be found in the modern textbooks

Pupil Activities

1. The pupil will become familiar with the terms radicand, index, radical and exponent, and with the basic principles of radicals by working such exercises as are to be found in the modern textbooks
2. Pupils will work exercises in the addition, subtraction, multiplication, and division of imaginary expressions as are to be found in the modern textbooks
3. Pupils will extract the square root of such expressions as
   a. 2; 3; 10; 165; 723.61
   b. $49x^2$; $900a^2b^2$; $2304m^4$
   c. $y^4 - 6y^3 + 13y^2 - 12y + 4$
   d. $4a^2 + 9b^2 + c^2 + 12ab - 4ac - 6bc$
4. Pupils should memorize $\sqrt{2} = 1.414$; $\sqrt{3} = 1.7321$
5. Pupils will work exercises in the addition, subtraction, multiplication, and division of imaginary expressions as found in the modern textbooks
6. Pupils will solve equations involving radicals as found in the modern textbooks
7. Pupils will extend and apply the laws for positive and integral exponents to exercises involving negative, zero, and fractional exponents as found in the modern textbooks
8. Pupils will apply the laws governing radicals and exponents to exercises found in modern textbooks

Evidences of Mastery

1. Ability to apply the laws for simplifying radical expressions, such as
   a. $\sqrt{\frac{121\ a^6x^{2m}}{81y^{2n+2}}}$
   b. $\sqrt{147a^5y^6}$
   c. $\frac{2a}{b}\sqrt{\frac{8b^2}{27a}}$
   d. $\sqrt[4]{100a^2x^8}$
2. Ability to apply the four fundamental operations to such radical expressions as
   a. $5a\sqrt{12ab^2} - 36\sqrt{27a^3} + 2\sqrt{300a^3b^2} - 40ab\sqrt{\frac{3}{4}a}$

b. $\sqrt{2ax^2} \cdot \sqrt{6x}$
c. $(2\sqrt{3} - 3\sqrt{2})(4\sqrt{3} + 5\sqrt{2})$
d. $a^3\sqrt{ab^3} \div \sqrt{a^3b}$
e. $(12\sqrt{7} - 60\sqrt{5}) \div 4\sqrt{3}$
f. $\dfrac{3\sqrt{a} - 4\sqrt{b}}{2\sqrt{a} - 3\sqrt{b}}$
g. $\left(\sqrt[4]{\sqrt[3]{x^4y^8}}\right)^6$

3. Ability to extract the square root of numbers and algebraic expressions, such as
   a. 396.482
   b. 3.1
   c. $9a^2 - 24ab + 30ac + 16b^2 - 40bc + 25c^2$
4. Ability to handle imaginary expressions, such as
   a. $\sqrt{\dfrac{-5}{10}}$; $\sqrt{-25x^2}$
   b. $(3 + \sqrt{-2})(3 - \sqrt{-2})$
   c. $[1/3\ (-1 + \sqrt{-2})]^2$
   d. $\dfrac{\sqrt{28}}{\sqrt{-14}}$
   e. $\dfrac{2}{1 + \sqrt{-3}}$
   f. $\dfrac{4 + \sqrt{-2}}{5 - \sqrt{-2}}$
5. Ability to solve equations involving radicals, such as
   a. $2\sqrt{x} - \sqrt{4x - 22} = \sqrt{2}$
   b. $\sqrt{9x + 35} = 7\sqrt{5} - 3\sqrt{x}$
   c. $\dfrac{10x}{\sqrt{10x - 9}} + \sqrt{10x + 2} = \dfrac{2}{\sqrt{10x - 9}}$

   Check all results
6. Ability to use the laws of exponents; the relations between exponents and radicals; to simplify such expressions as
   a. $\left(\dfrac{1}{36}\right)^{-3/2}$; $\dfrac{3\sqrt{a^{-1}}\ \sqrt[3]{x^2}}{2\sqrt{a^3}\ \sqrt[3]{x^{-1}}}$
   b. $\left(\dfrac{x^{-1}y^2}{a^3b^{-4}}\right)^{-2} \div \left(\dfrac{xy^{-1}}{a^{-2}b^3}\right)^3$

   a^3d^{-4} $a^{-2}b^3$

   c. $(a^{2/3} - a^{1/3} + 1)(a^{1/3} + 1)$
   d. $(5x + 2x^{2/3} - 2x^{1/3} + 1) \div (x^{1/3} + 1)$

IX. QUADRATIC EQUATIONS

Unit Objective

To acquire skill in handling and applying quadratic equations

Specific Objectives

1. To be able to handle quadratic equations
   a. To solve a quadratic equation by factoring
   b. To solve a quadratic equation by completing the square
   c. To solve a quadratic equation by the formula
2. To be able to solve equations quadratic in form
3. To be able to solve verbal problems leading to quadratic equations
4. To be able to understand the theory of the quadratic equation
5. To be able to make a graph of a quadratic equation

Teacher Procedures

1. Teacher will present a problem which leads to a quadratic equation and raises the question of solving such an equation
   a. Present the method of solution by factoring
   b. Present the method of completing the square
   Note: Teach only one method such as adding the square of $\frac{1}{2}$ the coefficient of x when the equation is in the form $x^2 + px = q$
   c. Present the formula method of solving a quadratic equation
2. The teacher will present such equations as:
   a. $x^6 - 26x^3 = 27$
   b. $(2x + 3)^2 - 7(2x + 3) - 18 = 0$
   c. $x^{2/3} + x^{1/3} - 6 = 0$
   and show how they are quadratic in form and may be solved by the methods for solving quadratics
3. The teacher will present many verbal problems leading to quadratics and develop skill in the pupils in solving such problems
4. The teacher will present and explain the relation between the sum and product of the roots with the coefficients of the quadratic equation. Also present the test for equal, real, and imaginary roots with explanations of their meanings
5. The teacher will train the pupil to make graphs of quadratic equations, pointing out the fact that the curve crosses the x-axis at the values of x which are roots of the equation

Pupil Activities

1. Pupils will solve by the factoring method quadratic equations as found in the modern textbooks
2. Pupils will solve by the method of completing the square, quadratic equations as found in the modern textbooks
3. Pupils will use the formula to solve a miscellaneous list of quadratic equations, as found in the modern textbooks
4. Pupils will solve miscellaneous quadratic equations and equations quadratic in form, such as
   a. $9a^4 - 10a^2 + 1 = 0$
   b. $(2x^2 - 1)^2 - 7(2x^2 - 1) - 144 = 0$
   c. $a + 2 + 3\sqrt{a + 2} = 18$
   d. $m^2 - 17m + 40 - 2\sqrt{m^2 - 17m + 69} = -26$

e. $3y^6 - 57y^3 - 648 = 0$
f. $\sqrt[3]{7x - 6} - 4\sqrt[6]{7x - 6} + 4 = 0$

5. The pupil will solve problems leading to quadratic equations found in the modern textbooks
6. The pupil will work exercises like the following
   a. Form the sum and product of the roots of the following equations without solving the equations
      $6x^2 + 4x + 2 = 0$
      $2 + 2x^2 = 4x$
   b. Form the equations whose roots are
      $3, -7;\ -5, -6;\ \sqrt{5}, -\sqrt{5}$
      $b, c,;\ a, -b;\ -b + k, -b - k$
      $3 + 2\sqrt{-1},\ 3 - 2\sqrt{-1};\ 1, \frac{1}{2}, \frac{1}{3}, 3$
7. Pupils will determine the character of the roots of equations such as
   $x^2 - 4x + 8 = 0$ $\qquad 20 - 13b - 15b^2 = 0$
   $5x^2 - 4x - 5 = 0$ $\qquad 1 - a^2 = 3a$
8. The pupil will graph equations such as
   a. $x^2 + x - 12 = 0$
   b. $x^2 - 3x = 0$
   c. $4x^2 - 25 = 0$

Evidences of Mastery

1. Ability to solve quadratic equations, such as
   a. $7x^2 + 11x = 6$
   b. $\frac{5}{2x + 3} + \frac{7}{3x - 4} = \frac{8x^2 - 13x - 64}{6x^2 + x - 12}$
   c. $\frac{1}{x - a} - \frac{2a}{x^2 - a^2} = b$
   d. $\frac{x - a}{\sqrt{x} - \sqrt{a}} = \frac{\sqrt{x} + \sqrt{a}}{2} + 2\sqrt{a}$
2. Ability to solve such equations as
   a. $y^6 - 19y^3 = 216$
   b. $5\sqrt[3]{x^2} = 8\sqrt[3]{x} + 4$
   c. $(x^2 + 2)^2 - 9(x^2 + 2) = -18$
3. Ability to solve problems leading to quadratic equations
4. Ability to understand the relations existing between the roots of a quadratic equation and the coefficients of the equation so that an equation may be formed when its roots are known
5. Ability to determine the character of the roots of a quadratic equation without solving the equation
6. Ability to make graphs of quadratic equations and understand their application to the character of the roots of a quadratic equation

X. QUADRATIC EQUATIONS WITH TWO VARIABLES

Unit Objective

To acquire ability to solve, and use quadratic equations in systems with two unknowns

Specific Objectives

1. To be able to make a graph of a system of two quadratic equations in two variables

2. To be able to solve the more simple systems of two quadratic equations in two variables

3. To be able to solve problems leading to quadratic equations

Teacher Procedures

1. The teacher will present methods of graphing such systems of quadratic equations as

   a. $x^2 + y^2 = 25$
   $y - x = 1$

   b. $y = x^2 - 2x - 1$
   $y = x + 3$

   c. $x^2 + y^2 = 25$
   $x^2 - y^2 = 9$

   d. $x^2 + y^2 = 25$
   $x^2 + 2y^2 = 34$

   Note: If the solution of these systems is presented at the same time as the graphing, a definite connection can be drawn between the solution and the graph

2. The teacher will present methods of solving such systems of quadratic equations as

   a. $x + y = 3$
   $2x^2 + y^2 = 9$

   b. $x^2 + 6y^2 = 15$
   $2x^2 - 9y^2 = 9$

   c. $x^3 + y^3 = 35$
   $x + y = 5$

   (Have all exercises checked unless the results are too large a fraction)

3. The teacher will present methods of solving problems leading to systems of quadratic equations in two unknowns, such as

   a. The sides of a triangle are 11 inches, 25 inches, and 30 inches. Find the altitude to the side 11 inches

   b. The square of the length of the diagonal of a rectangle is 40 and the area of the rectangle is 12 square feet. Find the dimensions of the rectangle

   c. The sum of the volumes of two cubes is 72; and an edge of one cube plus an edge of the other cube is 6. Find the length of the edges of each cube

Pupil Activities

1. The pupil will make graphs of systems of quadratic equations, such as

   a. $x^2 + y^2 = 10$
   $x + 4 = x^2$

   b. $x^2 - y^2 = 16$
   $x + 4 = x^2$

   c. $x^2 + xy + y^2 = 63$
   $x^2 - y^2 = -27$

2. The pupil will solve systems of quadratic equations, such as

a. $4x^2 - y^2 = 3\frac{3}{4}$
$6x^2 + 3y^2 = 6\frac{3}{4}$

b. $ax^2 + by^2 = c$
$x^2 - y^2 = d$

c. $\frac{16}{x^2} - \frac{9}{y^2} = 3$
$\frac{4}{x} - \frac{3}{y} = 1$

Check all results

3. The pupil will solve problems in systems of quadratic equations, such as
   a. The cost a trip for a number of men was $72 and was to be shared equally. Because 6 men failed to appear, however, the cost per man was increased $2. What was the original number of men?
   b. Some books were purchased for $150. If the price per book had been .10 less, 50 more books could have been bought for the same price. What was the price per book?

Evidences of Mastery

1. Ability to make graphs of the simpler types of systems of quadratic equations in two variables
2. Ability to solve the simpler systems of quadratic equations in two variables
3. Ability to solve problems leading to systems of quadratic equations studied in this course

XI. VARIATION

Unit Objective

To acquire ideas of dependence strengthened through mastery of direct, inverse and joint variation

Specific Objectives

1. To be able to use direct, inverse, and joint variation

Teacher Procedures

1. The teacher will ascertain whether pupils are prepared with the necessary understanding and skill in using ratio and proportion. If this work needs review, see page 00 of the section on plane geometry
2. The teacher will explain that there are various ways of expressing the idea of direct variation: y varies directly as x; y is proportional to x; the quotient of $x + y$ is constant; $y = kx$; the graph which shows the relation of y and x to be a straight line through the origin
3. The teacher will do the same for inverse and joint variation
4. It is suggested that the teacher begin with simple illustrations such as: $rt = 100$, in which r is the rate and t is the time; $25t = d$, in which t is the time and d is the distance; $pr = 200$, in which p is the principal and r is the rate; $250r = 1$, in which r is the rate and i is the interest
5. The teacher will recall from geometry that in similar figures any two lines or perimenters vary as corresponding lines, areas as squares of lines, and volumes as cubes of lines

6. It is suggested that the teacher prepare a supplementary list of problems such as:
   a. The records of two workmen show that A can join the parts of 400 automobile spark plugs in 36 hours, and B can do the same in 45 hours. If A and B are the only available men, what is the smallest number of working days of 8 hours each in which the employing company can guarantee to finish 2400 spark plugs?
   b. If a and b vary directly, and if a = 15 when b= 27, what is the value of b when a = 35?
   c. If a varies inversely as b, and if a = 27 when b = 9, what is the value of b when a = 51?

Pupil Activities

1. Pupils will cite examples of direct variation in everyday life, business, science, and the arts; also for inverse and joint variation
2. Pupils will set up a table of values; write the equation; get the constant of variation; substitute the value of the constant of variation and find values of either variable corresponding to the values of the other: graph the equation; and state the idea in several ways
3. Pupils will perform activities corresponding to the above for variation of the type; $x = yk^2$, $x = \frac{k}{y^2}$ $x = kyz$
4. Pupils will solve many verbal problems from various sources, such as: (The folowing are from physics)
   a. The distance a body falls from rest varies directly as the square of the number of seconds taken in falling. Put into an equation
   b. The weight times the distance from the fulcrum (as on a teeter board) equals the weight times the distance when there is a balance
   c. When a gas in a cylinder is placed under pressure, the volume is reduced as the pressure is increased
   d. The distance that sound travels varies directly as the time it takes to hear it
   e. "The safety load, L, of a horizontal beam supported at both ends varies jointly as the width, W, as the square of the depth, D, and inversely as the two supports."

Evidences of Mastery

1. Ability to write a proportion for statements like the following:
   a. The illumination of an object varies directly as the intensity of the light and inversely as the square of the distance from the source of light
   b. The area of the curved surface of a right circular cylinder varies jointly as the radius and altitude
2. Ability to solve by proportion, problems similar to these
   a. A steam shovel can dig 1500 cubic yard of earth in 9 hours. At the same rate, how many cubic yards can it dig in 4½ hours?
   b. A candle is placed 1 foot from a screen and another candle is placed 5 feet from the screen on the other side. How many times as strong is the illumination on the brighter side of the screen?

XII. LOGARITHMS

Unit Objective

To acquire an understanding of and ability to use logarithms

Specific Objectives

1. To be able to understand the meaning of a logarithm
2. To be able to apply logarithms
   a. To simple exercises in computation
   b. To problems in compound interest

Teacher Procedures

1. The teacher will build a table of the powers of 2, such as

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8} = .125$$

$$2^{-2} = \frac{1}{2^2} = \frac{1}{4} = .25$$

$$2^{-1} = \frac{1}{2} = .50$$

$2^0 = 1$	$2^{11} = 2048$
$2^2 = 4$	$2^{12} = 4096$
$2^3 = 8$	$2^{13} = 8192$
$2^4 = 16$	$2^{14} = 16384$
$2^5 = 32$	$2^{15} = 32768$
$2^6 = 64$	$2^{16} = 65536$
$2^7 = 128$	$2^{17} = 131072$
$2^8 = 256$	$2^{18} = 262144$
$2^9 = 512$	$2^{19} = 524288$
$2^{10} = 1024$	$2^{20} = 1048576$

2. The teacher will use the above table to point out
   a. How the operations of multiplication and division can be accomplished through addition and subtraction
   b. How the raising of a number to a given power can be accomplished through multiplication and the extraction of the root of a number through division
   c. That a logarithm is an exponent and obeys the laws of exponents
   d. That the above is a table of logarithms with the base 2
   e. That the logarithmic notation for $a^x = b$ is $\log_a b = x$
   f. That other numbers may be used as a base, and other similar tables built up
   g. That when a number is an exact power of the base, the logarithm is an integer, but otherwise a decimal

   For example

   $10^{.5} = 10^{1/2} = \sqrt{10} = 3.162$

   $\therefore \log_{10} 3.162 = .5$

3. The teacher will build up a table of the powers of 10 and point out the advantages of this system

The following may be helpful in explaining the working rule for determining the characteristic

$10^{.5} = 3.162$ hence $\log_{10} 3.162 = 0.5$

Multiply by 10 $10 \times 10^{.5} = 10^{1.5} = 31.62$ ∴ $\log 31.62 = 1.5$

Multiply by 10 $10^{2.5} = 316.2$ ∴ $\log 316.2 = 2.5$

Multiply by 10 $10^{3.5} = 3162$ ∴ $\log 3162 = 3.5$

Only the characteristic is changing by moving the decimal point. This gives the rule for finding the characteristic. It is numerically one less than the number of significant places to the left of the decimal point. This rule is for numbers greater than one

For numbers less than one: $10^{.5} = 3.162$ ∴ $\log 3.162 = 0.5$

Divide by 10 $\frac{10^{.5}}{10} = 10^{.5} \times 10^{-1} = 10^{\bar{1}.5} = .3162$ ∴ $\log 0.3162 = \bar{1}.5$

Divide by 10 $\frac{10^{\bar{1}.5}}{10} = 10^{\bar{1}.5} \times 10^{-1} = 10^{\bar{2}.5} = 0.03162$

∴ $\log 0.03162 = \bar{2}.5$, etc.

In $10^{.5} \times 10^{-1} = 10^{\bar{1}.5}$ the minus sign is placed above the 1 to show that only the one is negative, the .5 being positive. In this way the characteristic of all numbers less than one is seen to be negative while all mantissas are positive. This makes all calculation work more simple and all tabulation mantissas positive. This gives the following rule:—the characteristic of a number less than one is negative and numerically equal to the position of the first significant figure in the decimal. For example the characteristic of the logarithm

of .0035 is $\bar{3}$

of .0148 is $\bar{2}$

of .269 is $\bar{1}$

The teacher will show that in actual practice it is inconvenient to add numbers part positive and part negative and it facilitates the work if all the numbers are positive. Hence, the following device has been resorted to:

$\log 0.3162 = \bar{1}.5$

add and subtract 10 10.000–10

9.500–10

The characteristic is now 9–10 again in

$\log 0.003162 = \bar{3}.5$

add and subtract 10 10.000–10

7.500–10

The characteristic being 7–10, and again in

$\log 0.03694327 = \bar{2}.5675$

add and subtract 10 10.0000–10

8.5675–10

The characteristic is 8–10

4. The teacher will now explain the table of logarithms and how to use it. Simple interpolation should be explained and drilled upon

5. The teacher should insist upon neatness and form in all computations. All work to be done should be written before turning to the table

Example

To find $\dfrac{365 \times 28.4}{\sqrt{249} \times 568}$

log 365 = 2.	½ log 249 =
log 28.4 = 1.	log 568 =
(add)	(add)
log numerator =	log denominator =

log numerator =

log denominator =

(subt)

log fraction =

∴ value of fraction =

Turning to the table all logarithms should be looked up and placed in the form

6. The teacher will develop the formula for finding compound interest

Amount, $A = P(1 + r)^n$ (Interest) $I = A - P$

and explain its use in such examples as

Find compound amount (A) of:

a. $500 at 4% compounded annually for 10 years
b. $1000 at 5% compounded annually for 5 years
c. $400 at 4% compounded semiannually for 15 years
d. $1500 at 3% compounded semiannually for 12 years

In how many years will

a. $300 amount to $500 at 4% compounded annually?
b. $400 amount to $600 at 5% compounded annually?
c. $250 amount to $500 at 4% compounded semiannually?

Pupil Activities

1. Pupils will work exercises such as the following
   a. Using the table of the powers of two, perform the following operations

128 × 4096	$\sqrt{262144}$
256 × 1024	$\sqrt[3]{32768}$
65536 × 16	$\sqrt[5]{1048576}$
524,288 ÷ 1024	$(128)^2 : (32)^4$
65536 ÷ 262144	$2^{.5} = 2^{\frac{1}{2}} = \sqrt{2} = 1.414$
	hence $\log_2 1.414 = .5$

(1.414 lies between 1 and 4; hence its logarithm lies between 0 and 2 as it should from the table)

$2^{.333\text{——}} = 2^{\frac{1}{3}} = \sqrt[3]{2} = 1.259$

∴ log 1.259 = 0.3333

$8^{.6666\text{——}} = 8^{\frac{2}{3}} = \sqrt[3]{8^2} = 4$

hence $\log_8 4 = 0.6666$——

$\log_5 25 = ?$

$\log_3 27 =$

$\log_{11} 121 =$

$\log_6 216 =$

$\log_7 343 =$

$\log_{10} 10 =$

$\log_{10} 100 =$

$\log_{10} 1 =$

$\log_{10} 1000 =$

$\log_{10} .01 =$

b. What is the characteristic of the logarithm of 36; 248; 9; 5672; 21.96; 2.348

c. If the characteristic is 3, 2, 5, 1, 0, how would you point off the number?

d. Take the sequence of figures in the number 3694327 and write the characteristic of the logarithm of the numbers formed by moving the decimal point about, as in

369.4327	.3694327
36943.27	.003694327
36.94327	3694327
3694.327	3694327.000
3.694327	.00003694327000
369432.7	

e. Is the mantissa the same for the same sequence of figures?

f. If log 3,694,327 = 6.5675, write the logarithm of all the numbers in d

g. Add the following logarithms

$\bar{3}.5628$	7.5628–10
4.3216	4.3216
$\bar{1}.9863$	9.9863–10

Compare results

h. How to take ½ of $\bar{1}.3627$

$$\begin{array}{r} \bar{1}.3627 \\ 20.0000-20 \\ \hline 2\ \big)\ 19.3627-20 \\ 9.6814-10 \end{array}$$

Find ½ of $\bar{1}.7586$; ⅓ of $\bar{1}.6329$

i. Find the logarithm of 46; 548; 96.20; .0486; .00329; .0006; 3.8

j. Find the numbers whose logarithms are 3.4048; 1.8609; $\bar{2}.9795$; 9.5353–10; 0.7701

2. The pupil will use logarithms in working such examples and problems as:

a. $\sqrt[3]{275}$; $\sqrt[4]{72.40}$; $0.0493 \div .327$; $\sqrt{196 \times 255}$; $\sqrt[5]{15} \times \sqrt[4]{67}$

b. $(518)^{1/4} \div (381)^{1/3}$; $(91125)^{1/3} \cdot \sqrt[6]{54324}$; $\frac{5334 \times .02374}{27.43 \times 3.246}$

c. Find the area of a circle whose radius is 3.546 inches

d. The radius of a sphere is 5.236. Find the volume ($V = \frac{4}{3}\pi r^3$)

e. Find the area of a triangle whose sides are a = 302.4 feet, b = 453.7 feet, and c = 393.9 feet. [Area $A = \sqrt{s(s-a)(s-b)(s-c)}$]

3. The pupil will use logarithms in working problems in compound interest

a. To find the compound interest on $100.00 at 5% for 4 years, compounded annually. By the formula the amount of $100 for 4 years at 5% = 100 (1.05)4

$$\log 100 = 2{,}0000$$
$$4 \log 1.05 = 0.0848$$
$$\log A = 2.0848$$
$$\therefore A = \$121.56$$

b. Find the compound interest on $875 at 5% for 15 years compounded annually

c. How much will $1500 compounded semiannually for 15 years amount to at 3%

d. In how many years will $300 amount to $500 at 4% compounded annually

e. Find the compound interest on $875 at 5% compounded semiannually for 7 years

f. Find the amount of $350 invested at 4% compounded semiannually for 10 years

g. In how many years will $100 double itself at 4% compounded annually? Semiannually? Quarterly?

Evidences of Mastery

1. Ability to use a logarithm as an exponent and to show what it will accomplish in the way of simplifying operations
2. Skill in the use of logs in such examples as

$$\frac{3659 \times (342)^3}{25.67 \times 2.386} \qquad \sqrt[5]{3965.42}$$

3. An understanding of the formula for compound interest and its use in such examples as

a. Find the amount of $3500 invested for 5 years at 6% compounded annually

b. Find the amount of $2500 invested for 8 years at 4% compounded semiannually

c. What sum of money put aside now at 3% will amount to $2500 in 5 years?

XIII. TRIGONOMETRIC FUNCTIONS

For the work in trigonometry see the geometry outline page 94.

XIV. THE BINOMIAL THEOREM

Unit Objective

To acquire an understanding of and ability to use the binomial theorem

Specific Objectives

1. To be able to comprehend and use the binomial theorem

a. To understand its development

b. To find any term

c. To use the binomial theorem for fractional exponents

Teacher Procedures

1. The teacher will make an explanation of the binomial theorem being careful to distinguish between the coefficients and exponents of the binomial and those of the expansion theorem
2. The teacher will expand $(a + b)^n$ to six terms and develop the formula for any term
3. The teacher will show how the theorem becomes useful in the extraction of roots, assuming that the theorem holds for negative and fractional exponents
4. Pupils find Pascal's triangle very interesting. It is suggested that the teacher acquaint himself with this device and present it to the pupils. It is easy to see from this triangle why the coefficients should repeat themselves in reverse order after the middle term is reached

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \text{—} & & \text{—} & & \text{—} & & \text{—} & & \text{—} & & \text{—} \end{array}$$

This was first discovered by a Frenchman, Pascal, in 1653. Each row of numbers gives the coefficients of an expansion of a + b beginning with

$$\begin{array}{lccccccc} (a + b)^0, & & & & 1 & & & \\ (a + b)^2, & & & 1 & & 2 & & 1 \\ (a + b)^3 & & 1 & & 3 & & 3 & & 1 \end{array}$$

The numbers in each row can be found from the two numbers above it, one to the left, and one to the right

Example: 1 2; 1 4; 4 6
3 5 10

Pupil Activities

1. The pupil will find by actual multiplication
   a. $(a + b)^2$; $(a + b)^3$; $(a + b)^4$; $(a + b)^5 =$
   b. Without multiplying, write: $(a + b)^6 =$
   $(a - b)^6 =$
   c. Apply to $(x + 3y)^4$
   $(x + 3y)^4 = (x)^4 + 4(x)^3(3y) + 6(x)^2(3y)^2 + 4(x)(3y)^3 + (3y)^4 = x^4 + 12x^3y + 54x^2y^2 + 108xy^3 + 81y^4$

   Also
   $(x - 2y)^5$
   $(x^2 - 3y^2)^4 =$
   d. Write the first four terms of
   $(x + \frac{1}{y})^8$

   $(\frac{x}{y} - x)^6$

   $(3x + 4y^2)^7$

e. The pupil will write the
   5th term of $(a - 1)^8$
   4th term of $(x + 3y^2)^7$
   5th term of $(x^2 - y)^{12}$
   7th term of $(\frac{1}{2}a - b)$

f. Expand $(a + b)^{\frac{1}{2}}$ to 5 terms
   Apply to $\sqrt{18}$. $\sqrt{18} = \sqrt{16 + 2} = (16 + 2)^{\frac{1}{2}} =$
   Find to four decimal places: $\sqrt{17}$; $\sqrt[3]{28}$; $\sqrt{32}$; $\sqrt{35} = \sqrt{36 - 1} =$
   $(62)^{\frac{2}{3}}$

Evidences of Mastery

1. Ability to make such expansions as
   a. $(ax^2 + b)^8$
   b. $(2x - 3)^6$
   c. $(\frac{a}{2} - \frac{b^2}{3})^7$

2. Ability to write any term of any expansion such as
   a. Find the 5th term of $(b^3 - a)^7$
   b. Find he 7th term of $(\frac{1}{3}x - 3)^8$
   c. Find the middle of $(x + \frac{1}{x})^6$

3. Ability to extract roots by the binomial theorem and work such exercises as
   a. Write the 1st four terms of $(c - 4)^{\frac{1}{3}}$
   b. Write the 1st four terms of $(a - x)^{\frac{2}{3}}$
   c. Write the 1st four terms of $(a^2 + x)^{-4}$
   d. Find the approximate values to four decimal places of
      $\sqrt{51}$; $\sqrt[3]{60}$; $\sqrt[5]{245}$

XV. PROGRESSIONS

Unit Objective

To acquire ability in the use of the progressions

Specific Objectives

1. To be able to use arithmetic progressions
2. To be able to use geometric progressions

Teacher Procedures

1. The teacher will present the theory of arithmetic progressions through a problem such as
   a. A man is earning $1000 a year. For eight successive years he gets an increase of $100 a year. How much did he earn in the eighth year?
2. The teacher will develop the formula for the last term, the sum of n terms, and the arithmetic mean
3. The teacher will present the theory of geometric progression through a problem such as
   a. If a man deposits $2 in a savings bank on the 2nd of January, $4 on the

2nd of February, \$8 on the 2nd of March and so on, how much will he have deposited by the end of the year?

4. The teacher will develop the formula for the last term, the sum of n terms and the geometric mean
5. The teacher will develop the formula for the sum of an infinite decreasing geometric progression and its application to exercises such as:
   Find the value of .666- - -; of .24333- - -; of .3555- -

Pupil Activities

1. The pupil will work such exercises as
   a. If a denotes the first term and d the common difference, write four terms of the arithmetic progression, where

a = 2, d = 3	a = 8, d = —3
a = 4, d = 5	a = x^2, d = x
a = 6, d = 6	a = a, d = d

2. The pupil will work such examples and problems as
   a. Find the 16th term of —3, —8, —13 - - - - - -
   b. What term is —80 of the progression 7, 4, 1,- - -
   c. A man is paying for a house on the installment plan. His payments during the 1st three months are \$20.00, \$20.10 and \$20.20. What will his 20th and 30th payments be?
   d. A lot line is 165 feet long. Fencing it, a man wants to place the posts about 12 feet apart
      (1) Not counting end posts, how many posts would he place?
   e. A pile of fence posts has 40 in the first layer, 39 in the second, 38 in the third. There are 20 layers. How many fence posts are there in the pile?
3. The pupil will work such examples and problems as
   a. If *a* denotes the first term and *r* the common ratio write four terms of the geometric progressions, where

a = 1, r = 3	a = 6, r = ⅓
a = 2, r = 3	a = x^3, r = —x
a = 5, r = —2	a = a. r = r

4. The pupil will work such examples and problems as
   a. Find the 7th term of $\frac{3}{2}$, 3, —6, - - -
   b. What term of the progression 5, 10, 20, 40, is 640?
   c. Find the ratio of the geometric progression when a = 4, and the fifth term is $\frac{1}{64}$
   d. Find the 30th term when a = 50, r = 1.02
   e. Find the sum of the first six terms of the geometric progression 4, — 12, + 36 - - -
   f. Insert 2 geometric means be tween $\frac{1}{16}$ and 4
5. The pupil will work such exercises as
   a. Find the sum to infinity of

      8, 2, ½ - - - - -

      1, $-\frac{1}{3}$, + $\frac{1}{9}$ - - - - -

      — 4, — ½, $-\frac{1}{16}$ - - - -

b. Find the value of the following repeating decimals

.444- - - .409090- - -

.636363 - - - .52222- - -

6. Pupils will work miscellaneous exercises and problems involving the progressions
   a. What sort of aprogression is

      $-3, 3, 9, 15---$

      $3, 10, 17, 24---$

      $m, -m^2, m^3, -m^4---$

      $\sqrt{2}, 2 + \sqrt{2}, 4 + \sqrt{2}---$

      $\sqrt{2}, 2, 2\sqrt{2}, 4---$

   b. Rearrange so as to form an A.P. or a G.P.

      $3, 15, -3, 9$

      $\sqrt{2}, \sqrt{8}, 3\sqrt{2}$

      $4, -4, 8, 0$

      $x^{-4}, 1, x^{-2}, x^2$

      $\sqrt{3}, \sqrt{12}, \sqrt{48}$

      ½, ¼, 1, 2

   c. A vacation club calls for payments of 3 cents the first week; 5 cents the second week; 7 cents the third week and so on. How much will be collected from one person in five weeks?
   d. In a peanut race each player starts from a mark and brings back 10 peanuts, one at a time. The first peanut is 5 feet from the mark, and each of the others 5 feet farther than the preceding. How many feet does each player travel in gathering 10 peanuts?

Evidences of Mastery

1. Ability to use arithmetic progressions in the solution of such exercises as
   a. Find the arithmetic mean of 3 and 27
   b. In a Christmas savings plan, the payments to be made for 50 weeks are 5c, 10c, 15c, etc. What will be the fortieth payment?
   c. In a certain school system, a teacher is paid $950 for her first year's work, and is given an increase of $50 per year each year thereafter. What will be the teacher's total income during ten years of service?
   d. If the fifth term of an arithmetic progression is $7/15$, the twenty-first term is $\frac{11}{3}$, and the last term is $\frac{17}{3}$. Find the number of terms
2. Ability to use geometric progression in the solution of such problems as
   a. Insert 4 geometric means between 1 and 243
   b. Find the sum of the first 10 powers of 2
   c. Each year a man saves half as much again as he saved the preceding year. If he saved $128 the first year, to what sum will his savings amount at the end of seven years?

PLANE GEOMETRY

FIRST SEMESTER

I. THE INTRODUCTION

Unit Objective

To acquire (1) information about the origin, history, and importance of geometry ;(b) ability to use some of the fundamental notions and some of the tools of geometry

Specific Objectives

1. To answer the question, "Why study geometry?"
2. To have a clear understanding of the new vocabulary and of the more exact manner in which certain familiar words are used generally
3. To be able to use the protractor
4. To be able to use the compass accurately in a few fundamental constructions without knowledge of the proofs of their accuracy
5. To have an appreciation of the meaning and uses of assumptions
6. To realize the inadequacy of eyesight tests as proofs
7. To learn what is meant by a formal proof

Teacher Procedures

1. The teacher will give a short talk on the importance of the study of mathematics with special reference to geometry. This talk should show the teacher's enthusiasm for the subject but only those claims should be made which can be accomplished to a degree which will be recognized by the pupils as said claims or objectives are again referred to during the year Illustrations from the teacher's own experience, and comments of other persons regarding the value of the study of geometry to them, are more convincing to the pupil than general comments. This talk may include a discussion of

   The increasing dependence of civilization upon mathematics as it becomes more scientific and consequently more mathematical

   The high school pupil's need of planning for more mathematics than his father used

   The use of the fundamental principles of geometry by the constructing engineer, the sailor, the machinist, the carpenter, the mechanic, the draftsman, the electrician, landscape gardener, etc.

   The geometric formations in nature, in architecture, and about us everywhere

   The importance of geometry in the culture of the past

   The study of geometry as the only formal training in logic in the high school, training in connected exact thinking and against jumping at conclusions and over dependence upon eyesight tests

   A collection of pictures of gothic architecture, modern engineering feats, etc., hung on the bulletin board for a few days or dispayed in small groups throughout the year.

2. The teacher will request the pupils to put a light line through those words or paragraphs in the introduction which are not to be studied until they are needed later in the year

   Discussions about most of the geometric words before the pupils read the discussion of the text, *e.g.*, to explain the difference between the geometric plane and the Iowa plains and why the spelling for the tool called a plane is applicable; to show an empty box and ask if it is a geometric solid; to mark out in space with the arms a figure of three dimensions so the pupils get the image and then ask if it is a geometric solid; to put a chalk line on the board and ask if it is a geometric line and if necessary use the length of the chalk to make a wide mark so the pupils see that what has been drawn is really in the general shape of a rectangle

   It should be made very clear that complementary refers to two angles whose sum is 90° and supplementary to two angles whose sum is 180°
3. The teacher will demonstrate the use of the board protractor to measure and to make angles. To emphasize the fact that the center point must be located and that point put on the vertex of the angle and the same point and the one marked zero should be used to determine the initial side. Considerable emphasis is needed here to avoid the assumed use of the lower edge of the protractor
4. The teacher will begin constructions with compass and straightedge at once, along with the reading material of the text so that the learning of terms does not become tedious; and will supervise most of the construction work and require it to be very neat and accurate
5. The teacher will explain the use of the word assumption and of axiom as a special kind of assumption
6. The teacher will direct the pupil's attention to the uncertainty of eyesight, or of the current notion that "seeing is believing." A number of optical illusions drawn with India ink on cardboard and shown as flash cards can be filed away for use year after year
7. The teacher will prove a simple theorem, explaining the meaning of a proof and writing out the steps of the proof. The first theorem may well be one whose conclusion is uncertain as more interest will be aroused
8. The teacher will accumulate a list of all technical terms met, make clear their exact meaning, and hold pupils for ability to use, define, and illustrate them

Pupil Activities

1. The pupil will read the history of the origin of geometry in histories of mathematics, encyclopaedias, introductions in plane geometry texts

   Roll call answer by the pupil by naming triangles observed on the way to school as, in advertising signs, on a trellis, on the gable-end of a house. On other days rectangles, squares, rhombuses, trapezoides, and circles seen in use in different ways or the number of them counted, furnish the response to roll call

   Three pupils with the aid of a knotted cord or a tape measure imitate the early Egyptians in constructing a right triangle with the sides 3, 4, and 5 and also 6, 8, and 10 feet

   Pictures from magazines, and other sources showing mosaic floors, linoleum patterns, and other geometric designs posted on the bulletin board

2. The discussion of the mathematical vocabulary is read in the text and followed by class discussion

   Examples of activities in class: One pupil rotates a ruler from an initial line through angles of 90°, 180°, 270°, 360°, 450°, etc. Right, acute, and obtuse angles are pointed out in the room. The clock is observed in discussing angles of more than 360°. The pupils discuss together ways of remembering whether the word complementary is used for two angles whose sum is 90° or whether the word is supplementary

   Exercises on complementary and supplementary angles

   Some pupils remember that "c" comes before "s" in the alphabet and 90 comes before 180. They discuss the difference between intersect and bisect. A true horizontal line and a true vertical line are obtained with the ordinary equipment of the room, with a decision as to which can be obtained first

   Short daily objective quizzes on the understanding of geometric words which have been discussed thoroughly in class
3. To make and measure angles in all sorts of positions, exchanging protractors of different types and comparing results
4. Pupils will make constructions of perpendicular bisectors of lines; bisectors of angles; and of equal angles in all sorts of positions, alone, and when parts of a given figure. If possible each pupil should be seated at a table and have the use of a drawing board and T-square in addition to his compass and straight-edge. Unruled paper, preferably drawing paper, to be used.

   Supplementary activities:

   Practice of the required constructions may be carried on and symmetry learned at the same time by constructing symmetric figures.

   Practice may be secured and a future topic may be introduced here to advantage, by the construction of the three bisectors of angles, perpendicular bisectors of sides, medians, and altitudes of triangles. After this has been done all four sets of lines may be put on one triangle with a key as to the color or markings of each set. One or more pupils may make a large drawing on cardboard of all four sets of lines on an acute triangle and also on an obtuse triangle. One of these drawings on the bulletin board will very often be referred to throughout the year
5. Pupils will make a list of assumptions which are mathematical and another list which are not mathematical. They will decide which could be proved if desired. They will study the list of assumptions in the text and illustrate their use with numbers, etc.
6. Pupils will react to optical illusions shown by the teacher or seen in the text, and find others to show the class
7. Pupils will reason through the first theorem (or theorems) under direct guidance. The need for assembling all essential data before attempting the proof should be emphasized

Evidences of Mastery

1. Abliity to write a 300-word theme or floor talk on the origin of geometry. The names Egypt, Greece, Thales, and Euclid should be included

   The pupfls attacking the first few weeks of the work in geometry with interest and few pupils continuing to ask, "Why do we study geometry?"
2. Ability to define words and expressions such as acute, obtuse, right, straight,

adjacent, complementary, supplementary, and vertical angles; acute, right obtuse, equilateral, and isosceles triangles, altitude, and hypotenuse of a triangle; intersect and bisect; line segment; radius and arc; vertical and horizontal lines; vertex of an angle and of a triangle; square and rectangle; congruent triangles: words and expressions not to be defined but to be understood as evidenced by explanations requiring several sentences, or by their use, or by responses to questions about them; angle, assumption, axiom, circle, corresponding part, geometric figures, line, plane, point, solid, surface

Symbols to understand and use:

$=, \angle, \perp$

3. Ability to measure or make any angle with the protractor to within one degree of accuracy
4. Ability to make constructions in any position: bisector of a line and of an angle; at a point on a line and from a point off a line construct a perpendicular; to make one angle equal to another; to construct an isosceles and an equilateral triangle; to make an exact copy of a triangle by making respectively equal two angles and the included side, two sides and the included angle, and three sides
5. Ability to understand assumptions such as

Through two points it is possible to draw one straight line, and only one

A line segment may be produced to any desired length

The shortest path between two points is the line segment joining them

One and only one perpendicular can be drawn through a given point to a given line

The shortest distance from a point to a line is the perpendicular distance from the point to the line

Any figure may be moved from one place to another without changing its size or shape

All right angles and all straight angles are equal

If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line

All radii of the same or equal circles are equal

Two straight lines cannot intersect in more than one point

Equal angles have equal complements and equal supplements

Vertical angles are equal

If equals are added to, subtracted from, multiplied by, or divided by, equals, the results are equal

The whole is equal to the sum of all its parts and greater than any part

Like powers or like roots of equals are equal

6. Ability to go deeper than such expressions as "Any one can see that", and "I can tell it is true just by looking at the figure"
7. Ability to appreciate the exactness of formal proofs

Note: For geometry tests see bibliography

II. STRAIGHT LINE FIGURES
CONGRUENCE AND PARALLELS
A. CONGRUENCE

Unit Objective

To acquire ability to understand and use congruence and parallelism

Specific Objectives

1. Acquire the ability to
   a. Explain the word coincide
   b. Explain the words superposition and superimpose
   c. Explain the word congruent
   d. Prove the first theorem on congruent triangles by actual superposition
   e. Use first theorem as a tool to prove other triangles congruent
   f. Give further application of the theorem on congruence of triangles
   g. Prove the second theorem on congruent triangles by actual superposition and its use as a tool to prove other triangles congruent
   h. Choose the proper theorem to work out original exercises
   i. Prove the theorem: The base angles of an isosceles triangle are equal
   j. Use the theorems thus far proved in exercises such as are found in modern textbooks
   k. Prove the third theorem on congruent triangles and use it as a tool in original exercises

Teacher Procedures

1. The teacher will hold a general discussion with the class on how to determine whether two sticks are of the same length or not and if not which one is the greater (by placing them together with one end of each on the table or floor)
   Do the same with other objects, calling attention to the fact that, if of equal length, they are said to coincide. This is common practice in the comparison of things (use squares, triangles, dimes, quarters)
2. The teacher will call attention to the fact that in the above test for equality, the act of placing the objects together is called superposition as the one object is placed upon the other. The objects are then said to be superimposed
3. The teacher must make clear to the pupils the use of the words equal and congruent. Equal figures are not necessarily congruent, but congruent figures are necessarily equal. This distinction must be made clear
4. The teacher acts as a guide in the discussion of the first theorem on congruent triangles to lead the pupil to see the results desired. Corresponding parts of congruent triangles should be identified. Pupils and teacher will read the proof using the triangles, placing them together, as directed, actually doing what is usually imagined
5. Assign for discussion in class: I have a triangle here and my friend fifty miles away has one. By using the phone how can we test the congruence of these two triangles without bringing them together?
   Pupils must be able to use as a tool the proposition: If two sides of one triangle are equal respectively to the two sides of another and the included angles equal, the triangles are congruent.
   The teacher should give many exercises in applying this theroem such as

are found in the modern texts. It is more important at this time that the pupil be able to apply this theorem in original exercises than to repeat the proof word for word

6. Take up the second proposition on congruent triangles in the same manner as the first. Construction from data. Superposition in the classroom
   Results discussed
   Parts given
   Parts resulting

   Application of the theorem to simple exercises. Pupil not asked to give proof of the theorem itself
7. Applications to problems where either of the two theorems may be used, to develop power in the pupil to choose the right theorem

   A mimeographed list of simple exercises should be given to class. Few texts supply a sufficient number to give the pupils that power and confidence so necessary at this stage of the work. Many of the exercises should be in question and answer form.
8. Let the theorem about the base angles of an isosceles triangle be suggested through the construction of the triangle. Pupil is led to the proof before assigning it in the book. Also led to see why the auxiliary line is drawn to bisect the angle at the vertex. At first let it be drawn otherwise
9. A picture cord makes an isosceles triangle with a line connecting the points of suspension. A line from the picture hook bisecting the angle is the line of simple support or suspension for the picture. Is an equilateral triangle also isosceles?
10. By the use of isosceles triangles and the first theorem on congruent triangles prove the third theorem on congruent triangles. Could this theorem be proved by superposition as the first two were?

    Two right triangles are congruent if a side and hypotenuse are equal

    Two right triangles are congruent if a side and an acute angle are equal

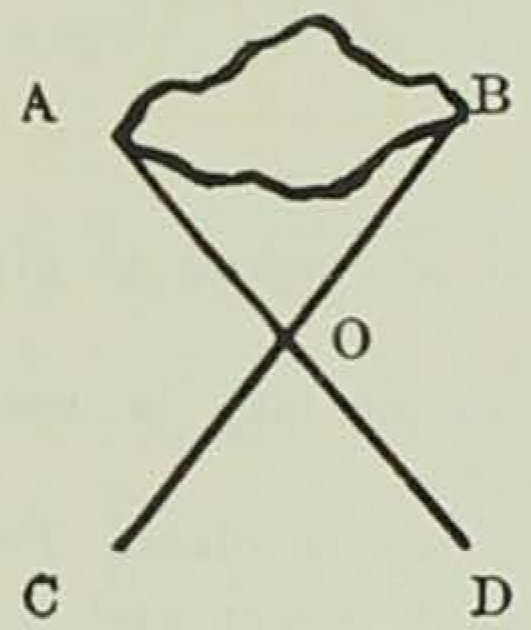
    Note 1. Proofs of the early constructions may be brought in here as simple applications and they at the same time will verify the correctness of the construction work

    Note 2. Attention should be called to the five theorems on congruent triangles and constant application made. Pupil should be drilled on the choice of five ways of proving triangles congruent by the use of exercises to be found in current textbooks

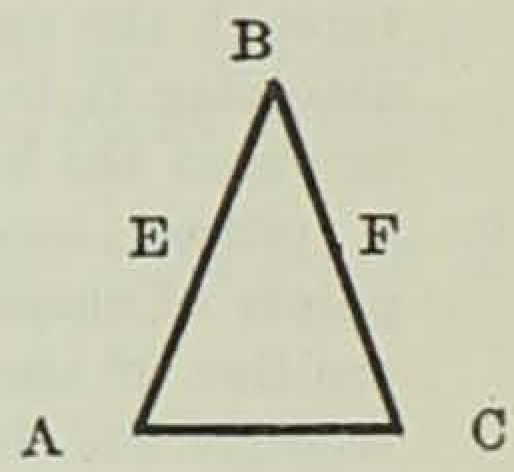
Pupil Activities

1. Pupils will find situations in which equality or inequality are determined by the principles of this unit
2. Pupils will find illustrations of superposition such as: When a design is used as a pattern it is placed upon a piece of paper and the next one cut from it. This is superposition. Even the carpenter uses this method. In cutting out rafters, he will make one very carefully and test it out using it as a pattern to cut out others
3. Pupils will cut out and test various figures for congruence by the above method
4. Pupils will construct a triangle given two sides and the included angle (AB = 3″, AC = 5″ and angle BAC = 45°). Use ruler and protractor. In class these triangles are tested by superposition and results discussed

5. Pupils will measure the sides and angles of the triangles to find if they satisfy the test for congruence
6. Pupils will solve such problems as
   To find the distance from A to B across a swamp



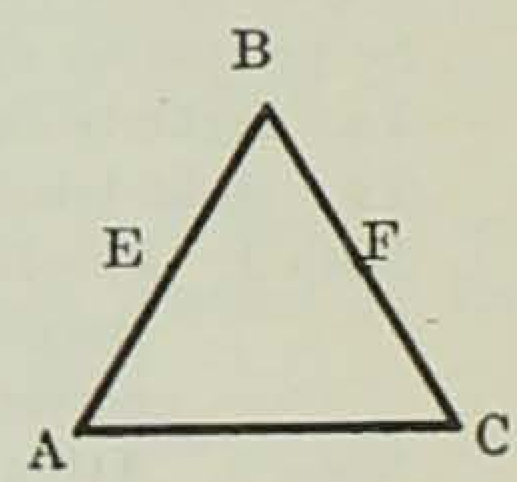
7. Pupils will present Thales' method of determining the distance of a ship at sea
8. Pupils will work such a list of exercises as
   a. In the isosceles triangle ABC, AB = CB, and AE = CF. Does BE = BF? Why?



   b. In the equilateral triangle ABC, E is the midpoint of AB and F is the midpoint of CB.

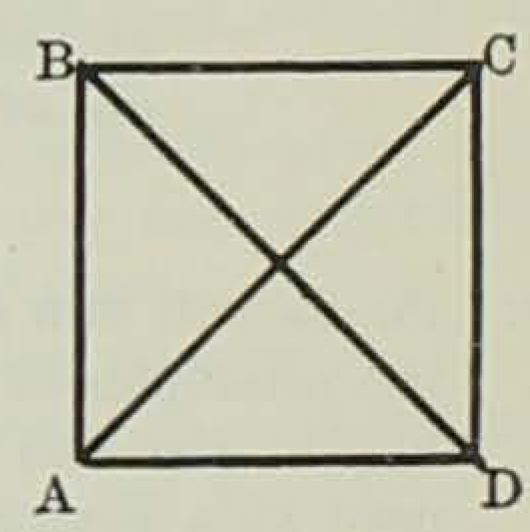
   Does AE = CF? Why?

   Does BE = BF? Why?

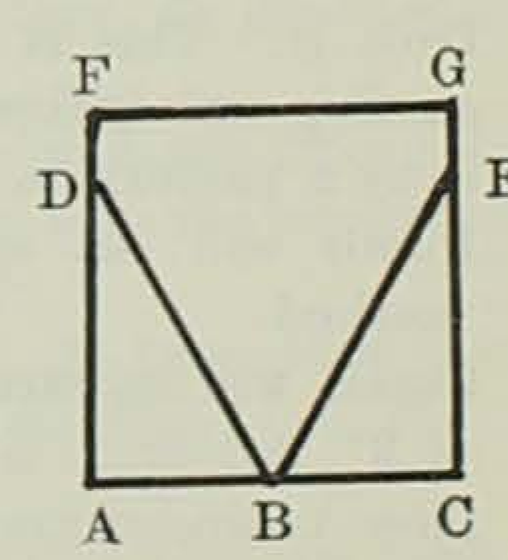


   c. In the square ABCD is triangle ABC ≅ BCD? Why?

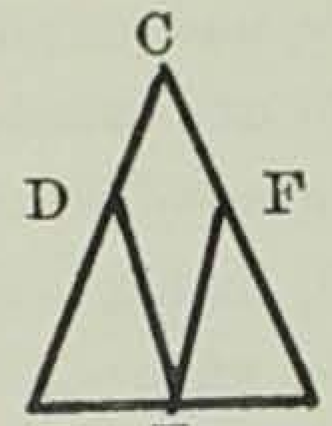
   Does BD = AC? Why?



   d. In the square AFGC, B is the midpoint of AC. BE and BD are drawn making angles ABD and CBE each = 60°. Are triangles ABD and CBE congruent? Why? Does BD = BE? Why?



e. In the triangle ACB, $\angle A = \angle B$. E is the midpoint of AB and AD = BF. Does ED = EF? Why?



f. Given a triangle with two sides equal, construct the triangle. Write a theorem about two of the angles of this triangle. Prove it.

9. Pupils will apply the first three theorems to the cases of congruence of right triangles

Evidences of Mastery

1. Ability to compare geometric figures in size, shape, and design, by the use superposition and coincidence
2. Ability to use accurately the terms equal and congruent
3. Ability to select the proper theorem by which lines and angles may be proved equal if they can be shown to be corresponding parts of congruent triangles
4. Ability to state the test for congruence, given in the second theorem.
5. Ability in handling the exercises given
6. By this time the pupil should be able to use the following elements of a formal proof: 1. Statement of the theorem; 2. Construction of the figure; 3. Hypothesis (or given); 4. To prove; 5. Proof and reasons. Pad paper with the form printed is an aid to this as the pupil is then more concerned about what to put down than how to put it down. After a week or two the pad form may be taken away and the pupil will be found to have acquired the form. Sometimes the understanding of a proof is clouded by insisting too early on the form. Most of this work should be done during the class periods so that pupils will not practice in error nor become unduly discouraged and get a wrong mental set toward the course
7. Ability to prove the usual set of theorems and problems found in the modern textbooks, such as

   The bisector of the vertex angle of an isosceles triangle is perpendicular to the base and bisects it

   An equiangular triangle is also equilateral

   If the three sides of one triangle are equal respectively to the two sides of another the triangles are congruent
8. The most important evidence of mastery is ability to recognize in a gross situation what proposition or theorem applies. This is far more important than ability to prove again propositions whose proof is given in the text. This ability must be shown by success in working new problems which demand the ability to select and apply the proper principle

B. PARALLEL LINES

Unit Objective

To acquire an appreciation of parallelism and its applications

Specific Objectives

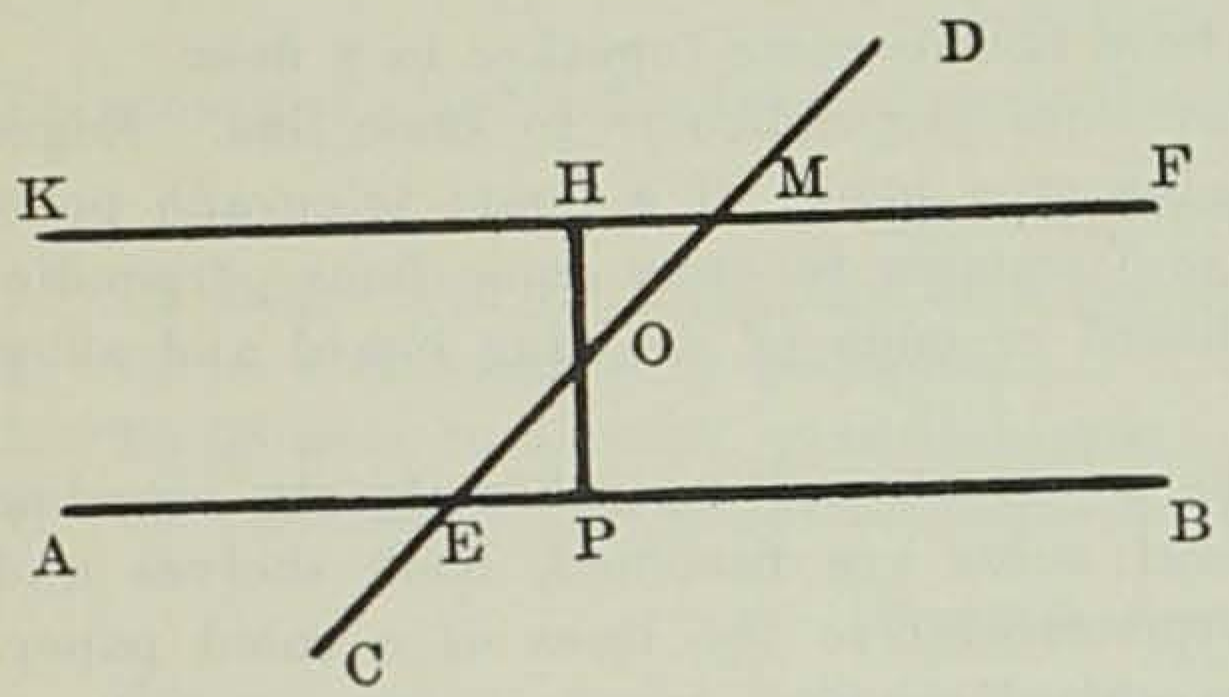
1. To acquire an understanding of parallel lines by definition

2. To understand the situations brought in by the transversal
3. To understand the parallel postulate
4. To understand other assumptions on parallel lines
5. To understand the first theorem on parallel lines
6. To understand other ways of proving lines parallel by use of the first theorem
7. To acquire the technique for constructing parallel lines
8. To see certain relations about the angles when lines are parallel
9. To understand other theorems on relations of angles about parallel lines cut by a transversal

Teacher Procedures

1. Develop and make clear the reasons why the lines have to be in the same plane
2. Present the definition of a transversal. Make clear that a transversal is a line cutting or crossing any two lines whether parallel or not. Have pupils give illustrations such as: frog in tracks, any crossing of streets, wires, cross-bars in windows, screen doors, trusses, etc.
3. Present the cutting of parallel lines by a transversal
   Cutting more than two parallel lines
   More than one transversal used
   Any figures formed in the above
4. Discuss the axiom: Through a given point one and only one line can be drawn parallel to a given line
5. Discuss the theorem: If two straight lines are parallel to a third straight line, they are parallel to each other
6. Present the relation: If two straight lines in the same plane are perpendicular to the same straight line, they cannot meet, that is they are parallel. Through illustrations and careful discussion of this theorem this truth should be made clear and then set down as a fact
7. Present the relation: If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also
8. Present the theorem: If two straight lines in the same plane are cut by a transversal, making the alternate interior angles equal, the lines are parallel. This makes a good construction problem and later is one of the methods of constructing parallel lines

   It is suggested that the teacher lead the pupil to prove this theorem by the use of the preceeding material on parallel lines and congruent triangles, thus making a more continuous development than is to be found in the majority of textboks. The following is given by way of illustration.



Draw the line AB. Through some point D not on the line AB draw the transversal DC cutting AB in E. Through M, a point on CD, construct the line KF making ∠KME = ∠BEM. Bisect ME at O and through O construct OP perpendicular to AB, extending it to cut KF at H. Prove the triangles HMO and PEO congruent. ∠OHM = ∠EPO and is a right angle. Therefore the lines KF and AB are parallel

9. Introduce analysis and train the pupil in its use. The power of analysis is a strong and effective method of investigation and its use should be encouraged. After the analysis, put the work into the usual synthetic form

10. Present the theorem: If two lines are cut by a transversal so that:
    a. A pair of corresponding angles are equal the lines are parallel
    b. A pair of interior angles on the same side of the transversal are supplementary, the lines are parallel

    Note: Applications to show parallelism should be introduced here

11. Present the theorem: If two parallel lines are cut by a transversal the alternate interior angles are equal. Lead the pupil to construct the parallel lines using the theorem: Two lines perpendicular to the same straight lines are parallel. Bisect this common perpendicular by a transversal and prove the right triangles congruent. This method avoids indirect proof and applies the preceding material on congruent triangles

12. Use analysis in developing these theorems

    Two lines are parallel if when cut by a transversal
    a. A pair of corresponding angles are equal
    b. A pair of interior angles on the same side of the transversal are supplementary

Pupil Activities

1. Pupils should bring in illustrations from their environment such as:
   The two rails of a track
   The two edges of a two by four
   The two edges of any straight board
   The corners of the room
2. Pupils will draw lines (oblique) and cut across them by a straight line or transversal, naming all the angles around the points of intersection
   Pupils to bring in illustrations
3. Pupils will name all the angles formed. Discovering relations of equality that may exist. Also some look supplementary. Test out with protractor
4. Through construction and discussion pupils will see the truth of the parallel postulate and that it means that two intersecting lines cannot both be parallel to the same line
5. By construction and discussion and perhaps the informal introduction of indirect proof (if the class is one that can understand it this early in the

work) pupils will show the truth of the statement regarding lines parallel to a third line

Illustration: telephone wires; where boards come together in a floor

6. Pupils will present illustrations of lines perpendicular to same line. Edges of straight edge; of T-square; edge of a book; of a door; telegraph poles if set in correctly. They will use T-squares to draw some lines perpendicular to a line or edge of blackboard or edge of drawing board and show what is revealed to them
7. Pupils will give such illustrations as streets cutting other streets, crossbar on a telephone pole where several wires are fastened, book shelves and edge of door frame. A line perpendicular to the lines of a ruled paper. Cross lines on graph paper. Township lines on a map
8. Pupils will find illustrations and draw them and use protractor to check them
9. Pupils will state the principle involved
10. Pupils will show how a carpenter draws parallel lines by means of his square
11. Pupils will draw parallel lines by use of T-square and by measurements with the protractor

Evidences of Mastery

1. Ability to explain and use the concept of parallel lines
2. Ability to apply a transversal to other conditions than parallel lines
3. Ability to use the basic theorems for parallel lines
4. Ability to see and use the truth in the parallel postulate
5. Ability through construction and analysis to present the first theorem on parallel lines in such a way as to give evidence that it can be used as a tool to solve original exercises
6. Ability to use theorems about angles and parallel lines, such as:
   a. The sum of the interior angles of a triangle is equal to two right angles (This should be proved in several ways)
   b. An exterior angle of a triangle is equal to the sum of the two remote interior angles of the triangle
   c. Angles with their sides parallel each to each are either equal or supplementary
   d. Angles with their sides perpendicular each to each are either equal or supplementary
7. Ability to handle new problems which involve the use of these principles

III. QUADRILATERALS

Unit Objective

To acquire an understanding of quadrilaterals

Specific Objectives

1. To understand the definition of a polygon
2. To be able to classify polygons
3. To be able to classify quadrilaterals
4. To understand the usual theorems on quadrilaterals and parallelograms

Teacher Procedures

1. Present convex and concave polygons
2. Present names, such as hexagons, etc.
3. Present the meaning of the word and the classification of quadrilaterals. Draw figures to illustrate
4. The teacher will present the following for discussion:
   Are screen doors rectangles or parallelograms? Are they always rectangles? How can they be made stable? Why is a strip sometimes nailed across the corner of a screen door?
   Are window frames rectangles, squares, or parallelograms?
   In delivering window and door frames, why is a strip nailed diagonally across the frame from top to side?
   How long is this strip left in place?
5. The teacher will present the usual theorems on quadrilaterals and their use as tools in working original exercises

Pupil Activities

1. Pupils will draw different regular and irregular figures. They will find illustrations in snow crystals; star- shaped figures
2. Pupils will draw free-hand sketches of such figures
3. Pupils will group different quadrilaterals to show the types from general to particular
4. Pupils will prove the theorems having to do with quadrilaterals
5. Pupils will use the theorems as tools to solve problems

Evidences of Mastery

1. Ability to define polygon
2. Ability to understand and use the terminology of quadrilaterals
3. Ability to draw and name all kinds of quadrilaterals
4. Ability to demonstrate the important theorems of quadrilaterals
5. Ability to use the theorem of quadrilaterals as tools of thinking in the solution of new problems
   Note. For geometry tests see bibliography

IV. INEQUALITIES

Unit Objective

To acquire ability to use inequalities

Specific Objectives

1. To understand the meaning of the symbol for inequality
2. To understand the inequality axioms
3. To understand the theorems on inequalities

Teacher Procedures

1. Present for discussion that mathematics deals mostly with equalities, *i.e.*, things that are equal, but that more things are unequal in the world about us than are equal
2. Similar to the axioms for equals there are the axioms for unequals
3. Have on hand sticks of different lengths to illustrate the possible and impossible cases of the construction of triangles
4. Present for discussion such questions as
   Why do you cut street corners and cut across the park?
   Are there any advantages to diagonal streets in a ctiy?
   Can anyone tell how Washington, D. C. is laid out?
   What definition of height or altitude makes it impossible for us to find that the height of anyone of you is ten feet?
   What do we mean by the expression "as the crow flies"?

Pupil Activities

1. The pupils will use and illustrate the following relations, being able to substitutes realities for a and b
   a. a equals b is expressed $a = b$
      a is unequal to b is expressed $a \neq b$
      a is greater than b is expressed $a > b$
      a is less than b is expressed $a < b$
   b. Equals added to equals give equals
      Equals added to unequals give unequals in the same order
      Equals substracted from equals give equals
      Equals subtracted from unequals give unequals in the same order
      Unequals subtracted from equals give unequals in the reverse order
   c. Give other illustrations of the inequality axioms
2. Pupils will work such problems and theorems as
   a. Construct triangles whose sides are 3, 5, and 7 inches; 7, 8, and 10 inches; etc.
   b. The sum of two sides of a triangle is greater than the third side; or any side of a triangle is less than the sum of the other two sides (This constitutes a condition that a triangle can be drawn if the three sides are given)
   c. If two angles of a triangle are unequal the sides opposite these angles are unequal, and the greater side is opposite the greater angle (hypotenuse of right triangle)
   d. If two sides of triangle are unequal the angles opposite these sides are unequal and the greater angle is opposite the greater side
   e. If two triangles have two sides of one equal respectively to two sides of the other and the included angles unequal the triangle that has the greater included angle has the greater third side
   f. If two triangles have two sides of one equal respectively to two sides of the other and the third sides unequal, the triangle that has the greater third side has the greater angle opposite

Evidences of Mastery

1. Ability to define and use the terminology and symbols of inequality
2. Ability to give illustrations

3. Ability to demonstrate and use the axioms of inequality
4. Ability to apply a knowledge of the fundamental relations between the sides of a triangle to inequalities
5. Ability to solve new problems which involve the use of these theorems

V. CIRCLES

Unit Objective

To acquire ability to use the properties of circles

Specific Objectives

1. To be able to construct circles
2. To be able to prove the common theorems of circles
3. To be able to use these theorems as further tools
4. To understand the simple relation of chords and arcs of a circle or of equal circles
5. To understand the relations of angles in and about a circle and their measurement
6. To have a working knowledge of the relations of circles
7. To see and apply the relationships of the circle in the every-day world
8. To use the theorems as tools in the working of miscellaneous exercises

Teacher Procedures

1. Explain the use of the compass at board; making of accurate circles
2. Discuss circles through one point; through two points; through three points; about a triangle
3. Take up concentric circles; intersecting circles; tangent circles; common chord; common tangents; line of centers; designs using circles
   Bring in illustration from direct and transverse belting
4. Explain the inscription of a hexagon in a circle; an equilateral triangle; polygon of 12, 24, 48, 96 sides, etc.
5. Have the pupils draw equal chords in same or equal circles. Have them write a theorem about what they see. Can they prove it? Suggest the converse. Have the pupils draw two chords equally distant from the center in same or equal circles. What theorem is suggested? Can they prove it? Refer to text when necessary in these proofs. Same treatment with other theorems
6. With the pupils at board or seats, teacher will direct class to draw a circle with radii making a central angle. Definition. Draw two perpendicular diameters. Make four central angles dividing circumference into four arcs. Notice if the angles are equal. Notice if the arcs are equal. Notice the size of the central angles. Notice the length in degrees of the arcs. Notice the relation between arc and corresponding central angle. Bisect these arcs and draw radii. Notice the relation between angles and arcs now. For every angular degree at the center there is an arc degree in the intercepted arc. Try same as above with inscribed hexagon and angles at the center. How are central angles measured? What is meant by the statement "a central angle is measured by its intercepted arc". What relation exists about equal central angles in the same or equal circles? No proof necessary. Draw an inscribed angle. How can it be measured? Proof. Do the same with angles formed by two intersecting chords; chord and tangent; two

tangents; tangent and secant; two secants. Try writing the truth of these theorems in one statement connected by moving the point of intersection of two chords from center of circle to non-central position; to circumference; one chord becoming a tangent; the point moving without the circle with one line a tangent and one a secant; with both lines secants; with both lines tangents. This can be illustrated by using two rulers

7. Take up the usual theorems and applications which are of interest, and of use in later chapters
8. Take up and present such examples and illustrations as:
   Circles of latitude with angles at center of earth, circles of longitude, circumference of earth, all wheels as circles—spokes as radii, circular ponds, circles in landscape gardening, circular flowers, circular windows, circular pipes in cross section, semicircular windows, arches in bridge construction

Pupil Activities

1. Pupils will draw a circle, the radius, diameter, chord, arc, tangent, and secant at board and properly designate each
2. Class at board with compass, take a point, draw a circle through it, another, another. They see how many circles can be drawn through one point. They take two points, draw a circle through them, another, another. They see how many circles can be drawn through two points. They take three points, draw a circle through them, another, another. They see how many circles can be drawn through three points
3. Pupils draw two circles having same center, another, another, Name them. Draw two circles intersecting each other. Use circles of same size and of different size to do this
4. Pupils draw two circles touching each other. They see in how many ways this can be done. They use circles of same size and of different size
5. At board or seats, pupils draw a good sized circle. With radius as a length cut off an arc and see how many times this will go around the circle. (Teacher will have to direct this) Name the polygon. Connect alternate points. Name the triangle. Bisect the original arcs, connect. Continue this as far as possible in construction
6. Pupils discuss what is happening. (Perimeter approaches circumference. Area of polygon approaches area of circle. Line from center of circle perpendicular to a side of polygon approaches radius. Polygon then approaches circles as near as we care to carry on the doubling of the sides)
7. Pupils show by drawings that equal chords are subtended by equal arcs and conversely. Equal chords are equally distant from the center and conversely. A line through the center of a circle perpendicular to a chord bisects the chord and the two subtended arcs and conversely. Tangents to a circle from an exterior point are equal. A line perpendicular to a radius at its outer extremity is tangent to the circle. Parallel lines intercept equal arcs on a circle and conversely
8. Pupils will present the theorems about central angles, such as
   Equal central angles intercept equal arcs and conversely
   Measurement of central angles; of inscribed angles;
   Angles inscribed in a semi-circle are right angles
   Angles inscribed in the same arc are equal

9. Pupils will work out these theorems. Discuss ways and means of handling the more difficult exercises through supervised study work
10. Pupils will observe and bring in examples of circles in their journeys about town, from their readings, about their school, from the shops, from the house

Evidences of Mastery

1. Ability to use these theorems in the solution of new problems
2. Ability to demonstrate and use the knowledge of the parts of a circle
3. Ability to show through construction that three points not in the same straight line determine a circle
4. Ability to show what is meant by the various terms used in the relation of two or more circles; to use this terminology
5. Ability to show the various relations of lines, triangles, and polygons in circles through construction
6. Ability to establish fundamental relations between chords, arcs, etc., in a circle
7. Ability to use angles in and about a circle with their measurements
8. Ability to show where the geometry of the circle appears in the world at large

Note: For geometry tests see bibliography

SECOND SEMESTER

VI. LOCI

Unit Objective

To acquire ability to understand and use loci

Specific Objectives

1. To understand the general meaning of the word loci
2. To understand the geometric meaning of loci
3. To work out exercises on loci to gain a mastery of the idea
4. To understand the directions for determining loci
5. To understand the usual theorems concerning loci as found in the modern texts

Teacher Procedures

1. Take up loci as place or location (plural). The teacher is a leader in this discussion
2. Take up loci as the position of all points satisfying a geometric condition, or set of conditions. Make these conditions real by questions about the location of homes with reference to school, church, post office, etc. Also discuss location of towns at a given distance from your town
3. Present illustrations by drawing a line AB and taking a point C on it. Draw several circles tangent to AB at C. What is the locus of their centers? Take two points A and B. Construct several isosceles triangles with AB as a base. What is the locus of the vertices of these triangles? etc.
4. Locate three or more points satisfying the given condition
   a. Decide what the locus is
   b. Prove that the decision is correct
5. Two things must be proved
   a. Every point on the locus must satisfy the given condition
   b. Every point that satisfies the given condition must be on the locus

Pupil Activities

1. Pupils will begin with illustrations not necessarily geometric such as telling
   a. Where are all places one block east of .... street? west of .... street? three blocks north of .... street?
   b. What postal zones are within certain distances from central position
   c. Where are all places in a room half way between two of its opposite walls? adjacent walls? ceiling and floor?
2. Pupils will discuss such questions as
   a. What is the locus of all points one inch from a given point? two inches from a given line?
   b. What is the position of all points equidistant from two given points? etc. (Apply this in space)
   c. What is the path made by the hub of an automobile wheel as the car moves down the street? of a point (or piece of mud) on a tire? on the rim? on a spoke half way between the hub and rim?

d. A point starts at the vertex of an angle and moves so that it is always equidistant from the sides of the angle. What kind of a line does it generate?
3. Pupils will use compass, ruler, and protractor in these constructions
4. Pupils will construct the figures as far as possible
5. Pupils will devise illustrations to show that these conditions are necessary
   a. Locus of all cities in Lat. 40° E.; Long. 90° W.
   b. Why does the carpenter's gauge draw a line parallel to the edge along which it is drawn?

Evidences of Mastery

1. Ability to use the word loci correctly
2. Ability to use the word as a geometric concept
3. Ability to use the concept later when it occurs
4. Ability to handle ordinary tests for loci
5. Ability to handle the geometric tests for loci
6. Ability to prove and use theorems such as
   a. The locus of a point (or all points) equidistant from two given points is the perpendicular bisector of the line joining the two points
   b. The locus of a point (or all points) equidistant from the sides of an angle is the bisector of the angle
   c. The locus of the vertex of a right angled triangle with a given fixed hypotenuse is a circle with the hypotenuse as a diameter
   d. The locus of a point at a given distance from a given point is a circle with the given point as a center and the given distance as a radius
   e. The locus of a point at a given distance from a given line is two lines, one on each side of the given line, parallel to it and at the given distance
   f. The locus of a point equally distant from two parallel lines is a line parallel to both lines half way between the two lines
7 Ability to prove and use the usual theorems on concurrent lines
   a. The bisectors of the angles of a triangle are concurrent (or pass through a single point) at a point that is equidistant from the sides of the triangle
   b. The perpendicular bisectors of the sides of a triangle are concurrent at a point that is equidistant from the vertices of the triangle
   c. The three altitudes of a triangle, or their prolongations, are concurrent
   d. The medians of a triangle are concurrent at a point that is twice as far from the vertex as from mid-point of the opposite side

VII. PROPORTION

Unit Objective

To acquire ability to understand and use proportion

Specific Objectives

1. To understand the idea of ratio
2. To grasp ratio as a fraction
3. To understand the idea of proportion
4. To know when four quantities are in proportion
5. To know how to test a proportion
6. To understand and use the usual first two theorems in proportion
7. To understand two laws of proportion

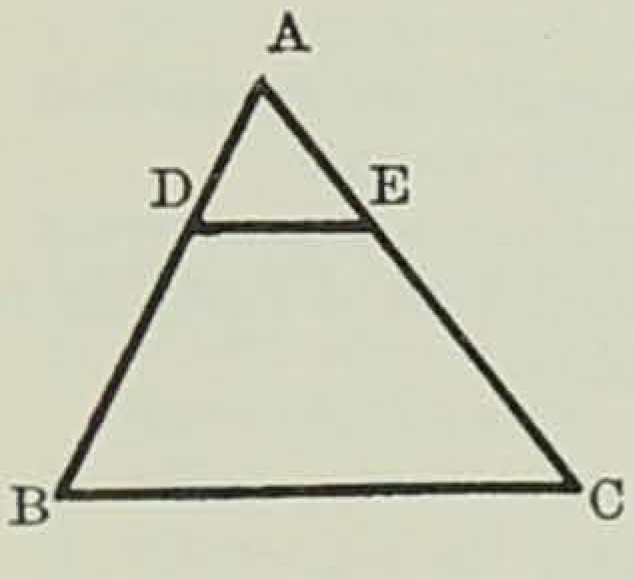
8. To understand the meaning of proportionality
9. To understand the proof of the theorem: A line drawn through two sides of a triangle parallel to the third side divides those sides proportionally
10. To know the proof of the therom in 9 for the ratio of m to n
11. To extend the above theorems. Also to bring out some of the properties of a proportion

Teacher Procedures

1. Present measurements. Comparison of measurements
2. Present ratio as a fraction and so express the operation of division
3. Present proportion as equal ratios

   If certain magnitudes can be measured so that equal ratios are found, the idea of a proportion as an equality of ratios can be developed. With the class, measure sidewalks, yards, gardens, room dimensions or magnitudes within the room. If one ratio is $\frac{2}{3}$ and another $\frac{4}{6}$ then $\frac{2}{3} = \frac{4}{6}$ and thus form an equality of ratios or a proportion. Develop by other illustrations
4. Present how four quantities form a proportion
5. Present the test for a proportion
6. Present the theorem: The product of the means equals the product of the extremes
7. Present the converse of the theorem in 6
8. By the use of the previous theorem present the theorem: A line drawn through the midpoint of one side of a triangle parallel to the base bisects the other side
9. Lead up to the theorem: A line drawn through two sides of a triangle parallel to the third side divides those sides proportionally, by construction and measurement
10. Present the proof of the theorem in 9
11. Lead up to what is usually given as a corollary

   $\frac{AB}{AD} = \frac{AC}{AE}$, etc.



12. Present the usual theorems on proportional lines

Pupil Activities

1. Using a yard stick, the pupil will find the relative length of the teacher's desk and his own desk and express the comparison
2. Pupils will reduce ratios by the same laws as fractions
3. Pupils will write a group of proportions using the following numbers: 3, 5, 6, 8, 9, 16, 10, 14, 18, 27, 36, 4, 7, 15, 20, 1, 35
4. Pupils will show that four quantities are required for a proportion

5. Pupils will test proportions to see if they are true. Is $5/6 = 11/12$ a true proportion?
6. Pupils will show the test for a true proportion
7. Pupils will prove the usual theorems and exercises in proportion

Evidences of Mastery

1. Ability to use readily the word and idea of ratio
2. Ability to show that proportion is a renaming of an old idea
3. Ability to use proportion in the solution of problems
4. Ability to test a proportion
5. Ability to use the terminology
6. Ability to use certain laws of proportion
7. Ability to use the word proportionality with meaning

VIII. SIMILARITY

Unit Objective

To acquire ability to understand and use the principle of similarity

Specific Objectives

1. To know the meaning of similar
2. To formulate a definition of similar figures
3. To acquire ability to apply the knowledge of similar figures to polygons
4. To understand the applications of the usual set of theorems
5. To understand the use of similarity of figures to prove lines proportional

Teacher Procedures

1. Develop comparisons of different figures
2. Develop the definition of similar figures
3. Take up the theorems recommended by the National Committee and a few others

Pupil Activities

1. Pupils will answer such questions as the following and give reasons
   a. Do all triangles look alike in some respects? what? all right triangles?
   b. What is the mathematical term used for "look alike"?
2. Pupils will give tests for similarity of geometric figures
3. Pupils will discuss and prove: Two triangles are similar if they are mutually equiangular. Two triangles are similar if two angles of one are equal respectively to the two angles of another. Two right triangles are similar if an acute angle of the one is equal to an acute angle of another. If two triangles have an acute angle of the one equal to an acute angle of the other and the including sides are proportional, they are similar. If two triangles have their corresponding sides proportional they are similar. If two parallel lines are cut by three or more transversals passing through a common point, the corresponding segments of the parallels are proportional
4. Application of the tool theorems to the usual sets of theorems and exercises to be found in the modern textbooks

Evidences of Mastery

1. Ability to demonstrate the idea of similarity through simple illustrations
2. Ability to apply the tests for similarity

3. Ability to use the term similarity, with the simpler theorems on similar figures
4. Ability to apply the ideas of similar figures to other things
5. Ability to prove independently the theorems on similarity
6. Ability to recognize and use the principle of these theorems in a total situation, whether a verbal exercise, a drawing, or an aspect of real experience

IX. TRIGONOMETRIC RATIOS AND THEIR APPLICATIONS

Unit Objective

To acquire ability to understand and use trigonometric ratios

Specific Objectives

1. To acquire ability
   a. To find heights by means of length of shadows
   b. To represent these relations by a geometric drawing
   c. To develop and understand the use of the tangent in measuring the relations which have been discovered
   d. To develop and understand the use of sine
   e. To develop and understand the use of the cosine

Teacher Procedures

1. The teacher will present the problem: To find the height of a tree from its shadow. He will tell them the following:

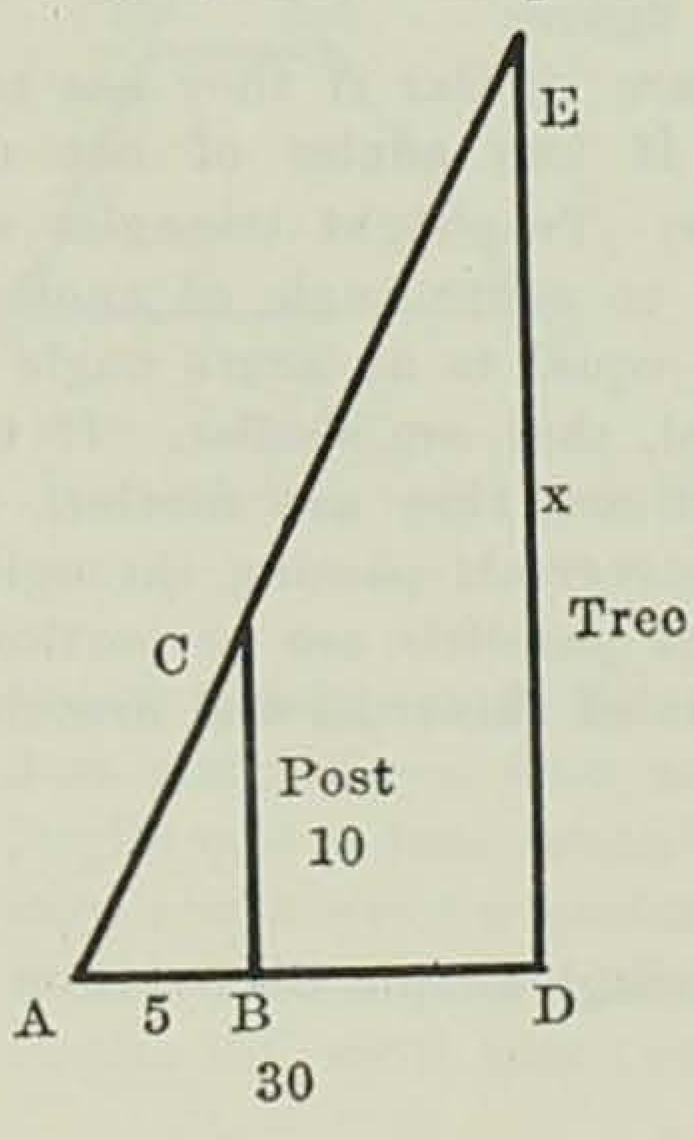
   In 500 B.C. a Greek mathematician named Thales surprised the Egyptian kings by measuring the heights of the pyramids by means of their shadows. This was a great event in its time. But any boy or girl may do as well in this age of algebra and geometry. In your arithmetics you will find problems similar to the following: How high is a tree that casts a shadow 30 feet long if at the same time a post 10 ft. high casts a shadow five feet long?

   Thales measured the height of the pyramids by measuring their shadows at the same time that he measured the shadow of a stick of a known length. But let us set up this example about the tree and post and look over the geometry of the situation for we will have occasion to use this principle in a new and useful way

2. The teacher will present this drawing and these problems: What is true geometrically? Two right triangles. Why? What kind of right triangles? (similar) Why? What relation does this give? x/30 = 10/5 or x = 60)

   If AE were extended where would it go?

   That is why this angle EAD is called the angle of elevation of the sun



3. Suppose at this time a certain smoke stack were to cast a shadow 50 feet long. How high would it be? Why? Would all objects at that particular time and place be twice as high as their shadows are long? What makes this true? (All the triangles formed by shadows and object are similar) What really makes them similar? (That the triangles are right triangles or that this angle EAD is constant? Both, but the latter will be considered) Then it is true that for this particular angle EAD the ratio of $\frac{BC}{AB}$ or $\frac{DE}{AD}$ or the ratio of the side opposite the angle to the side adjacent to the angle is always equal to 2. As the sun rises this angle will be larger and the ratio for each angle also larger. This ratio is called the tangent of the angle A, and is defined as the ratio of the side opposite the angle to the side adjacent to the angle

   The teacher may now have the pupils construct this figure to scale and measure the angle EAD with the protractor. Compare this value with that to be found in the table of tangents

   The teacher may have the pupils construct a 30°–60° right triangle and lead them to find the tangents of 30° and 60°. By the use of an isosceles right triangle the tangent of 45° may be found

   This work will make the value of tangents more real to the pupil

   Follow with exercises in heights and distances where the tangent of an an angle is to be used
4. The teacher should develop the sine and cosine of an angle and its use in a manner similar to that of the tangent. The 30°–60° right triangle and the isosceles right triangle can be used in this situation

   Follow with exercises using the new function and finally with a miscellaneous list of exercises where the pupil must choose the function necessary to solve the problems

Pupil Activities

1. Pupils will make problems involving trigonometric ratios from observations of their own
2. Pupils will work many exercises involving the use of the tangent for finding height and distance
3. Pupils will make trips and use the transit and tape to measure heights and distances of objects in the immediate neighborhood

Evidences of Mastery

1. Ability to use the methods of finding heights and distances by means of the right triangle and the use of the sine, cosine, and tangent of an acute angle
2. Ability to solve problems like the following:
   a. The brace wire of a post makes with ground an angle of 50°. It reaches the ground 10 feet from the foot of the post. Find the height of the post
   b. A 16-foot ladder is leaning against a wall. What angle does it make with the wall if the foot of the ladder is 6 feet from the base of the wall?
   c. A kite string is 80 feet long and makes an angle of 40° with the horizontal. How high is the kite above the ground? Assume that the end of the string is 3 feet above the ground and that it is kept straight

   Note: It is recommended that the slide rule be used to check trigonometric computation

X. AREAS OF POLYGONS

Unit Objective

To acquire ability to find the areas of polygons

Specific Objectives

1. To understand units of measurements
2. To understand congruency *vs.* equivalency
3. To understand the areas of simple figures

Teacher Procedures

1. Teacher will take up the development of the units

   The question of measurement goes back to early history. When man first became interested in how much ground he owned he had to devise a unit of surface measurement. Before this he was undoubtedly interested in distance and so the linear unit was in use. Various races of people had different units of length. Let us suppose they were out in the woods and wished to know how far two trees were apart. How would they do it? (Step it off, or take any convenient stick and use it as a unit of measurement?) If they desired to tell how far apart these trees are it becomes necessary to talk in terms that all will understand. So a recognized standard of measurement becomes necessary. What unit is used in our country? In France? Show the pupils that to find the length of a pole we divide by the unit of length; to find the area of a rectangle we divide by the unit of surface, etc.
2. Present the comparison of figures

Pupil Activities

1. Pupils will define a unit of length; a unit of surface; a unit of volume, and show the relation existing between these units and show in each case measurement is the result of division

   They will measure the sides of the room. Lay the yard stick end for end along the side, *i.e.*, see how many times the side of the room will contain the yard stick. Measure the area of a rectangle by laying the unit of measure (a square unit) upon the rectangle to be measured and determine how many times it will lie along a side
2. Pupils will discuss such questions as
   a. Are two congruent figures also equal in area?
   b. Are two figures that are equal in area necessarily congruent? (Examples)
3. Pupils will show the following
   a. Any two rectangles are to each other as the products of their bases and altitudes. Rectangles having equal bases are to each other as their altitudes. (Areas of the two rectangles is what is meant by this theorem) Also rectangles having equal altitudes are as their bases
4. Pupils do numerical exercises involving the theorems studied

Evidences of Mastery

1. Ability to show the meaning of "unit" and the origin of units
2. Ability to explain the idea of area
3. Ability to use the principles involved in areas of polygons

XI. REGULAR POLYGONS AND CIRCLES

Measurement of the Circle

Unit Objective

To acquire ability to understand and apply the theory of measurement of the circle

Specific Objectives

1. To know the meaning of regular polygon
2. To acquire the principles involved in circles inscribed in and circumscribed about regular polygons
3. To know how to make a comparison of the areas of similar polygons
4. To know how to find the area of a polygon
5. To know the relation between the inscribed polygon and the circumscribed circle when the number of sides of the polygon is indefinitely increased
6. To know and apply the principles involved in comparison of circles

Teacher Procedures

The teacher will develop this unit in accordance with the textbook used

Pupil Activities

1. Pupil activity for this unit will include
   a. Study of the theorems
   b. Presentation of proofs for the theorems
   c. Solution of problems in the text and others set by the teacher to illustrate the principles
   d. Construction of problems of their own

Evidences of Mastery

1. Ability to show the relation between regular polygons and their related circles
2. Ability to inscribe circles in regular polygons, and ability to circumscribe circles about regular polygons
3. Ability to compare areas of polygons
4. Ability to find the area of a polygon
5. Ability to establish a reasonable basis for the important relation between area and circumference of a circle and the radius or diameter
6. Ability to compare circles

   Note: For tests of achievement in geometry see bibliography

SOLID GEOMETRY

Objectives

The main objectives in solid geometry are: (1) to develop more fully spatial perception and the ability to picture spatial relations by means of a drawing on a plane; (2) to give further knowledge of the fundamental spatial relationships and the power to work with them; (3) to review in *actual* practice algebra and plane geometry. To assist in developing a space concept, it is desirable to have some simple apparatus such as knitting needles or pieces of small wire, some corks, pieces of cardboard and of glass to be used in making models. The pupil should be encouraged to build up models for a few propositions at the beginning of the course but he should not use these too much or too frequently. He must learn to visualize a solid from a flat outline picture and in turn to represent a solid figure by means of a plane drawing. Stereoscopic views, home made and commercial models, and colored chalk are helpful but a teacher must guard against too extensive use of them. All definitions should be introduced when needed. Constant reference should be made to corresponding theorems in plane geometry; and all formulas connected with various geometric solids should be stated algebraically; and, by applying these to the solution of many practical problems, the previous algebraic work should come to seem more real, useful, and vital.

The development of theorems which seem nearly self-evident should be minimized and emphasis should be given to problems of construction and computation.

Theorems

* On list of College Entrance Examination Board
** On list of "Fundamental Theorems" in Report of National Committee
# On list of "Subsidiary Theorems" in Report of National Committee

BOOK VI

Lines and Planes in Space

** *1. The intersection of two planes in a straight line

** *2. If a line is perpendicular to each of two lines at their intersections, it is perpendicular to their plane

** 3. All the perpendiculars to a given line at a point on the line lie in a plane perpendicular to the line at that point

** 4. Through a given point either on or external to a plane, only one line can be passed perpendicular to the plane

*5. Through a given point either on or external to a line, only one plane can be passed perpendicular to that line

\# 6. If from a point without a plane a perpendicular and oblique line-segment are drawn to the plane

a. The perpendicular is shorter than any oblique line-segment

b. Oblique line-segments cutting off equal distances from the foot of the perpendicular are equal

c. Oblique line-segments cutting off unequal distances from the foot of the perpendicular are unequal, the one cutting off the greater distance being the greater

** *7. Two lines perpendicular to the same plane are parallel

** * 8. If a plane contains one of two parallel lines, the plane and the other line are parallel

9. If one of two parallel lines is perpendicular to a plane, the other is also

** 10. If two parallel planes are cut by a third plane, their lines of intersection are parallel

** 11. Two planes perpendicular to the same line are parallel

12. A line perpendicular to one of two parallel planes is perpendicular to the other also

** *13. If two angles in different planes have their sides parallel and extending in the same direction from their vertices, the angles are equal and their planes are parallel

#14. If two lines intersect three or more parallel planes, their corresponding segments are in proportion

Dihedral Angles

15. Two dihedral angles are equal if their plane angles are equal

16. If two planes are perpendicular to each other, a line perpendicular to one of them at any point of their intersection will lie in the other

** *17. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other

** *18. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane

** *19. If two intersecting planes are each perpendicular to a third plane, their line of intersection is perpendicular to that plane

20. The locus of a point equidistant from the faces of a dihedral angle is the plane bisecting the angle

#21. Between any two skew lines there is one and only one common perpendicular

Polyhedral Angles

#22. The sum of any two face angles of a trihedral angle is greater than the third

#23. The sum of the face angles of any convex polyhedral angle is less than four right angles

BOOK VII

Prisms, Parallelepipeds, Pyramids, Cylinders, and Cones

** *24. The sections of a prism made by two parallel planes cutting all the lateral edges are congruent polygons

** *25. The lateral area A of a prism is equal to the product of a lateral edge l and the perimeter, p, of a right section that is, A = lp

** *26. An oblique prism is equivalent to a right prism having for its base a right section of the oblique prism and for its altitude a lateral edge of the oblique prism

** 27. The opposite faces of a parallelepiped are congruent and lie in parallel planes

** *28. The plane that passes through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equal triangular prisms

29. The volume of a rectangular parallelepiped is equal to the product of its three dimensions

** 30. The volume of any parallelepiped is equal to the product of its altitude and the area of its base

*31. The volume of a triangular prism is equal to the product of its altitude and the area of its base

** 32. The volume of any prism is equal to the product of its altitude and the area of its base

Pyramids

** *33. The lateral area, A, of a regular pyramid is equal to one-half the product of the slant height, s, and the perimeter, p, of the base, that is $A = \frac{1}{2} sp$

** *34. If a pyramid is cut by a plane parallel to the base
- a. The altitude and lateral edges are divided proportionally
- b. The section is a polygon similar to the base
- c. The areas of the section and of the area of the base have the same ratios as the squares of their distances from the vertex

** *35. The volume, V, of a triangular pyramid is equal to one-third of the product of its altitude, h, and the area of its base, b; that is $V = \frac{bh}{3}$

#36. The volume of the frustum of a regular pyramid of bases with areas B and B′ and altitude h is given by the formula $V = \frac{(B + B' + \sqrt{BB'})h}{3}$

#37. Two triangular pyramids having equal bases and equal altitudes are equal

Similar Solids

#38. The volumes of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other have the same ratio as the products of the edges including the equal angles

#39. The volumes of two similar polyhedrons have the same ratio as the cubes of their corresponding edges

#40. The volumes of two similar tetrahedrons have the same ratio as the cubes of their corresponding edges

41. There are only five regular polyhedrons

Cylinders

#42. The bases of a cylinder are congruent

** 43. The lateral area, A, of a circular cylinder is equal to the product of an element, e, and the perimeter, p, of a right section; that is $A = ep$

** 44. The volume, V, of a cylinder is equal to the product of the area, B, of the base and the altitude, h; that is, $V = Bh$

45. The lateral areas, or the total areas, of two similar polyhedrons have the same ratio as the squares of their corresponding dimensions. Their volumes have the same ratio as the cubes of their corresponding dimensions

Cones

*46. In a circular cone, a section made by a plane parallel to the base is a circle

** 47. The lateral area, A, of a right circular cone is equal to one-half the product of the circumference, C, or the base and the slant height, l; that is $A = \frac{lC}{2}$

** 48. The volume, V, of a circular cone in equal to one-third of the product of the altitude, h, and the area of its base, B, that is $V = \frac{Bh}{3}$

#49. The volume of the frustum of a circular cone with bases of areas B and B′ and with altitudes h is given by $V = \frac{(B + B' + \sqrt{BB'})h}{3}$

BOOK VIII

The Sphere

** 50. All points of a circle of a sphere are equi-distant from the poles of the circle

** 51. A line on a plane perpendicular to a radius of a sphere at its extremity is tangent to the sphere

** *52. One sphere and only one can be inscribed in (circumscribed about) a given tetrahedron

*53. If a plane intersects a sphere, the intersection is a circle

54. The number of degrees in a spherical angle is equal to the arc of the great circle that has the vertex of the angle for its pole and which is intercepted by the sides of the angle, produced if necessary

** *55. On any sphere, a point which is at the distance of a quadrant from each of two other points, not the extremities of a diameter, is a pole of the great circle passing through these points

*56. An angle formed by arcs of two great circles is measured by the arc of a great circle described from its vertex as a pole and included between its sides, produced if necessary

Spherical Triangles and Polygons

#57. The sum of any two sides of spherical triangle is greater than the third side

#58. Two spherical triangles on the same sphere are either congruent or symmetric if (a) two sides and the included angle of one are equal to the corresponding parts of the other; (b) two angles and the included side of one are equal to the corresponding parts of the other; (c) they are mutually equilateral; (d) they are mutually equiangular

** *59. If one spherical triangle is the polar of a second then the second is the polar of the first

** *60. In two polar triangles each angle of the one is the supplement of the opposite side of the other

#*61. The sum of the angles of a spherical triangle is more than two and less than six right angles

** 62. The area, S, of the surface of a sphere is equal to the product of the diameter, 2 r, and the circumference $2 \pi r$ of a great circle, that is $A = 4 \pi r^2$

63. If two solids lie between parallel planes and are such that the two sections made by any plane parallel to the given planes are equal in area, the solids are themselves equal in volume

** 64. The volume, V, of a sphere is equal to the product of the area of its surface, $4 \pi r^2$ and one-third of the radius, r; that is $V = \frac{4 \pi r^3}{3}$

I. LINES AND PLANES IN SPACE

BOOK VI

A. DIHEDRAL ANGLES

Unit Objective

To acquire ability to understand and use the relations between lines and planes in space

Specific Objectives

1. To acquire the ability to visualize lines and planes in relation to each other and to represent these relations by means of drawings
   Note: These outlines afford a basic study and should be used in connection with a modern textbook
2. To understand the various conditions that determine a plane
3. To understand the nature of the intersection of two planes
4. To understand the conditions that make a line perpendicular to a plane
5. To acquire the conception of the locus of points in space as treated in solid geometry. Also the idea of projection
6. To determine the locus of a point in space relative to two given points and to the circumference of a circle
7. To determine the relations of parallel lines and planes and of perpendicular lines and planes

Teacher Procedures

1. The teacher will show how these relations are involved in architecture, machinery, works of art, etc. She should assist the pupil to develop ability to visualize these lines and planes in relation to each other and to represent these relations by means of a drawing in a plane; to develop informally preliminary definitions and theorems such as
   a. Preliminary theorem: A given straight line lies on an infinite number of planes. Example: The leaves of a book. A line is fixed by two points in space. Corollary: At a given point in a given line an infinite number of perpendiculars can be drawn to that line. Example: The top or bottom edges of the leaves of a book. Note that the corresponding theorem in plane geometry is a special case of this corollary. Definition of phrase "to determine (or fix) a plane"
2. The teacher will give many simple problems involving location of points, lines, and planes, in space. She should define skew lines and illustrate them by needles and show the difference between parallel and skew lines
3. The teacher will call attention to the intersection of two planes as in corner of a room; to the intersection of more than two planes in various positions
   a. In one line
   b. In three lines

4. The teacher will develop the definition that: A line is perpendicular to a plane if it is perpendicular to every line in the plane that passes through the point at which it meets the plane. That the plane is also perpendicular to the line

   Note: Here use is made of the important idea that in order to prove that a line is perpendicular to *every* line on the plane passing through its foot, it is necessary to prove that it is perpendicular to *any* line on the plane that passes through its foot. Pupils' attention should be called to the various facts that are illustrated in this theorem
5. The teacher will present the theorem: All the perpendiculars to a given line at a point on the line lie in a plane perpendicular to the line at that point. She will cite illustrations such as
   a. Swinging a sling rapidly around an arm as an axis
   b. Spoke of a wheel
   c. Contrasts with the theorem in plane geometry: At a point on a line only one line can be drawn to the given line, which is a special case of theorem 3.

   She will develop the concept of an angle that a line makes with a plane. The close relationship between theorems, 3, 4, 5, should be emphasized

   Discussion of the projection lantern and movie machine will fit in at this point
6. The teacher will develop theorems 6 and corollaries: The locus of a point equally distant from two given points is the plane perpendicular to the line—segment joining the two given points at its midpoint. The locus of a point equally distant from all points on a circle is a line perpendicular to the plane of the circle at its center. In connection with this work, the idea of the projection of a point upon a plane and of a line upon a plane should be developed
7. The teacher will take up
   a. Discussion of lines perpendicular to the same plane by use of needles and cork pad
   b. Planes perpendicular to the same plane
   c. Planes perpendicular to the same line

Pupil Activities

1. The pupil will locate three points, A, B, C, in space by sticking three knitting needles of different lengths into a cork pad and then laying a glass or cardboard plane on these points to illustrate the theorem: A plane is determined by three points that are not in the same straight line. He will investigate the case when the three points are in a straight line, and compare with the preliminary theorem. He will take up the following questions:
   a. Why is a tripod used to support a camera, or a surveyor's transit?
   b. Why do the four legs of a chair sometimes not all rest on the floor?
   c. How many planes are determined by four points not all lying in the same plane or in the same straight line?
2. The pupil will investigate in a similar way whether a plane is determined by a line and a point without the line; by two intersecting straight lines; by two parallel lines and state the theorems that result from the investigation. He will discuss the following questions

a. Why does a mason use a trowel with long straight edges when "truing up" a wall?
b. Are lines in space that make equal angles with a given line always parallel to one another?
c. How does a carpenter determine whether a floor is level?
d. Must a triangle be in a plane?
e. Must a parallelogram be in a plane? Must any quadrilateral?

3. The pupil will prove the theorem: The intersection of two planes is a straight line. He will discuss the following
   a. Why does folding a piece of paper give a straight line?
   b. What is the locus of all points common in two straight lines?
4. The pupil will prove the theorem: If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of these lines. It is suggested that the pupil
   a. Make an adequate model or figure which shows the given facts clearly
   b. Analyze what procedure is necessary in the light of the above definition
   c. Add to the model already made all the lines and planes that must be added in the light of this analysis
   d. Make a drawing of this model. This leads naturally to the question as to what conditions are sufficient for a line to be perpendicular to a plane and hence to the proof for Theorem 2.
   e. Show how a perpendicular to a plane can be determined by two carpenters' squares
   f. Whether the hands of clock revolve in a plane or not and why
   g. How a carpenter may set up a piece of timber perpendicular in the floor of a building. How many braces does he need to keep the piece in place?
5. The pupil will discuss and prove the corollaries given in teacher procedures and the following
   a. What is the locus of a line that passes through a given point, and is parallel to a given plane?
   b. What is the locus of all points equidistant from the vertices of a triangle?
   c. What is the length of the locus of a point that is 5 inches from each of two given points that are 8 inches apart?
   d. Where are all the points on a given plane that are equally distant from 2 given points in space?
6. The pupil will prove Theorem 6. Here the pupil should develop ability to select the theorems from plane geometry that are of use and to see that the facts of plane geometry are sufficient for proving the new theorem in solid geometry. The converse of each part of this theorem should be stated and validity investigated. Solve such problems as
   a. Some boys build a shack in the wooks. They use for its frame six 10-foot poles which are so placed as to meet at a common vertex and rest on the ground 6 feet apart. What is the height of the shack and the area of the floor?
   b. How does the length of the projection of a line upon a plane compare with the length of the line when (a) the line is parallel to the plane, (b) the line is perpendicular to the plane, (c) neither parallel nor perpendicular to the plane
   c. Find the projection of a line 16 inches long upon a plane, if the angle it makes with the plane is 45°, 60°, 30°

d. Show how a line 6 inches long and one 10 inches long may have equal projections on a given plane

7. The pupil will prove Theorem 7. In considering Theorem 7 he will pass a plane through the parallel lines and compare with the theorems of plane geometry. Then prove Theorem 10, and work such exercises as: If one saws diagonally through a board the cross section is a parallelogram

Evidences of Mastery

1. Ability to recognize the difference between two dimensional and three dimensional objects
2. Ability to draw solids such as table top and chalk box according to perspective
3. Ability to explain and apply ways of determining a plane
   a. Three points not in the same line
   b. Two intersecting lines
   c. Two parellel lines
4. Ability to distinguish between parallel and skew lines
5. Ability to prove: The intersection of two planes is a straight line; The intersection of more than two planes may be a line or several lines
6. Ability to visualize planes intersecting at various angles. To put this visualization into a drawing in two dimensions. To apply the theorems of plane geometry as needed. To construct a perpendicular to a given plane through a given point
7. Ability to apply this theorem to practical problems such as locating the guy wires for a stack. Where should these wires be fastened? Definition of (1) distance from a point to a plane, (2) locus of a point satisfying a given condition, same as in plane geometry
8. Ability to determine the locus of points satisfying various conditions. For example, five feet from one of the walls of the room, ten inches from the tip of a light bulb, four feet from the curb in front of the building, equally distant from two given points in the room
9. Ability to understand the relations of perpendicular and oblique lines drawn from a point to a plane
10. Ability to recognize this principle in objects around us. For example, apposite corners of a room are coplanar
11. Ability to answer such questions as:
    a. What are the relations of the following to each other?
       lines parallel to the same line
       planes parallel to the same plane
       lines parallel to the same plane
       planes parallel to the same line
12. Ability to answer: What are the relations of the following to each other?
    lines perpendicular to the same line
    lines perpendicular to the same plane
    planes perpendicular to the same line
    planes perpendicular to the same plane

B. POLYHEDRAL ANGLES

Unit Objective

To acquire ability to understand the definitions of dihedral, trihedral, and polyhedral angles, their uses and relations to space conceptions

Specific Objectives

1. To be able to understand the definition of a dihedral angle; the definition of a trihedral angle; the definition of a polyhedral angle; and the plane angle of a dihedral angle
2. To be able to prove the theorems involving dihedral angles
3. To be able to handle the simple theorems on trihedral and polyhedral angles

Teacher Procedures

1. The teacher will discuss derivation of words, "di" meaning two, "tri" meaning three, "hedral" meaning face. Discuss way of measuring a dihedral angle and cite many simple illustrations of dihedral angles. Show how this can be brought under a theorem in plane geometry
2. The teacher will take up the theorem on dihedral angles and lead the pupil to get these concepts fixed firmly in his mind before going on in the general polyhedral angle
3. Using a pyramid, compare the sum of the plane angles of the triangles in the base with the sum of the plane angles in the sides of the pyramid, apply Theorem 22 to each trihedral at the base
   Note: For solid geometry tests see biblography

Pupil Activities

1. The pupil will pick out illustrations of dihedral, trihedral, and polyhedral angles in the classroom; will prove Theorem 15; will make models; will prove Theorems 16, 17, 18, 19, 20, 21. He should make a model or careful drawing and give the proof for Theorem 20
2. The pupil will make a model, draw a figure, and prove Theorem 22
3. The pupil will prove Theorem 23 and take up the following questions
   How many equilateral triangles could be placed together at one vertex, so to make a polyhedral angle? How many squares, regular pentagons, hexagons, etc.?

Evidences of Mastery

1. Ability to recognize each type of polyhedral angles and to define the plane angle of a dihedral angle
2. Ability to correlate the bisection of a plane angle with the bisecting plane of a dihedral
3. Ability to determine upper and lower limits for number of degrees in each face angle of a trihedral
4. Ability to prove that there are only five regular polygons, and to recognize and name each. Also to prove Theorem 41

II. PRISMS, PARALLELEPIPEDS, PYRAMIDS, CYLINDERS, CONES

BOOK VII

A. PRISMS AND CYLINDERS

Unit Objective

To acquire ability to understand and use some of the solids most commonly observed in nature and very frequently used in architecture, engineering, and the arts

Specific Objectives

1. To acquire an understanding of the prism and the cylinder
2. To acquire an understanding of the relation of the parts of a prism and of a cylinder
3. To understand the nature of parallel sections of a prism and of a cylinder
4. To be able to find the value of the lateral area of a prism
5. To be able to find the relation between a right and an oblique prism
6. To be able to find the lateral area of a circular cylinder

Teacher Procedures

1. The teacher will discuss informally the following facts:
   a. Lateral edges of a prism or the elements of a cylinder are equal
   b. The lateral faces of a prism are parallelograms
   c. A right section of a prism, or cylinder is perpendicular to all the lateral edges or elements
   d. The section of a prism made by a plane parallel to a lateral edge is a parallelogram
   e. A section of a cylinder made by a plane containing two elements is a parallelogram
2. The teacher will develop the proof for theorem 24: Show that every section of a prism parallel to the bases is congruent to them, and develop corresponding theorem for cylinder, Theorem 42
3. The teacher will discuss the shape of the face of any prism; how the area of this plane figure is determined; how the area of all the faces can be determined; have the proof of Theorem 25 given
4. The teacher will explain how you can make a right prism, cut it up, and rearrange the two pieces in the form of an oblique prism. He will have the pupils prove Theorem 26, and compare with the theorem in plane geometry: Any parallelogram is equivalent to a rectangle that has its base and altitude equal respectively to the base and altitude of the parallelogram
5. In taking up the lateral area it is suggested that the teacher consider it as that of the rectangle formed by straightening out the cylindrical surface. Make drawings to illustrate

Pupil Activities

1. The pupil will follow the definitions and generate in turn a plane surface, a prismatic surface, a cylindrical surface; and discuss the essential differences in each

2. The pupil will give examples of a right prism, oblique prism, and regular prism. When is a prism called a parallelepiped?
3. The pupil will make a model or good drawing to show parallel sections; will investigate the relation between the lateral edge and altitude of right prism; will show that the lateral faces of a right prism are rectangles and the corresponding relations for cylinder
4. The pupil will make a good drawing and analyze the method of finding the area of each face
5. The pupil will make a careful drawing of the figures, used in work explained in teacher procedures
6. The pupil will discuss the idea of inscribed and circumscribed prism and discuss relations between such regular prisms and the cylinder as the number of sides is increased; will develop informally such theorems as: If prisms whose bases are regular polygons are inscribed in and circumscribed about a circular cylinder and if the number of faces of the prisms is indefinitely doubled (1) The perimeter of a right section of the cylinder is the common limit of right sections of the prisms, (2) The lateral area of the cylinder is the common limit of the lateral area of the prisms, (3) The volume of the cylinder is the common limit of volumes of the prism. Prove Theorem 43, 44

Evidences of Mastery

1. Ability to define plane surface, cylindrical surface, and prismatic surface, section of a closed surface
2. Ability to pick out prisms from any group of polyhedrons, and to classify them as to type and number of sides of the bases
3. Ability to prove Theorem 24; to define right section; right prism; regular prism; and altitude of prism. Ability to prove
   a. The bases of prisms are congruent
   b. Every lateral edge of a prism is parallel to the plane determined by any other two lateral edges
   c. What is the locus of all points equidistant from a straight line?
4. Ability to state and prove theorem concerning lateral area of a right prism. To solve such exercises as: Find the lateral area of a regular hexagonal prism each side of whose base is 4 inches and whose altitude is 16 inches. Also find the total area
5. Ability to distinguish between terms equal, congruent, and equivalent as applied to solids; to prove: In a truncated prism having a parallelogram for base, the sum of two opposite lateral edges is equal to the sum of the other two opposite lateral edges
6. Ability (1) to develop formula for lateral area of right circular cylinder and to solve such problems as:
   a. Find the lateral area and the total area of a right circular cylinder having altitude 10 and radius of base 8.
   b. A steam boiler has a diameter of 72 inches, is 18 feet long, and contains 70 tubes each having a diameter of 4 inches, extending length-wise of the boiler. Find the heating surface of the boiler, using "one-half" in

the following rule. In finding the heating surface for a horizontal boiler it is customary to take one-half to two-thirds the lateral area of the shell,, the lateral area of the tubes, one-half to two-thirds the area of the ends of the boiler, and subtract the area of both ends of the tubes

B. PARALLELEPIPEDS

Unit Objective

To acquire the ability to explain, use, and illustrate the principles involved in parallelepipeds

Specific Objectives

1. To be able to show that parallelepipeds are special forms of prisms
2. To be able to find the volume of a prism
3. To be able to find the volume of a circular cylinder
4. To be able to determine the relation of similar polyhedrons to each other

Teacher Procedures

1. The teacher will have the pupils prove Theorems 27 and 29; and consider Theorem 29 as a special case of Theorem 30
2. The teacher will have the pupils prove Theorems 28, 31, 32. Theorem 32 is one of the basal theorems of solid geometry and has many applications. This theorem was known to the ancients, as one finds in the Ahmes papyrus directions for measuring bins
3. The teacher will develop the formula for the volume of a cylinder and give examples for the pupil to work
4. The teacher will develop the proof of Theorems 38, 39, and 40

Pupil Activities

1. The pupil will discuss the definitions of
   a. right parallelepiped
   b. oblique parallelepiped
   c. rectangular parallelepiped
      1) cube
2. The pupil will investigate the relation between prisms and parallelepipeds
3. The pupil will make applications to the computation of the volumes of bins, silos, etc.
4. The pupils will recall the idea of similarity and obtain the definition for similar polyhedrons

Evidences of Mastery

1. Ability to solve such problems as:
   a. A box car that is 36 $\frac{2}{3}$ feet long and 8 feet wide, inside measurements, can be filled with wheat to a height of $4\frac{1}{2}$ feet. How many bushels of wheat will it hold if there are 5/4 cubic feet in a bushel?

b. If the length of one of the diagonals of a cube is known, can the edge be found? the area? the volume?
c. Find the diagonal of a cube whose volume is 512 cubic inches

2. Ability to solve problems like:
   a. A flow of 300 gallons per second will supply water for a stream of what depth, if the stream is 4 feet wide and flows 5 miles an hour?
   b. An iron casting shrinks 1/8 inch per linear foot in cooling down to 70 degrees Fahrenheit. How many cubic inches is the shrinkage per cubic foot?
   c. A rectangular solid with a square base has a volume of 80 cubic feet and a surface of 112 square feet. Find its dimensions
3. Ability to solve such problems as
   a. Find the height of a 10 gallon wash boiler whose base is 10 inches wide with semi-circular ends, the length of the straight part of the sides being 9¼ inches
   b. The outer diameter of an iron pipe is one inch and the inner is ¾ inch. What will ten feet of the pipe weigh if the specific gravity of the iron is 7.2?
   c. What is the locus of all points at a given distance from a cylindrical surface whose right section is a circle?
   d. Why is there such a close relation between the theorems concerning prisms and cylinders?
4. Ability to solve such problems as
   a. Find the ratios of the volumes, the lateral areas and the total areas of two similar right prisms having hexagonal bases two of whose corresponding edges are 2 and 5 inches respectively
   b. Two similar right circular cylinders have bases whose areas are 20.25 square inches and 100 square inches respectively. Find the ratio of their volumes
   c. What effect does it have upon the volume of a prism or cylinder if the base is doubled? If the altitude is doubled? If both base and altitude are doubled?
   d. The volume of an irregular shaped body may be found by immersing it in water and determining the amount of water displaced. A cylindrical vessel that has a diameter of 4 inches is partly filled with water. A stone immersed in the water raises its level 3½ inches. What is the volume of the stone?

C. PYRAMIDS AND CONES

Specific Objectives

1. To be able to clearly define pyramidal surface, conical surface, pyramid, cone, regular pyramid, right circular cone or cone of revolution
2. To be able to find the lateral area of a regular pyramid and a right circular cone
3. To be able to find the volume of a triangular pyramid and of the frustum of a regular pyramid
4. To be able to find the volume of circular cone and of a frustum of a circular cone

Teacher Procedures

1. The teacher will have the pupils prove informally
   a. The lateral faces of a regular pyramid are congruent isosceles triangles
   b. The lateral edges of a regular pyramid are equal
   c. The elements of a right circular cone are equal
   d. The section of a circular cone made by a plane containing an element is a triangle
   e. The section of a pyramid made by a plane through its vertex is a triangle
   f. Theorem 46
2. The teacher will develop the proofs for Theorems 33 and 47
3. The teacher will call attention to the fact that foundations of heavy machinery are usually frustums of pyramids
4. The teacher will review the relation of a cone to the inscribed and circumscribed regular pyramids and prove Theorems 48 and 49
   Note: For solid geometry tests see bibliography

Pupil Activities

1. The pupil will contrast these surfaces with those developed by a straight line which passes through a fixed point and always intersects a given straight line. Prove Theorem 34
2. The pupil will investigate the relations between the volumes of two triangular pyramids having equivalent bases and equal altitudes. Prove Theorem 37. Construct a model for Theorem 35 and give proof. Also prove Theorem 36. Read about pyramids in Egypt and work problems such as
   a. The pyramid of Cheops has a square base 720 feet on a side and an altitude of 480 feet. Find the number of cubic yards in it
   b. Hard coal dumped in a pile lies at an angle of 30° with the horizontal. Estimate the number of tons in a pile in the shape of a right circular cone having an altitude of ten feet. Large egg size weighs 38 pounds per cubic foot
   c. The diameter of the top of a water pail is 12 inches, the diameter of the bottom 10 inches and the altitude 10½ inches. How many quarts will the pail hold? One quart is 57.75 cubic inches
   d. A glass is in the shape of a frustum of a cone, the radii of the bases being 1½ inches and 2 inches. What must be the depth of the glass in order that it may hold 15 cubic inches?

Evidences of Mastery

1. Ability to state and prove the fundamental theorems for a cone, and pyramids
2. Ability to derive and apply the formula for the lateral area of a regular pyramid and of a right circular cone, and work problems such as
   a. A farmer has a cylindrical shaped silo, diameter 16 feet and height 28 feet. It has a roof cap with a slant height of 10 feet. If a gallon of paint covers 300 cubic feet, how many gallons will be required to paint the outside of the silo including the roof?

3. Ability to see clearly that any triangular prism may be divided into three equal triangular pyramids. Ability to derive and apply the formula for the volume of a pyramid

III. THE SPHERE

Unit Objective

To acquire ability to understand and use the fundamental properties of a sphere

Specific Objectives

1. To be able to have a knowledge of spherical angles and ways of measuring them
2. To be able to have a knowledge of spherical triangles and polygons and the the relations of their various parts to the parts of the polyhedral angle formed at the center of the sphere
3. To be able to have a knowledge of polar triangles
4. To be able to understand and use congruent and symmetrical spherical triangles
5. To be able to find the formula for the area of a sphere
6. To be able to find the volume of a sphere

Teacher Procedures

1. The teacher will explain the definition of a sphere, and show how a sphere may be generated by revolving a circle about a diameter. He will develop the difference between great and small circles and the way of measuring distance between two points on a sphere; develop, from the definition, ways of determining a sphere; extend the idea of tangent line to that of tangent plane; develop the relation between a spherical angle and the plane angle that measures the dihedral angle formed by the great circles that form the spherical angle
2. The teacher will emphasize the fact that the sides of a spherical polygon are arcs of great circles in contrast to sides of a plane triangle
3. The teacher will give the definition of a polar triangle and of the spherical excess
4. The teacher will explain the difference between congruent and symmetric spherical triangles and develop the conditions for symmetry and congruency
5. The teacher will discuss various solids generated by revolving a line about another as an axis

Pupil Activities

1. The pupil will prove Theorem 53 and give some examples of great and small circles; give some examples of great and small circles on the earth; prove Theorems 50 and 55; prove Theorem 52
2. The pupil will determine how the sides of a spherical polygon are measured by the face angles of the polyhedral angle at the center of the sphere, and thus lead to proof of Theorems 57 and 61. Prove Theorem 58

3. The pupil will construct on a spherical blackboard or globe a triangle and its polar; measure the distance of each vertex from the opposite side of the other triangle, and prove Theorems 59 and 60; contrast the number of right angles that a spherical triangle may have with the number in a plane triangle
4. The pupil will compare these conditions with the corresponding ones for plane triangles, and review Theorem 58
5. The pupil will use this point of view and derive formulas for the area of a cylinder in terms of the projection on the axis and the perpendicular erected at the midpoint of the line. Similarly derive proof for Proposition 62
6. The pupil will prove Theorems 63 and 64, and work the problem: to double the volume of a sphere by what per cent must the radius be increased?

Evidences of Mastery

1. Ability to show knowledge of the definition of a sphere, great circle, small circle, poles of a circle, polar distance, tangent lines and planes and ways in which a sphere may be determined. Ability to solve such problems as:
   a. What parallels of latitude are each equal to half the equator?
2. Ability to recognize spherical angles and to find the number of degrees in them
3. Ability to define and construct polar triangles and to solve simple exercises that call for application of theorems such as:
   a. The sides of a spherical triangle are arcs of 70, 85, and 90 degrees. Find the angles of the polar triangle
   b. The angles of a spherical triangle are 77, 93, and 107 degrees. Find the sides of the polar triangle
   c. What spherical triangle is its own polar?
4. Ability to derive formula for area of a sphere and to apply it to problems in computation, such as:
   a. The ratio of the radii of two spheres is 1:2. If the area of the smaller sphere is 20π, find the area of the larger. If the area of the larger is 20π, what is the area of the smaller?
   b. Find the ratio of the area of the surface of the moon to that of the earth assuming the diameter of the moon is 2,162 miles
5. Ability to apply the formula for volume in computation, such as A cone has a radius equal to the radius of a sphere and an altitude equal to the diameter of the sphere. What is the ratio of their volumes?
   Note: For solid geometry tests see bibliography

STUDY HABITS IN MATHEMATICS

In high school mathematics a most important part of the teacher's work is to assist the pupil in forming correct study habits. Experimentation with pupils will show that most of their difficulties are due to wrong study habits. There is too much assigning of lessons in the simple faith that in some mysterious way the work assigned will enter into the intellectual life of the pupil and inspire him to greater efforts. Until very recently slight attention has been paid to the important matter of teaching pupils to study.

Following are some study helps which should be kept before pupils:

1. Have proper study conditions—a comfortable room properly lighted and adequately furnished

2. Provide necessary materials such as pencils, rulers, protractors, compasses, and paper. The necessary equipment is neither elaborate nor expensive

3. Note assignments carefully before beginning to study. If in doubt about any point consult your teacher

4. Work alone as a general policy. There is a tendency to waste time in group study. Learn to think independently. You will be compelled to settle your own individual difficulties in later life

5. Plan your study time carefully. Do not lose any time in getting to work. Work steadily after you have started and stop when the task is finished

6. Use the textbook as a guide and helper. Do not make it your master to be followed blindly

7. Go over the assignment for the first time for a general impression of its requirements. A second reading will go into details as to number relations and possible mathematical expression of such relations. Later readings will complete your preparation

8. Be interested in your work. Enthusiastic interest is a powerful solvent of difficulties in mathematics or any other subject. Discuss interesting items with other members of the class or with your teacher. Enter into your study without knocking and go out the same way

9. Be on the lookout for practical applications of the subject matter which you are studying

10. Keep your work up to date. An efficient workman does not let his work pile up ahead of him

11. If after continued effort you find some apparently insurmountable difficulties, lay aside the work for a time, or do some reviewing. Talk over the difficulty with your teacher. There is always a way out

12. Get the habit of wider vision. Note the relationships of algebra and geometry to the whole field of mathematics and of mathematics to the whole field of human knowledge

(See *The Teaching of Mathematics in Secondary Schools*, Breslich, Chapter IV, for an excellent discussion of this topic with references)

DEBATE RECITATIONS IN GEOMETRY

The debate recitation has been used a number of years and has proved much worthwhile because it is not just another device, but a method which is corrective of the teacher-pupil dialogue recitation which has been so outstanding a fault of geometry teaching.

Early in the year the class is divided into two sections, equal in number and as nearly equal in ability as can be ascertained. The two sections, or "sides" should be seated separately, as in the south side of the room and the north side. This plan should not necessarily change the original seating, except that one or two good pupils from one side may exchange seats with poorer ones on the opposite side in order to equalize the sections. This arrangement lasts throughout the year. The debate recitations are held once or twice a week. On other days there is the usual type of recitation or a laboratory period.

The names of the pupils in each group are written on a pack of cards. The teacher shuffles the cards and draws a card from one pack and then from the other until she has selected the two teams of two or three pupils to whom she assigns four or six theorems or problems. She also draws an extra name from each pack and these pupils serve as critics for their respective sides.

Each side has a permanent section at the board where they put their work. At the first of the year the teams drawn write out the proofs completely and later in the year they usually put on the board only the theorem, the figure, the given, and what is to be proved; the proof is given orally. It counts against an individual to receive help from the critic but if the critic helps him to get a problem he could not get alone, of course the critic is helping his side to win. A definite time limit is given to this work at the board. Meanwhile the rest of the class may be working at something else or they may be watching the work at the board in order to be ready with criticisms of the work of the opposing side, or be ready to help for their own side.

When the time alloted to the work at the board is up, first a member of one team and then of the other, gives his proof, if the proofs have not been written out at the board. The opposing side then criticizes. If there is anything left for the side whose representative gave the proof to criticize, or for the teacher to criticize, it counts against the side whose representative gave the proof.

The teacher is the judge of whether a proof is won or lost. A *W* or an *L* is put on the board beside a problem after it is finished in order to help keep track of the score. Sometimes the teacher finds it necessary to count a problem as neutral or *O* when it was only partly right and the opposing side corrects it. If the one assigned the problem does not give a correct proof, then no one on his side has a chance to do it until every one on the opposing side who wishes has tried it. This regulation is made so that the burden of winning the debate will be on the team and not primarily on their side as a whole.

The teacher is the judge of the contest, deciding which side won or in some cases pronouncing the decision as a tie. She also decides who won individual honors. The ones on the team have a little better chance to win this honor, yet it often goes to someone not on the team, who was a very good critic. An *S* (south) or an *N* (north) in the grade book will keep the record of which side won and an *I* will keep the record of who received the individual honors. The number of individual honors received may be the deciding factor as to whether a pupil is worth an A grade. Sometimes it is announced that no one has won individual honors. That serves as a tonic to the good pupils. There is a chance for all to participate in the criticisms. The poorest pupil can criticize the form in which the work is written up, or the omissions of reasons in an oral proof, etc.

The poorer pupils who volunteer criticisms should, in general, be called on first in order to encourage all to participate.

When the pupils have learned how to go ahead with a debate recitation, they will even remind the teacher that she is taking away their opportunity to criticize, if she speaks up before they have had their chance. The teacher has a good spur to use. She can say to the opposing side that there is an important criticism which has not been given and for which she will wait a few seconds.

Added zest is given to the debate recitations if the pupils decide that the losing side will give the winners a picnic just before the end of the year. If there are several sections, the losing sides of each may combine to entertain the winners of all sections.

Of course the details of how the debate recitation is carried out are not essential. They are given as suggestions to show how the pupils may be induced to carry the load in an oral recitation.

MATHEMATICS CLUBS

Purpose

1. To bring together teachers and pupils who are interested in mathematics
2. To afford an opportunity for discussion of a wider range of inspiring supplementary material than can be covered in the classroom

Organization

The organization should be as simple as is consistent with effective work. An enthusiastic faculty adviser is indispensable.

Programs may include

1. Historical and biographical topics

   The lives of noted mathematicians may be studied. The history of arithmetic, algebra, or geometry may include the development of present symbols and the reasons for introduction of new topics and exclusion of others. Various number systems other than our present one based on the number ten are not only interesting but are actually enjoyed by the pupils computing in them.

   Studies of the mathematics of the American Indian, and other primitive peoples and of various countries, including both the past and present methods of computing and their system of mensuration, are enlightening. A pupil from almost any foreign county can be useful in this regard.

   A program on Egyptian mathematics can be made particularly interesting by including the early surveying with the help of rope stretchers (pupils should actually use the knotted ropes). A poster with a pyramid on it and early characters for various numbers may announce the meeting. A report may be given about the pyramids and a contest staged between two sections of the club, or individual members, on information about these pyramids.

   Napier's Rods, the history of pi and magic squares furnish material for other programs. Pupils enjoy not only the history of the magic squares, but like to learn how to fill the squares.

2. Famous problems

   There are many famous problems which never grow old such as the trisection of an angle, various proofs of the theorem of Pythagoras, squaring the circle, the golden section, duplication of the cube, and the nine point circle. Considering motion possible in a one, two, or four dimensional world is stimulating. Very simple discussions of the fourth dimension are available.

3. Fallacies

Proving that 1 = 2 and similar fallacies, and optical illusions, including such geometric puzzles as changing a square into a rectangle of unequal area, illustrate the many possibilities of worth-while entertainment.

4. Geometric forms

Uses of similar triangles in practical situations such as in photography, the study of physics and scale drawing are almost numberless.

A program on ''Curves other than Circles'' proves exceptionally interesting. The pupils will all be surprised to learn that there are many different curves. The parabola, ellipse, hyperbola, catenary, and cycloid curves may be discussed in a non-technical manner and simple ways of constructing the ellipse and parobola may be explained. The members of the club will be most interested to learn about the curve of the headlight on an automobile, the airplane search light, a chain swing, and the path of a stone which has been thrown.

Roll call at a number of meetings may be responded to by describing triangles, trapezoids, circles, and curves other than circles seen outdoors.

5. Recreations

Catch problems and puzzles are innumerable and many of them are instructional. If they are not used in a miscellaneous group but are grouped so that there is a central idea for the program, they are more interesting. For instance circle puzzles may be the topic at one meeting and catch problems in algebra or arithmetic at another meeting. If these problems are worked in a group contest rather than in an individual contest the interest is greater because the method is farther removed from class procedure.

A guessing contest on estimation of lengths, areas, volumes, or weights is good practice. The length of the room, the blackboard, the door, and objects seen outdoors may be used.

Paper folding may be enjoyed for one or more programs and may involve considerable geometry.

A treasure hunt may be so conducted as to apply the ideas of the topic of locus. Directions may be given for finding a treasure hid outdoors. Pins may be hid within the room and locus problems which determine their location written on the board.

6. Special days

Programs may be made to fit special days. Just before Christmas a ''star'' program is appropriate. Club members may learn to construct accurate stars. Problems can be found relating to stars as this one: How can one plant 10 trees in 5 rows with 4 in each row? This program may include information concerning the mathematics used in astronomy.

A Hallowe'en program or a club party may feature a black cat contest in which queer cat-problems (found in *Boy's Own Arithmetic* and elsewhere) can be written in white ink on black cats. A bit of diversion may be furnished by guessing the number of seeds in a pumpkin.

7. Rapid calculation

Speed tests in computation are occasionally interesting. Short cuts may be learned. The advantages of the metric system may be brought out through debates, contest, or play as well as by a report of its history.

Calculating instruments of all sorts may be studied and used.

8. The use of mathematics for various school subjects and for various occupations

A program may be devoted to the mathematics needed in high school physics. The teacher of physics or a member of the club studying physics may give the report. Chemistry, shop, engineering, commerce, and other studies may also be included.

A lantern slide program may be made from graphs used in business. Some of these may be local. Opportunities other than in engineering or teaching, offered to pupils of mathematics, will be a surprise to all. The girls will be interested in women mathematicians. (See the *American Mathematical Monthly*, 1918)

Such material for the club may be gradually accumulated. To begin a club only several books and magazine articles are essential.

See

1. Breslich, *The Teaching of Mathematics in Secondary Schools*, Chapter III, "Arousing and Maintaining Interest"
2. Smith and Reeve, *The Teaching of Junior High School Mathematics*, Chapter XIII, "Mathematics Clubs and Contests"
3. Reed, Zula, "High School Mathematics Clubs," *The Mathematics Teacher*, October, 1925. Single copies 40c. An excellent discussion of mathematics clubs with sample programs, tricks, songs, games and a pageant, also a very complete bibliography
4. Woodring and Sanford, *Source Book*. This book contains a bibliography of material suitable for almost every program here suggested. It should be in the possession of every mathematics teacher

MATHEMATICS PLAYS

Appeal to the dramatic instincts of pupils by putting on a play. A number of excellent ones are available. The following list is suggestive

I. *School Science and Mathematics*, 1439 Fourteenth St., Milwaukee, Wis.
   1. Vol. 14, 1914, p. 583. "Flatland"
   2. Vol. 16, 1916, p. 39. "A Living Theorem"
   3. Vol. 17, 1917, p. 152. "A Fairy Tale"
   4. Vol. 17, 1917, p. 475. "A Mathematical Victory"
   5. Vol. 18, 1918, p. 611. "A Mock Trial of B vs. A" or "Solving a Personal Equation by Judicial Process"
   6. Vol. 20, 1920, p. 457. "The Mathematics Quest"
   7. Vol. 21, 1921, p. 381. "Euclid Dramatized"
   8. Vol. 23, 1923, p. 581, "Socrates Teaches Mathematics"

II. *The Mathematics Teacher*, 525 W. 120 St., New York
   1. Vol 17, 1924, p. 154, "Its Easy—If"
   2. Vol. 17, 1924, p. 336. "A Little Journey to the Land of Mathematics"
   3. Vol. 21, 1928, p. 92. "Geometry Humanized"
   4. Vol. 20, 1927, p. 389. "Falling in Love with Plain Geometry"
   5. Vol. 20, 1927, p. 459. "Mathesis"

6. Vol. 21, 1928, p. 305. "The Evolution of Numbers—An Historical Drama in Two Acts", H. E. Slaught
7. Vol. 22, 1929, p. 413. "A Mathematical Nightmare", Josephine Skerrett
8. Vol. 22, 1929, p. 472. "A Near Tragedy", Florence Brooks
9. Vol. 22, 1929, p. 472. "If", Ruth L. Snyder

See also Woodring and Sanford *Source Book*, p. 75

These plays are usually short, and easily costumed and presented, thus making them available for a club program. Some of them afford an evening's entertainment. All of them are suitable for arousing interest by presenting new possibilities in mathematics

COLLEGE ENTRANCE REQUIREMENTS IN MATHEMATICS
(Units)

Institution	Division	Algebra	Geometry	Geometry	Trigonometry
Amherst	Arts	2½	1	½	½
Buena Vista	Arts	1	1		
Coe	Arts	1	1		
Columbia College	Arts	1	1		
Columbia University	Arts	2½	1	½	½
	Eng.	2½	1	½	½
	Educ.	2½	1	½	½
Cornell College	Arts	1	1		
	Music	1½	1		
Cornell University	Agri.	1	1		
	Arts	1	1		
	Eng.	2	1	½	½
	Vet. Med.	1	1		
	Sciences	2	1	½	½
	(Not more than 4 units from the group)				
Drake University	Arts	1½	1	½	½ Adv. Arith ½
Grinnell	Arts	2½ or 3 units			
Harvard University	Arts	2	1		
Iowa State College	Agr.	1	1		
	Eng.	1½	1	½	
	H. Econ.	1	1		
Iowa State Teachers Col.	Educ.	1	1		
Kansas State	Agr.	1	1		
	Eng.	1½	1	½	
	H. Econ.	1	1		
	Sciences	1½	1		
Luther	Arts	Two units of mathematics			
Michigan State	Agr.	1	1		
	Arts	1	1		
	Eng.	1½	1	½	
Morningside	Arts	1	1		

Institution	Division	Algebra	Geometry	Geometry	Trigonometry
Northwestern University	Arts	1	1		
	Eng.	1½	1	One-half unit from college Alg., Sol. Geom., Pl. Geom.	
Ohio State	Agr.	1	1		
	Arts	1	1		
	Eng.	1½	1	½	
	Com. and Adm.	1	1		
	Educ.	1	1		
Parsons	Arts	Recommended but not required			
Princeton University	Arts	2	1		
	Eng.	2	1	½	½
	Sciences	2	1	½	½
Simpson	Arts	1	1		
Smith	Arts	4	2		
University of Chicago	For any Bachelor's Degree, three or more units See catalogue				
University of Colorado	Arts	1	1		
	Eng.	1½	1	½	
University of Illinois	Agr.	1	1		
	Arts	1	1		
	Eng.	1½	1	1	½
	H. Econ.	1	1		
	Com. and Adm.	1½	1	½	
	Vet. Med.	1	1		
	Sciences	1	1		
	Music	1	1		
University of Iowa	Arts	1	1		
	Eng.	1½	1	½	
University of Michigan	Arts	1	1		
	Eng.	1½	1½		
University of Minnesota	Arts	1	1		
	Eng.	1	1	½	
University of Nebraska	Agr.	1	1		
	Arts	1	1		
	Eng.	1½	1	½	
University of N. Dakota	Arts	One unit mathematics			
	Eng.	1½	1	½	
University of S. Dakota	Arts	1	1		
University of Wisconsin	Arts	1	1		
	Eng.	1½ or 2	1	½	
Upper Iowa	Arts	1	1		
	Eng.	1½	1	1	
Wesley	Arts	2	1		
Yale	Arts	2	1		
	Sciences	2	1		

BIBLIOGRAPHY

No effort has been made to include in this bibliography a complete list of titles. For additional items the teacher is referred to the complete lists found in some of the books listed below.

The files of periodicals like *The Mathematics Teacher* and *School Science and Mathematics* are filled with material of great value to which it has not been possible to refer. See the complete bibliographies for this information.

Such bibliographies are found in the Sanford and Woodring *Source Book;* Breslich's *The Teaching of Mathematics in Secondary Schools;* and in Smith and Reeve's *The Teaching of Junior High School Mathematics.*

The list published here has been selected with care as a representative group of recent publications helpful to high school teachers and pupils.

Sanford, Vera, and Woodring, Maxie N., *Enriched Teaching of Mathematics in the High School.* A source book for teachers of mathematics, listing chiefly free and low cost illustrative and supplementary materials. Bureau of Publications, Teachers College, Columbia University, New York City, 1928. This source book should be available for every teacher of high school mathematics. It includes: I. Materials for the study of special topics; II. Extra-curricular activities; III. Practice tests and exercises; IV. Materials primarily for teachers; V. Equipment for the mathematics classroom. There are suggestions for assembly programs, mathematics clubs, motion pictures, lantern slides and plays. It is a unique and invaluable teacher's assistant.

Breslich, Ernst R., *The Teaching of Mathematics in Secondary Schools,* Volume I, "Technique". The University of Chicago Press, 1930. Modern tendencies in the teaching of mathematics and in the selection and organization of the instruction materials. Excellent bibliographies at the end of each chapter

Volume II, 1931. Problems in Teaching Secondary School Mathematics

Smith, David Eugene, and Reeve, William David, *The Teaching of Junior High School Mathematics,* Ginn and Company, Chicago, 1927. Teaching of intuitive and demonstrative geometry. Model lessons, tests, mathematics clubs, mathematical reactions, complete bibliographies

Young, J. W. A., *The Teaching of Mathematics in Elementary and Secondary Schools* (Rev. ed.), Longmans Green and Company, New York City, 1924

Smith, David Eugene, and others, *A Brief List of Mathematical Books Suitable for Libraries in High Schools and Normal Schools,* Bureau of Publications, Teachers College, Columbia University, New York City, 1925

National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education,* Houghton Mifflin Company, Boston, Mass. This is a reprint of the essential parts of the 1923 report which has been the basis for curriculum revision since its publication

McCormick, Clarence, *The Teaching of General Mathematics in the Secondary Schools of the United States,* Bureau of Publications, Teachers College, Columbia University, New York City, 1929. A study of the reorganization of mathematics courses during the first quarter century

Yearbooks of the National Council of Teachers of Mathematics

a. First Yearbook, 1926. "A General Survey of Progress in the Last Twenty-five Years." Out of print

b. Second Yearbook, 1927. "Curriculum Problems in Teaching Mathematics." Chapters on Arithmetic, Junior High School Mathematics, and Senior High School Mathematics

c. Third Yearbook, 1928. "Selected Topics in the Teaching of Mathematics" Fourteen chapters on timely subjects by expert teachers

d. Fourth Yearbook, 1929. "Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910." Mathematics in thirteen of the leading countries of the world

e. Fifth Yearbook, 1930. "The Teaching of Geometry." Fourteen chapters by different writers who are authorities in their several fields. All phases of the subject are covered according to the most recent developments

f. Sixth Yearbook, 1931. Mathematics in Modern Life

These yearbooks are a library in themselves. All except the 1926 issue can be secured from The Bureau of Publications, Teachers College, Columbia University, New York City

GEOMETRY

National Council of Teachers of Mathematics, *Fifth Yearbook*, "The Teaching of Geometry", Bureau of Publications, Teachers College, Columbia University, New York City, 1930. New, vital, stimulating

Smith, David Eugene, *The Teaching of Geometry*, Ginn and Company, Chicago, 1911. "For the recognition of geometry for geometry's sake and not for the sake of a fancied utility that hardly exists"

ALGERBA

Thorndike, E. L., and others, *The Psychology of Algebra*, The Macmillan Company, Chicago, 1923. Should be read by every teacher of mathematics

Ligda, Paul, *The Teaching of Elementary Algebra*, Houghton Mifflin Company, Chicago, 1925. A practical and thought stimulating discussion by a teacher with industrial experience as well as that of the schoolroom

Everett, John Phelps, *The Fundamental Skills of Algebra*, Bureau of Publications, Teachers College, Columbia University, New York City, 1928. A Study of the value of "associative" as compared with "manipulative" skills in ninth year algebra. Whether one agrees with the author's conclusions or not this book will repay a careful reading

Sanford, Vera, *The History and Significance of Certain Standard Problems in Algebra*, Bureau of Publications, Teachers College, Columbia University, New York City, 1927. The reasons for the use of verbal problems; the history of certain typical problems; the place of problems in our present curriculum. A valuable contribution

Daily, Benjamin W., *The Ability of High School Pupils to Select Essential Data in Solving Problems*, Bureau of Publications, Teachers College, Columbia University,

New York City, 1925. To what extent are problems with superfluous and insufficient data found in our textbooks? What is the effect of such problems?

Periodicals

The American Mathematical Monthly, Mathematical Association of America, W. D. Cairns, Secretary, Oberlin, Ohio. Primarily for collegiate mathematics but abounds with materials of value to high school teachers

The Mathematics Teacher, 525 West 120th Street, New York City. The official organ of the National Council of Teachers of Mathematics. Eight issues each year. Devoted to mathematics in elementary and secondary schools. For high school teachers and pupils

School Science and Mathematics, 1439 Fourteenth Street, Milwaukee, Wisconsin. The official organ of the Central Association of Science and Mathematics Teachers. For high school teachers and pupils

History

Smith, David Eugene, *History of Mathematics,* Vol. I, 1923, Vol. II, 1925, Ginn and Company, Chicago. Should be in every high school library

Sanford, Vera, *A Short History of Mathematics,* Houghton Mifflin Company, Chicago, 1930. "The practical uses of mathematics as well as the history of the development of its theory. Of interest to teachers, pupils, and general readers

Smith, David Eugene, *Mathematics, Our Debt to Greece and Rome,* Longmans Green and Company, Chicago, 1923. Written in popular style and very interesting

Karpinski, Louis Charles, *The History of Arithmetic,* Rand McNally and Company, Chicago, 1925. Well illustrated and very readable

Smith, David Eugene, and Karpinski, Louis Charles, *The Hindu-Arabic Numerals,* Ginn and Company, Chicago, 1911

Smith, David Eugene, *Number Stories of Long Ago,* Ginn and Company, Chicago, 1919

Mathematical Monographs

Rupert, W. W., *Famous Geometrical Theorems,* D. C. Heath and Company, Chicago

Mathematical Recreations

Jones, S. I., *Mathematical Wrinkles,* Published by the author, Nashville, Tenn., 1923

Weeks, Raymond, *The Boy's Own Arithmetic,* E. P. Dutton and Company, New York City, 1924. A take-off on problems in arithmetic and algebra

Ball, W. W. R., *Mathematical Recreations and Essays,* The Macmillan Company, Chicago. A standard work on the subject

Abbott, E. A., *Flatland: A Romance of Many Dimensions*, Little Brown and Company, New York City. A classic speculation about existence in a world consisting of a line and a world consisting of a plane. Sets one to thinking about other than three dimension space

TESTS AND MEASUREMENTS

Reeve, William David, *A Diagnostic Study of the Teaching Problems in High School Mathematics*, Ginn and Company, Chicago, 1926. To encourage a better use of tests in mathematics, and particularly of the kinds of tests which can be used by teachers to improve their instruction

ALGEBRA

1. Practice exercises

   a. Sykes, Mabel, and Comstock, Clarence E., CBA Hurdle Tests, Rand McNally Company, Chicago. Nineteen booklets covering all essential features of algebra

   b. Smith, D. E.; Reeve, W. D.; Morse, E. L., Exercises and Tests in Algebra Through Quadratics, Ginn and Company, Chicago, 1926. Two hundred and twenty-four exercises

   c. Schorling, Raleigh; Clark, John R.; Lindell, Selma, Instructional Tests in Algebra, World Book Company, Chicago, 1925. Tests in ninth grade algebra

   d. Fowlkes-Kingsburg-Wallace-Goff, Algebra Word Book, The Macmillan Company, Chicago, 1928. A booklet of one hundred tests with full directions for checking. Covers all essential topics through quadratics

   e. Gonnelly-Huff-Studabaker, Practice Exercises in Algebra, Scott Foresman and Company, Chicago. On cards like the Studebaker Economy Practice. Exercises in arithmetic, exercises in self checking

2. Tests

   a. Otis, Arthur S., and Wood, Ben D., Columbia Research Bureau Algebra Test, World Book Company, Chicago, 1927. Part I, Mechanical Operations; Part II, Problem Solving

   b. Rugg, Harold O., and Clark, John R., Standardized Tests in First Year Algebra, World Book Company, Chicago. Twelve tests including radicals

   c. Hotz, Henry G., First Year Algebra Scales, Public School Publishing Company, Bloomington, Illinois, 1918. Five tests on fundamental operations; equations; problems; graphs

   d. Orleans, J. B., and Orleans, J. S., Algebra Prognostic Test, World Book Company, Chicago, 1928. Twelve lessons with tests to forecast a pupil's ability to succeed in algebra

   e. Illinois Standardized Algebra Test, Public School Publishing Company, Bloomington, Illinois. In four parts each containing twenty exercises. Based on the simple equation

Geometry

1. Diagnostic

   a. Smith, D. E.; Reeve, W. D.; Morse, E. L., Exercises and Tests in Plane Geometry, Ginn and Company, Chicago, 1928. One hundred and sixty tests covering a year's work

   b. Minnick, J. H., Geometry Tests. A. Drawing figures; B. Stating what is given and what is to be proved; C. Stating facts about given figures; D. Organizing proof from data

   c. Orleans, J. B., and Orleans, J. S., Geometry Prognosis Test, World Book Company, Chicago, 1928. Ten lessons with tests on each lesson. To show whether the pupil can learn geometry easily

2. Achievement

   a. Lane, Ruth O., and Greene, Harry A., Unit-Achievement Tests in Plane Geometry, Ginn and Company, Chicago, 1929. Six tests—two forms for each test. Class record, Key, and Manual

   b. Hawkes, H. E., and Wood, Ben D., Columbia Research Bureau Plane Geometry Test, World Book Company, Chicago, 1926. For a final examination in plane geometry. Manual of directions

   c. Schorling, Raleigh, and Sanford, Vera, Achievement Tests in Plane Geometry, Bureau of Publications, Teachers College, Columbia University, 1925. A new type final examination covering the five books of plane geometry

   d. Raudenbush, H. W., Siceloff, L. P., and Wood, Ben D., American Council Solid Geometry Test, World Book Company, Chicago, 1928. True-false and completion tests, Key and record sheet

3. Practice

   a. Welte, H. D., and Knight, F. B., Standard Service Geometry Work Book, Scott Foresman and Company, Chicago, 1929. Five preliminary lessons, thirty drills, five discussion units, progress chart

   b. Lane, Ruth O.; Knight, F. B.; and Ruch, G. M., Geometry Rapid Drill Cards, Scott Foresman and Company, Chicago, 1929. One hundred and forty-three cards for supplementary drill, with complete directions

Mathematics and Physics

Kilzer, L. R., and Kirby, T. J., Inventory Test for the Mathematics of High School Physics, Public School Publishing Company, Bloomington, Illinois, 1928